6. $\dfrac{\partial u}{\partial t} = L[u] + uf\left(t, \dfrac{u}{w}\right)\ln u + ug\left(t, \dfrac{u}{w}\right), \quad \dfrac{\partial w}{\partial t} = L[w] + wf\left(t, \dfrac{u}{w}\right)\ln w + wh\left(t, \dfrac{u}{w}\right).$

Here, L is an arbitrary linear differential operator (of any order) in the coordinates $x_1, \dots, x_n$, whose coefficients can depend on $x_1, \dots, x_n, t$. It is assumed that $L[\text{const}] = 0$.

Solution:

$$u = \varphi(t)\psi(t)\theta(x_1, \dots, x_n, t), \quad w = \psi(t)\theta(x_1, \dots, x_n, t),$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by solving the ordinary differential equations

$$\begin{aligned} \varphi'_t &= \varphi[g(t,\varphi) - h(t,\varphi) + f(t,\varphi)\ln\varphi], \\ \psi'_t &= \psi[h(t,\varphi) + f(t,\varphi)\ln\psi], \end{aligned} \tag{1}$$

and the function $\theta = \theta(x_1, \dots, x_n, t)$ is determined by the differential equation

$$\frac{\partial \theta}{\partial t} = L[\theta] + f(t,\varphi)\theta\ln\theta. \tag{2}$$

Given a solution of equation (1), the second equation can be solved with the change of variable $\psi = e^{\zeta}$ (the equation is then reduced to a linear one for ζ). If the operator L is one dimensional ($n = 1$) and constant-coefficient and if $f = \text{const}$, then equation (2) admits a traveling-wave solution $\theta = \theta(kx - \lambda t)$.

Reference

Polyanin, A. D., Exact solutions of nonlinear systems of reaction-diffusion equations and mathematical biology equations, *Theor. Found. Chem. Eng.*, Vol. 37, No. 6, 2004.

