Systems of Ordinary Differential Equations > Linear Systems of Two Equations

16. \[ t^2 x'' + a_1 tx' + b_1 ty' + c_1 x + d_1 y = 0, \quad t^2 y'' + a_2 tx' + b_2 ty' + c_2 x + d_2 y = 0. \]

Linear system homogeneous in the independent variable (Euler type).

1°. The general solution is determined by a linear combination of linearly independent particular solutions sought in the form of power-law functions,

\[ x = A|t|^k, \quad y = B|t|^k, \]

by the method of undetermined coefficients. On substituting these expressions into the original system and collecting coefficients of \( A \) and \( B \), one obtains a system for \( A \) and \( B \):

\[
\begin{align*}
[k^2 + (a_1 - 1)k + c_1]A + (b_1 k + d_1)B &= 0, \\
(a_2 k + c_2)A + [k^2 + (b_2 - 1)k + d_2]B &= 0.
\end{align*}
\]

The determinant of this system must vanish for nontrivial solutions to exist. Hence follows the characteristic equation for the exponent \( k \):

\[
[k^2 + (a_1 - 1)k + c_1][k^2 + (b_2 - 1)k + d_2] - (b_1 k + d_1)(a_2 k + c_2) = 0.
\]

If the roots of this equation, \( k_1, \ldots, k_4 \), are all distinct, then the general solution of the original system of differential equations is expressed as

\[
\begin{align*}
x &= -C_1(b_1 k_1 + d_1)|t|^{k_1} - C_2(b_1 k_2 + d_1)|t|^{k_2} - C_3(b_1 k_1 + d_1)|t|^{k_3} - C_4(b_1 k_4 + d_1)|t|^{k_4}, \\
y &= C_1[k_1^2 + (a_1 - 1)k_1 + c_1]|t|^{k_1} + C_2[k_2^2 + (a_1 - 1)k_2 + c_1]|t|^{k_2} \\
&\quad + C_3[k_3^2 + (a_1 - 1)k_3 + c_1]|t|^{k_3} + C_4[k_4^2 + (a_1 - 1)k_4 + c_1]|t|^{k_4},
\end{align*}
\]

where \( C_1, \ldots, C_4 \) are arbitrary constants.

2°. The substitution \( t = \sigma e^r \) (\( \sigma \neq 0 \)) leads to the system of constant-coefficient linear differential equations

\[
\begin{align*}
x'' + (a_1 - 1)x' + b_1 t y' + c_1 x + d_1 y &= 0, \\
y'' + a_2 x' + (b_2 - 1)y' + c_2 x + d_2 y &= 0.
\end{align*}
\]