5. $\dfrac{\partial u}{\partial t} = L[u] + uf\left(t, \dfrac{u}{w}\right) + \dfrac{u}{w}h\left(t, \dfrac{u}{w}\right), \quad \dfrac{\partial w}{\partial t} = L[w] + wg\left(t, \dfrac{u}{w}\right) + h\left(t, \dfrac{u}{w}\right).$

Here, L is an arbitrary linear differential operator (of any order) in the coordinates $x_1, \dots, x_n$, whose coefficients can depend on $x_1, \dots, x_n, t$. It is assumed that $L[\text{const}] = 0$.

Solution:

$$u = \varphi(t)G(t)\left[\theta(x_1, \dots, x_n, t) + \int \frac{h(t, \varphi)}{G(t)}\,dt\right], \quad G(t) = \exp\left[\int g(t, \varphi)\,dt\right],$$
$$w = G(t)\left[\theta(x_1, \dots, x_n, t) + \int \frac{h(t, \varphi)}{G(t)}\,dt\right],$$

where the function $\varphi = \varphi(t)$ is determined by the nonlinear first-order ordinary differential equation

$$\varphi'_t = [f(t, \varphi) - g(t, \varphi)]\varphi,$$

and the function $\theta = \theta(x_1, \dots, x_n, t)$ satisfies the linear heat equation

$$\frac{\partial \theta}{\partial t} = L[\theta].$$

Reference

Polyanin, A. D., Exact solutions of nonlinear systems of reaction-diffusion equations and mathematical biology equations, *Theor. Found. Chem. Eng.*, Vol. 37, No. 6, 2004.

