1. $\displaystyle \frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + uf(bu-cw) + g(bu-cw),$

$$\frac{\partial w}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + wf(bu-cw) + h(bu-cw).$$

1°. Solution:

$$u = \varphi(t) + c\exp\left[\int f(b\varphi - c\psi)\,dt\right]\theta(x,t),$$

$$w = \psi(t) + b\exp\left[\int f(b\varphi - c\psi)\,dt\right]\theta(x,t),$$

where $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the system of ordinary differential equations

$$\varphi'_t = \varphi f(b\varphi - c\psi) + g(b\varphi - c\psi),$$
$$\psi'_t = \psi f(b\varphi - c\psi) + h(b\varphi - c\psi),$$

and the function $\theta = \theta(x,t)$ satisfies the linear heat equation

$$\frac{\partial \theta}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial \theta}{\partial x}\right). \tag{1}$$

2°. Let us multiply the first equation by b and add it to the second equation multiplied by $-c$ to obtain

$$\frac{\partial \zeta}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial \zeta}{\partial x}\right) + \zeta f(\zeta) + bg(\zeta) - ch(\zeta), \qquad \zeta = bu - cw. \tag{2}$$

This equation will be treated in conjunction with the first equation of the original system,

$$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + uf(\zeta) + g(\zeta). \tag{3}$$

Equation (2) can be treated separately. Given a solution $\zeta = \zeta(x,t)$ of equation (1), the function $u = u(x,t)$ can be found by solving the linear equation (3), and the function $w = w(x,t)$ is determined by the formula $w = (bu - \zeta)/c$.

Note two important solutions of equation (2):

(i) In the general case, equation (2) admits stationary solutions $\zeta = \zeta(x)$; the corresponding exact solutions of equation (3) have the form $u = u_0(x) + \sum e^{\beta_n t}u_n(x)$.

(ii) If the condition $\zeta f(\zeta) + bg(\zeta) - ch(\zeta) = k_1\zeta + k_0$ holds, equation (2) is linear,

$$\frac{\partial \zeta}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial \zeta}{\partial x}\right) + k_1\zeta + k_0,$$

and is reduced to the linear heat equation (1) with the change of variable $\zeta = e^{k_1 t}\bar{\zeta} - k_0 k_1^{-1}$.

