8. $x''_{tt} = \dfrac{\partial F}{\partial x}, \quad y''_{tt} = \dfrac{\partial F}{\partial y}, \quad z''_{tt} = \dfrac{\partial F}{\partial z},$ **where** $F = F(r), \quad r = \sqrt{x^2 + y^2 + z^2}.$

Equations of motion of a point mass under gravitational force.

The equations can be rewritten in the vector form

$$\mathbf{r}''_{tt} = \operatorname{grad} F \qquad \text{or} \qquad \mathbf{r}''_{tt} = \frac{F'(r)}{r}\mathbf{r},$$

where $\mathbf{r} = (x, y, z)$.

1°. First integrals:

$$(\mathbf{r}'_t)^2 = 2F(r) + C_1 \quad \text{(law of conservation of energy)},$$
$$[\mathbf{r} \times \mathbf{r}'_t] = \mathbf{C} \quad \text{(law of conservation of areas)},$$
$$(\mathbf{r} \cdot \mathbf{C}) = 0 \quad \text{(trajectories are plane curves)}.$$

2°. Solution:

$$\mathbf{r} = \mathbf{a}\, r \cos\varphi + \mathbf{b}\, r \sin\varphi.$$

Here, the constant vectors **a** and **b** must satisfy the conditions

$$|\mathbf{a}| = |\mathbf{b}| = 1, \qquad (\mathbf{a} \cdot \mathbf{b}) = 0,$$

and the functions $r = r(t)$ and $\varphi = \varphi(t)$ are defined by the relations

$$t = \int \frac{r\,dr}{\sqrt{2r^2 F(r) + C_1 r^2 - C_3^2}} + C_2, \qquad \varphi = C_3 \int \frac{dr}{r\sqrt{2r^2 F(r) + C_1 r^2 - C_3^2}}, \qquad C_3 = |\mathbf{C}|,$$

where C_1, C_2, and C_3 are arbitrary constants.

Reference

Kamke, E., *Differentialgleichungen: Lösungsmethoden und Lösungen, I, Gewöhnliche Differentialgleichungen*, B. G. Teubner, Leipzig, 1977.

