**1.** $$\frac{\partial^2 u}{\partial t^2} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + uf(bu-cw) + g(bu-cw),$$
$$\frac{\partial^2 w}{\partial t^2} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + wf(bu-cw) + h(bu-cw).$$

$1^\circ$. Solution:
$$u = \varphi(t) + c\theta(x,t), \qquad w = \psi(t) + b\theta(x,t),$$
where $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the system of ordinary differential equations
$$\varphi''_{tt} = \varphi f(b\varphi - c\psi) + g(b\varphi - c\psi),$$
$$\psi''_{tt} = \psi f(b\varphi - c\psi) + h(b\varphi - c\psi),$$
and the function $\theta = \theta(x,t)$ satisfies the linear equation
$$\frac{\partial^2 \theta}{\partial t^2} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial \theta}{\partial x}\right) + f(b\varphi - c\psi)\theta.$$
For $f = \text{const}$, this equation can be solved by the method of separation of variables.

$2^\circ$. Let us multiply the first equation by $b$ and add it to the second equation multiplied by $-c$ to obtain
$$\frac{\partial^2 \zeta}{\partial t^2} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial \zeta}{\partial x}\right) + \zeta f(\zeta) + bg(\zeta) - ch(\zeta), \qquad \zeta = bu - cw. \tag{1}$$
This equation will be treated in conjunction with the first equation of the original system,
$$\frac{\partial^2 u}{\partial t^2} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + uf(\zeta) + g(\zeta). \tag{2}$$
Equation (1) can be treated separately. Given a solution $\zeta = \zeta(x,t)$ of equation (1), the function $u = u(x_1,\dots,x_n,t)$ can be found by solving the linear equation (2), and the function $w = w(x_1,\dots,x_n,t)$ is determined by the formula $w = (bu-\zeta)/c$.

Note three important cases:

(i) In the general case, equation (1) admits a space-homogeneous solution $\zeta = \zeta(t)$. The corresponding solution of the original system is given in Item $1^\circ$ in a different form.

(ii) In the general case, equation (1) admits a stationary $\zeta = \zeta(x)$; the corresponding exact solutions of equation (3) have the forms $u = u_0(x) + \sum e^{-\beta_n t}u_n(x)$ and $u = u_0(x) + \sum \cos(\beta_n t)u_n^{(1)}(x) + \sum \sin(\beta_n t)u_n^{(2)}(x)$.

(iii) If the condition $\zeta f(t,\zeta) + bg(t,\zeta) - ch(t,\zeta) = k_1\zeta + k_0$ holds, equation (1) is linear,
$$\frac{\partial^2 \zeta}{\partial t^2} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial \zeta}{\partial x}\right) + k_1\zeta + k_0,$$
and can be solved using the method of separation of variables.

 