11. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + uf\left(\dfrac{u}{w}\right)\ln u + ug\left(\dfrac{u}{w}\right), \quad \dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + wf\left(\dfrac{u}{w}\right)\ln w + wh\left(\dfrac{u}{w}\right).$

Solution:

$$u(x,t) = \varphi(t)\psi(t)\theta(x,t), \quad w(x,t) = \psi(t)\theta(x,t),$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by solving the first-order ordinary differential equations

$$\varphi'_t = \varphi[g(\varphi) - h(\varphi) + f(\varphi)\ln\varphi], \tag{1}$$

$$\psi'_t = \psi[h(\varphi) + f(\varphi)\ln\psi], \tag{2}$$

and the function $\theta = \theta(x,t)$ is determined by the differential equation

$$\frac{\partial\theta}{\partial t} = a\frac{\partial^2\theta}{\partial x^2} + f(\varphi)\theta\ln\theta. \tag{3}$$

The solution to the separable equation (1) can be represented in implicit form. Equation (2) is easy to integrate, since it is reduced, with the change of variable $\psi = e^\zeta$, to a linear equation. Equation (3) admits exact solutions of the form

$$\theta = \exp\bigl[\sigma_2(t)x^2 + \sigma_1(t)x + \sigma_0(t)\bigr],$$

where the functions $\sigma_n(t)$ are determined by the equations

$$\begin{aligned}\sigma'_2 &= f(\varphi)\sigma_2 + 4a\sigma_2^2,\\ \sigma'_1 &= f(\varphi)\sigma_1 + 4a\sigma_1\sigma_2,\\ \sigma'_0 &= f(\varphi)\sigma_0 + a\sigma_1^2 + 2a\sigma_2.\end{aligned} \tag{4}$$

This system can be integrated successively, since the first equation is a Bernoulli equation, and the second and the third ones are linear in the sought function. Note that the first equation has the particular solution $\sigma_2 = 0$.

Remark. Equation (1) admits a singular solution $\varphi = k = \text{const}$, where k is a root of the algebraic (transcendental) equation $g(k) - h(k) + f(k)\ln k = 0$.

Reference

Polyanin, A. D., Exact solutions of nonlinear systems of reaction-diffusion equations and mathematical biology equations, *Theor. Found. Chem. Eng.*, Vol. 37, No. 6, 2004.

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