1. $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=uf(au-bw)+g(au-bw), \quad \dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=wf(au-bw)+h(au-bw).$

1°. Solution:

$$u=\varphi(x)+b\theta(x,y), \quad w=\psi(x)+a\theta(x,y),$$

where $\varphi=\varphi(x)$ and $\psi=\psi(x)$ are determined by the system of ordinary differential equations

$$\varphi''_{xx}=\varphi f(a\varphi-b\psi)+g(a\varphi-b\psi),$$
$$\psi''_{xx}=\psi f(a\varphi-b\psi)+h(a\varphi-b\psi),$$

and the function $\theta=\theta(x,y)$ satisfies a linear Schrödinger equation of the special form

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=F(x)\theta, \qquad F(x)=f(au-bw).$$

Its solutions are constructed by the method of separation of variables.

2°. Let us multiply the first equation by a and add it to the second equation multiplied by $-b$ to obtain

$$\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}=\zeta f(\zeta)+ag(\zeta)-bh(\zeta), \qquad \zeta=au-bw. \tag{1}$$

This equation will be treated in conjunction with the first equation of the original system

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=uf(\zeta)+g(\zeta). \tag{2}$$

Equation (1) can be treated separately. An extensive list of exact solutions to this sort of equations for various kinetic functions $F(\zeta)=\zeta f(\zeta)+ag(\zeta)-bh(\zeta)$ can be found in the "Handbook of Nonlinear Partial Differential Equations" by A. D. Polyanin & V. F. Zaitsev (2004).

Note two important cases:

(i) In the general case equation (1) admits a traveling-wave solution $\zeta=\zeta(z)$, where $z=k_1x+k_2y$ (k_1 and k_2 are arbitrary constants).

(ii) If the condition $\zeta f(\zeta)+ag(\zeta)-bh(\zeta)=c_1\zeta+c_0$ holds, equation (1) is a linear Helmholtz equation.

Given a solution $\zeta=\zeta(x,y)$ of equation (1), the function $u=u(x,y)$ can be found by solving the linear equation (2), and the function $w=w(x,y)$ is determined by the formula $w=(bu-\zeta)/c$.

