3. $\dfrac{\partial u}{\partial t} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\left(x^n \dfrac{\partial u}{\partial x}\right) + uf\left(\dfrac{u}{w}\right), \quad \dfrac{\partial w}{\partial t} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\left(x^n \dfrac{\partial w}{\partial x}\right) + wg\left(\dfrac{u}{w}\right).$

1°. Multiplicative separable solution:

$$u = x^{\frac{1-n}{2}}[C_1 J_\nu(kx) + C_2 Y_\nu(kx)]\varphi(t), \quad \nu = \tfrac{1}{2}|n-1|,$$
$$w = x^{\frac{1-n}{2}}[C_1 J_\nu(kx) + C_2 Y_\nu(kx)]\psi(t),$$

where C_1, C_2, and k are arbitrary constants, $J_\nu(z)$ and $Y_\nu(z)$ are the Bessel functions, and the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the system of ordinary differential equations

$$\varphi'_t = -ak^2\varphi + \varphi f(\varphi/\psi),$$
$$\psi'_t = -ak^2\psi + \psi g(\varphi/\psi).$$

2°. Multiplicative separable solution:

$$u = x^{\frac{1-n}{2}}[C_1 I_\nu(kx) + C_2 K_\nu(kx)]\varphi(t), \quad \nu = \tfrac{1}{2}|n-1|,$$
$$w = x^{\frac{1-n}{2}}[C_1 I_\nu(kx) + C_2 K_\nu(kx)]\psi(t),$$

where C_1, C_2, and k are arbitrary constants, $I_\nu(z)$ and $K_\nu(z)$ are the modified Bessel functions, and the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the system of ordinary differential equations

$$\varphi'_t = ak^2\varphi + \varphi f(\varphi/\psi),$$
$$\psi'_t = ak^2\psi + \psi g(\varphi/\psi).$$

3°. Multiplicative separable solution:

$$u = e^{-\lambda t}y(x), \qquad w = e^{-\lambda t}z(x),$$

where λ is an arbitrary constant and the functions $y = y(x)$ and $z = z(x)$ are determined by the system of ordinary differential equations

$$ax^{-n}(x^n y'_x)'_x + \lambda y + yf(y/z) = 0,$$
$$ax^{-n}(x^n z'_x)'_x + \lambda z + zg(y/z) = 0.$$

4°. Let k is a root of the algebraic (transcendental) equation

$$f(k) = g(k).$$

Solution:

$$u = ke^{\lambda t}\theta, \quad w = e^{\lambda t}\theta, \quad \lambda = f(k),$$

where the function $\theta = \theta(x,t)$ satisfies the linear heat equation

$$\frac{\partial \theta}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n \frac{\partial \theta}{\partial x}\right). \qquad (1)$$

5°. Solution:

$$u = \varphi(t)\exp\left[\int g(\varphi(t))\,dt\right]\theta(x,t), \quad w = \exp\left[\int g(\varphi(t))\,dt\right]\theta(x,t),$$

where the function $\varphi = \varphi(t)$ satisfies the separable nonlinear first-order ordinary differential equation

$$\varphi'_t = [f(\varphi) - g(\varphi)]\varphi, \qquad (2)$$

and the function $\theta = \theta(x,t)$ satisfies the linear heat equation (1).

To the particular solution $\varphi = k = \text{const}$ to equation (2) there corresponds the solution of Item 4°. The general solution to equation (2) is written out in implicit form as

$$\int \frac{d\varphi}{[f(\varphi) - g(\varphi)]\varphi} = t + C.$$

