Systems of Ordinary Differential Equations > Linear Systems of Two Equations

18. \( x''_t = f(t)(tx'_t - x) + g(t)(ty'_t - y), \quad y''_t = h(t)(tx'_t - x) + p(t)(ty'_t - y). \)

The transformation
\[
\begin{align*}
u &= tx'_t - x, \\
v &= ty'_t - y
\end{align*}
\]
leads to the linear system of first-order equations
\[
\begin{align*}
u'_t &= tf(t)u + tg(t)v, \\
v'_t &= th(t)u + tp(t)v.
\end{align*}
\]

In order to obtain its general solution, it suffices to know a particular solution of this system (see equation 1.7 in the current section).

For solutions to some systems of the form 92), see equations 1.3–1.6.

If all the functions in (2) are proportional,
\[
\begin{align*}f(t) &= a\varphi(t), \\
g(t) &= b\varphi(t), \\
h(t) &= c\varphi(t), \\
p(t) &= d\varphi(t),
\end{align*}
\]
then the introduction of the new independent variable \( \tau = \int t\varphi(t)\,dt \) leads to a constant-coefficient system of the form 1.1.

2°. Suppose a solution of system (2),
\[
\begin{align*}u &= u(t, C_1, C_2), \\
v &= v(t, C_1, C_2),
\end{align*}
\]
where \( C_1 \) and \( C_2 \) are arbitrary constants, is known. Then, on substituting (3) into (1) and on integrating the resulting equation, one obtains a solution of the original system in the form
\[
\begin{align*}x &= C_3t + t\int \frac{u(t, C_1, C_2)}{t^2} \,dt, \\
y &= C_4t + t\int \frac{v(t, C_1, C_2)}{t^2} \,dt.
\end{align*}
\]
where \( C_3 \) and \( C_4 \) are arbitrary constants.