11. $x''_{tt} + a_1 x'_t + b_1 y'_t + c_1 x + d_1 y = k_1 e^{i\omega t}, \quad y''_{tt} + a_2 x'_t + b_2 y'_t + c_2 x + d_2 y = k_2 e^{i\omega t}.$

Similar systems are frequent in the theory of oscillations (e.g., movement of a ship or a seaborne gyroscope). The general solution of this linear nonhomogeneous system of constant-coefficient differential equations is given by the sum of its particular solution and the general solution of the corresponding homogeneous system (with $k_1 = k_2 = 0$).

1°. A particular solution is sought by the method of undetermined coefficients:

$$x = A_* e^{i\omega t}, \quad y = B_* e^{i\omega t}.$$

On substituting these expressions into the original system of differential equations, one arrives at a linear nonhomogeneous system of algebraic equations for the coefficients A and B.

2°. The general solution of the homogeneous system of differential equations is determined by a linear combination of linearly independent particular solutions sought by the method of undetermined coefficients in the form of exponentials:

$$x = A e^{\lambda t}, \quad y = B e^{\lambda t}.$$

On substituting these expressions into the original system and collecting the coefficients of the unknowns A and B, one obtains

$$(\lambda^2 + a_1\lambda + c_1)A + (b_1\lambda + d_1)B = 0,$$
$$(a_2\lambda + c_2)A + (\lambda^2 + b_2\lambda + d_2)B = 0.$$

The determinant of this system must vanish for nontrivial solutions A, B to exist. This requirement results in the following characteristic equation for λ:

$$(\lambda^2 + a_1\lambda + c_1)(\lambda^2 + b_2\lambda + d_2) - (b_1\lambda + d_1)(a_2\lambda + c_2) = 0.$$

If all roots, $k_1, \ldots, k_4$, of this equation are distinct, the general solution of the original system of differential equations has the form

$$\begin{aligned} x &= -C_1(b_1\lambda_1 + d_1)e^{\lambda_1 t} - C_2(b_1\lambda_2 + d_1)e^{\lambda_2 t} - C_3(b_1\lambda_1 + d_1)e^{\lambda_3 t} - C_4(b_1\lambda_4 + d_1)e^{\lambda_4 t}, \\ y &= C_1(\lambda_1^2 + a_1\lambda_1 + c_1)e^{\lambda_1 t} + C_2(\lambda_2^2 + a_1\lambda_2 + c_1)e^{\lambda_2 t} \\ &\quad + C_3(\lambda_3^2 + a_1\lambda_3 + c_1)e^{\lambda_3 t} + C_4(\lambda_4^2 + a_1\lambda_4 + c_1)e^{\lambda_4 t}, \end{aligned}$$

References

Matveev, N. M., *Methods of Integration of Ordinary Differential Equations* [in Russian], Vysshaya Shkola, Moscow, 1963.

Kamke, E., *Differentialgleichungen: Lösungsmethoden und Lösungen, I, Gewöhnliche Differentialgleichungen,* B. G. Teubner, Leipzig, 1977.

