2. $\dfrac{\partial u}{\partial t}=a\dfrac{\partial^2 u}{\partial x^2}+uf(bu-cw)+g(bu-cw),\qquad \dfrac{\partial w}{\partial t}=a\dfrac{\partial^2 w}{\partial x^2}+wf(bu-cw)+h(bu-cw).$

1°. Solution:

$$u=\varphi(t)+c\exp\left[\int f(b\varphi-c\psi)\,dt\right]\theta(x,t),$$
$$w=\psi(t)+b\exp\left[\int f(b\varphi-c\psi)\,dt\right]\theta(x,t),$$

where $\varphi=\varphi(t)$ and $\psi=\psi(t)$ are determined by the system of ordinary differential equations

$$\varphi'_t=\varphi f(b\varphi-c\psi)+g(b\varphi-c\psi),$$
$$\psi'_t=\psi f(b\varphi-c\psi)+h(b\varphi-c\psi),$$

and the function $\theta=\theta(x,t)$ satisfies linear heat equation

$$\frac{\partial\theta}{\partial t}=a\frac{\partial^2\theta}{\partial x^2}.$$

2°. Let add together the first equation multiplied by b and the second equation multiplied by $-c$ to obtain

$$\frac{\partial\zeta}{\partial t}=a\frac{\partial^2\zeta}{\partial x^2}+\zeta f(\zeta)+bg(\zeta)-ch(\zeta),\qquad \zeta=bu-cw. \tag{1}$$

This equation will be treated in conjunction with the first equation of the original system,

$$\frac{\partial u}{\partial t}=a\frac{\partial^2 u}{\partial x^2}+uf(\zeta)+g(\zeta). \tag{2}$$

Equation (1) can be treated separately. An extensive list of exact solutions to this sort of equations for various kinetic functions $F(\zeta)=\zeta f(\zeta)+bg(\zeta)-ch(\zeta)$ can be fount in the “Handbook of Nonlinear Partial Differential Equations” by A. D. Polyanin & V. F. Zaitsev (2004). Given a solution $\zeta=\zeta(x,t)$ of equation (1), the function $u=u(x,t)$ can be found by solving the linear equation (2) and the function $w=w(x,t)$ is determined by the formula $w=(bu-\zeta)/c$.

Note three important solutions of equation (1).

(i) In the general case, equation (1) admits a spatially homogeneous solution $\zeta=\zeta(t)$. The corresponding solution of the original system is presented in Item 1° in another form.

(ii) In the general case, equation (1) admits a traveling-wave solution $\zeta=\zeta(z)$ with $z=kx-\lambda t$; the corresponding exact solutions of equation (2) have the form $u=u_0(z)+\sum e^{\beta_n t}u_n(z)$.

(iii) If the condition $\zeta f(\zeta)+bg(\zeta)-ch(\zeta)=k_1\zeta+k_0$ is satisfied, equation (1) is linear,

$$\frac{\partial\zeta}{\partial t}=a\frac{\partial^2\zeta}{\partial x^2}+k_1\zeta+k_0,$$

and is reduced to a linear heat equation.

Reference

Polyanin, A. D., Exact solutions of nonlinear systems of reaction-diffusion equations and mathematical biology equations, *Theor. Found. Chem. Eng.*, Vol. 37, No. 6, 2004.