3. $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=uf\left(\dfrac{u}{w}\right),\quad \dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=wg\left(\dfrac{u}{w}\right).$

1°. Solution in the form of the product of functions with different arguments, periodic in the space variable (a similar solution can be obtained by swapping x and y):

$$u=[C_1\sin(kx)+C_2\cos(kx)]\varphi(y),$$
$$w=[C_1\sin(kx)+C_2\cos(kx)]\psi(y),$$

where C_1, C_2, and k are arbitrary constants, and the functions $\varphi=\varphi(y)$ and $\psi=\psi(y)$ are determined by the system of ordinary differential equations

$$\varphi''_{yy}=k^2\varphi+\varphi f(\varphi/\psi),$$
$$\psi''_{yy}=k^2\psi+\psi g(\varphi/\psi).$$

2°. Solution in multiplicative form:

$$u=[C_1\exp(kx)+C_2\exp(-kx)]U(y),$$
$$w=[C_1\exp(kx)+C_2\exp(-kx)]W(y),$$

where C_1, C_2, and k are arbitrary constants, and the functions $U=U(y)$ and $W=W(y)$ are determined by the system of ordinary differential equations

$$U''_{yy}=-k^2U+Uf(U/W),$$
$$W''_{yy}=-k^2W+Wg(U/W).$$

3°. Degenerate solution in multiplicative form:

$$u=(C_1x+C_2)U(y),$$
$$w=(C_1x+C_2)W(y),$$

where C_1 and C_2 are arbitrary constants, and the function $U=U(y)$ and $W=W(y)$ are determined by the system of ordinary differential equations

$$U''_{yy}=Uf(U/W),$$
$$W''_{yy}=Wg(U/W).$$

Remark. In Items 1°–3°, the functions f and g can be dependent on y.

4°. Solution in multiplicative form:

$$u=e^{a_1x+b_1y}\xi(z),\qquad w=e^{a_1x+b_1y}\eta(z),\qquad z=a_2x+b_2y,$$

where a_1, a_2, b_1, and b_2 are arbitrary constants, and the functions $\xi=\xi(z)$ and $\eta=\eta(z)$ are determined by the system of ordinary differential equations

$$(a_2^2+b_2^2)\xi''_{zz}+2(a_1a_2+b_1b_2)\xi'_z+(a_1^2+b_1^2)\xi=\xi f(\xi/\eta),$$
$$(a_2^2+b_2^2)\eta''_{zz}+2(a_1a_2+b_1b_2)\eta'_z+(a_1^2+b_1^2)\eta=\eta g(\xi/\eta).$$

5°. Solution:

$$u=k\theta(x,y),\qquad w=\theta(x,y),$$

where k is a root of the algebraic (transcendental) equation $f(k)=g(k)$, and the function $\theta=\theta(x,y)$ satisfies the linear Helmholtz equation

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=f(k)\theta.$$

For its exact solutions see the "Handbook of Linear Partial Differential Equations for Engineers and Scientists" by A. D. Polyanin (2002).

