4. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + uf\left(\dfrac{u}{w}\right), \quad \dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + wg\left(\dfrac{u}{w}\right).$

Suppose $u = u(x,t)$, $w = w(x,t)$ is a solution of the system. Then the functions

$$u_1 = Au(\pm x + C_1,\ t + C_2), \qquad w_1 = Aw(\pm x + C_1,\ t + C_2);$$
$$u_2 = \exp(\lambda x + a\lambda^2 t)u(x + 2a\lambda t,\ t), \quad w_2 = \exp(\lambda x + a\lambda^2 t)w(x + 2a\lambda t,\ t),$$

where A, C_1, C_2, and λ are arbitrary constants, are also solutions of this equations.

1°. Multiplicative separable solution:

$$u = [C_1 \sin(kx) + C_2 \cos(kx)]\varphi(t),$$
$$w = [C_1 \sin(kx) + C_2 \cos(kx)]\psi(t),$$

where C_1, C_2, and k are arbitrary constants, and the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the system of ordinary differential equations

$$\varphi'_t = -ak^2\varphi + \varphi f(\varphi/\psi),$$
$$\psi'_t = -ak^2\psi + \psi g(\varphi/\psi).$$

2°. Multiplicative separable solution:

$$u = [C_1 \exp(kx) + C_2 \exp(-kx)]U(t),$$
$$w = [C_1 \exp(kx) + C_2 \exp(-kx)]W(t),$$

where C_1, C_2, and k are arbitrary constants, and the functions $U = U(t)$ and $W = W(t)$ are determined by the system of ordinary differential equations

$$U'_t = ak^2U + Uf(U/W),$$
$$W'_t = ak^2W + Wg(U/W).$$

3°. Degenerate solution:

$$u = (C_1x + C_2)U(t),$$
$$w = (C_1x + C_2)W(t),$$

where C_1 and C_2, and the functions $U = U(t)$ and $W = W(t)$ are determined by the system of ordinary differential equations

$$U'_t = Uf(U/W),$$
$$W'_t = Wg(U/W).$$

This autonomous system can be integrated, since it is reduced, on eliminating t, to a homogeneous first-order equation (the corresponding systems of Items 1° and 2° can be integrated likewise).

4°. Multiplicative separable solution:

$$u = e^{-\lambda t}y(x), \qquad w = e^{-\lambda t}z(x),$$

where λ is an arbitrary constant and the functions $y = y(x)$ and $z = z(x)$ are determined by the system of ordinary differential equations

$$ay''_{xx} + \lambda y + yf(y/z) = 0,$$
$$az''_{xx} + \lambda z + zg(y/z) = 0.$$

5°. Solution(generalizes the solution of Item 3°):

$$u = e^{kx-\lambda t}y(\xi), \qquad w = e^{kx-\lambda t}z(\xi), \qquad \xi = \beta x - \gamma t,$$

where k, λ, β, and γ are arbitrary constants, and the functions $y = y(\xi)$ and $z = z(\xi)$ are determined by the system of ordinary differential equations

$$a\beta^2 y''_{\xi\xi} + (2ak\beta + \gamma)y'_\xi + (ak^2 + \lambda)y + yf(y/z) = 0,$$
$$a\beta^2 z''_{\xi\xi} + (2bk\beta + \gamma)z'_\xi + (bk^2 + \lambda)z + zg(y/z) = 0.$$

To the special case $k = \lambda = 0$ there corresponds a traveling-wave solution. The case of $k = \gamma = 0$ and $\beta = 1$ corresponds to the solution of Item 3°.

6°. Solution of point-source type:

$$u = \exp\left(-\frac{x^2}{4at}\right)\varphi(t), \quad w = \exp\left(-\frac{x^2}{4at}\right)\psi(t),$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the system of ordinary differential equations

$$\varphi'_t = -\frac{1}{2t}\varphi + \varphi f\left(\frac{\varphi}{\psi}\right),$$
$$\psi'_t = -\frac{1}{2t}\psi + \psi g\left(\frac{\varphi}{\psi}\right).$$

7°. Functional separable solution:

$$u = \exp\bigl(kxt + \tfrac{2}{3}ak^2t^3 - \lambda t\bigr)y(\xi), \quad \xi = x + akt^2,$$
$$w = \exp\bigl(kxt + \tfrac{2}{3}ak^2t^3 - \lambda t\bigr)z(\xi),$$

where k and λ are arbitrary constants, and the functions $y = y(\xi)$ and $z = z(\xi)$ are determined by the system of ordinary differential equations

$$ay''_{\xi\xi} + (\lambda - k\xi)y + yf(y/z) = 0,$$
$$az''_{\xi\xi} + (\lambda - k\xi)z + zg(y/z) = 0.$$

8°. Let k is a root of the algebraic (transcendental) equation

$$f(k) = g(k). \tag{1}$$

Solution:

$$u = ke^{\lambda t}\theta, \quad w = e^{\lambda t}\theta, \quad \lambda = f(k),$$

where the function $\theta = \theta(x,t)$ satisfies the linear heat equation

$$\frac{\partial\theta}{\partial t} = a\frac{\partial^2\theta}{\partial x^2}.$$

9°. Periodic solution:

$$u = Ak\exp(-\mu x)\sin(\beta x - 2a\beta\mu t + B), \quad \beta = \sqrt{\mu^2 + \frac{1}{a}f(k)},$$
$$w = A\exp(-\mu x)\sin(\beta x - 2a\beta\mu t + B),$$

where A, B, and μ are arbitrary constants, and k is a root of the algebraic (transcendental) equation (1).

10°. Solution:

$$u = \varphi(t)\exp\left[\int g(\varphi(t))\,dt\right]\theta(x,t), \quad w = \exp\left[\int g(\varphi(t))\,dt\right]\theta(x,t),$$

where the function $\varphi = \varphi(t)$ is determined by the separable nonlinear first-order ordinary differential equation

$$\varphi'_t = [f(\varphi) - g(\varphi)]\varphi, \tag{2}$$

and the function $\theta = \theta(x,t)$ satisfies the linear heat equation

$$\frac{\partial\theta}{\partial t} = a\frac{\partial^2\theta}{\partial x^2}.$$

To the particular solution $\varphi = k = \text{const}$ of equation (2) there corresponds the solution of Item 8°. The general solution of equation (2) is written out in implicit form as

$$\int\frac{d\varphi}{[f(\varphi) - g(\varphi)]\varphi} = t + C.$$

11°. The transformation

$$u = a_1 U + b_1 W, \quad w = a_2 U + b_2 W,$$

where a_n and b_n are arbitrary constants ($n = 1, 2$), leads to an equation of the similar form for U and W.

Reference

Polyanin, A. D., Exact solutions of nonlinear systems of reaction-diffusion equations and mathematical biology equations, *Theor. Found. Chem. Eng.*, Vol. 37, No. 6, 2004.

