4. $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=uf\left(\dfrac{u}{w}\right)+\dfrac{u}{w}h\left(\dfrac{u}{w}\right),\quad \dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=wg\left(\dfrac{u}{w}\right)+h\left(\dfrac{u}{w}\right).$

Solution:

$$u=kw,\qquad w=\theta(x,y)-\frac{h(k)}{f(k)},$$

where k is a root of the algebraic (transcendental) equation

$$f(k)=g(k),$$

and the function $\theta=\theta(x,y)$ satisfies the linear Helmholtz equation

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=f(k)w.$$

For its exact solutions see the “Handbook of Linear Partial Differential Equations for Engineers and Scientists” by A. D. Polyanin (2002).

