23. $\displaystyle \frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf(u^2 - w^2) + wg\left(\frac{w}{u}\right), \qquad \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + ug\left(\frac{w}{u}\right) + wf(u^2 - w^2).$

Solution:

$$u = r(x,t)\cosh\varphi(t), \quad w = r(x,t)\sinh\varphi(t),$$

where the function $\varphi = \varphi(t)$ is determined by the autonomous ordinary differential equation

$$\varphi'_t = g(\tanh\varphi), \tag{1}$$

and the function $r = r(x,t)$ is determined by the differential equation

$$\frac{\partial r}{\partial t} = a\frac{\partial^2 r}{\partial x^2} + rf(r^2). \tag{2}$$

The general solution of equation (1) is expressed in implicit form as

$$\int \frac{d\varphi}{g(\tanh\varphi)} = t + C.$$

Equation (2) admits a traveling-wave solution $r = r(z)$ with $z = kx - \lambda t$, where k and λ are arbitrary constants, and the function $r(z)$ is determined by the autonomous ordinary differential equation

$$ak^2 r''_{zz} + \lambda r'_z + rf(r^2) = 0.$$

For other exact solutions of equation (2) for various f, see the "Handbook of Nonlinear Partial Differential Equations" by A. D. Polyanin and V. F. Zaitsev (2004).

Reference

Polyanin, A. D., Exact solutions of nonlinear systems of reaction-diffusion equations and mathematical biology equations, *Theor. Found. Chem. Eng.,* Vol. 37, No. 6, 2004.

