1. $\dfrac{\partial u}{\partial t} = L[u] + uf(t, au - cw) + g(t, au - cw), \quad \dfrac{\partial w}{\partial t} = L[w] + wf(t, au - cw) + h(t, au - cw).$

Here, L is an arbitrary linear differential operator in the coordinates $x_1, \ldots, x_n$ (of any order in derivatives), whose coefficients can depend on $x_1, \ldots, x_n, t$:

$$L[u] = \sum A_{k_1 \ldots k_n}(x_1, \ldots, x_n, t) \frac{\partial^{k_1 + \cdots + k_n} u}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}}$$

It is assumed that $k_1 + \cdots + k_n \geq 1$, and hence $L[\mathrm{const}] = 0$.

1°. Solution:

$$u = \varphi(t) + c \exp\left[\int f(t, a\varphi - b\psi)\, dt\right] \theta(x_1, \ldots, x_n, t),$$
$$w = \psi(t) + a \exp\left[\int f(t, a\varphi - b\psi)\, dt\right] \theta(x_1, \ldots, x_n, t),$$

where $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the system of ordinary differential equations

$$\varphi'_t = \varphi f(t, a\varphi - b\psi) + g(t, a\varphi - b\psi),$$
$$\psi'_t = \psi f(t, a\varphi - b\psi) + h(t, a\varphi - b\psi),$$

and the function $\theta = \theta(x_1, \ldots, x_n, t)$ satisfies the linear equation

$$\frac{\partial \theta}{\partial t} = L[\theta].$$

2°. Let us multiply the first equation by a and add it to the second equation multiplied by $-b$ to obtain

$$\frac{\partial \zeta}{\partial t} = L[\zeta] + \zeta f(t, \zeta) + ag(t, \zeta) - bh(t, \zeta), \qquad \zeta = au - bw. \tag{1}$$

This equation will be treated in conjunction with the first equation of the original system,

$$\frac{\partial u}{\partial t} = L[u] + uf(t, \zeta) + g(t, \zeta). \tag{2}$$

Equation (1) can be treated separately. Given a solution $\zeta = \zeta(x, t)$ of equation (1), the function $u = u(x_1, \ldots, x_n, t)$ can be found by solving the linear equation (2), and the function $w = w(x_1, \ldots, x_n, t)$ is determined by the formula $w = (au - \zeta)/b$.

Note three important cases where equation (1) admits exact solutions:

(i) Equation (1) admits a space-homogeneous solution $\zeta = \zeta(t)$.

(ii) Let the coefficients of the operator L and the functions f, g, h be implicitly independent of t. Then equation (1) admits a stationary solution $\zeta = \zeta(x_1, \ldots, x_n)$.

(iii) If the condition $\zeta f(t, \zeta) + bg(t, \zeta) - ch(t, \zeta) = k_1 \zeta + k_0$ holds, equation (1) is linear. If L is a linear constant-coefficient operator, then solutions may be found using the method of separation of variables.

Reference

Polyanin, A. D., Exact solutions of nonlinear systems of reaction-diffusion equations and mathematical biology equations, *Theor. Found. Chem. Eng.*, Vol. 37, No. 6, 2004.

