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# New Reductions of the Unsteady Axisymmetric Boundary Layer Equation to ODEs and Simpler PDEs

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**Abstract:** Reductions make it possible to reduce the solution of a PDE to solving an ODE. The best known are the traveling wave, self-similar and symmetry reductions. Classical and non-classical symmetries are also used to construct reductions, as is the Clarkson–Kruskal direct method. Recently, authors have proposed a method for constructing reductions of PDEs with two independent variables based on the idea of invariance. The proposed method in this work is a modification of the Clarkson–Kruskal direct method and expands the possibilities for its application. The main result of this article consists of a method for constructing reductions that generalizes the previously proposed approach to the case of three independent variables. The proposed method is used to construct reductions of the unsteady axisymmetric boundary layer equation to ODEs and simpler PDEs. All reductions of this equation were obtained.

**Keywords:** steady-state plane boundary layer; unsteady axisymmetric boundary layer; self-similar solution; invariant solution; symmetry reduction; invariant auxiliary functions; one- and two-dimensional reductions

**MSC:** 35B06; 35B07; 35C06; 35-G20; 35Q35; 76D10; 76M60



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## 1. Introduction

### 1.1. Preliminary Remarks

Constructing reductions of nonlinear PDEs is of fundamental importance for finding exact solutions. Reductions make it possible to reduce the solution of a PDE to solving an ODE. The best known are self-similar reductions [1]

$$u = t^{\nu} \varphi(\zeta), \quad \zeta = xt^{-\delta}, \quad (1)$$

where  $u$  is a dependent variable,  $x$  and  $t$  are independent variables, and  $v$  and  $\delta$  are constants. Self-similar reductions are a special case of symmetry reductions. Symmetry reductions are found by applying group analysis methods [2]. Traveling wave reductions have the form [3]

$$u = \varphi(\zeta) + V_0(t), \quad \zeta = x - V(t).$$

In [4], the reductions

$$u = \lambda(t) + \mu(t)\varphi(\zeta), \quad \zeta = x p(t) + q(t), \tag{2}$$

were considered; they generalize the reductions found in (1).

A method for finding reductions of PDEs with two independent variables was proposed in [5]; this is also known as the Clarkson–Kruskal direct method. For the Boussinesq equation

$$u_{tt} + \frac{1}{2}(u^2)_{xx} + u_{xxxx} = 0 \tag{3}$$

all reductions of the form

$$u = U(x, t, w(z)) \tag{4}$$

were obtained, where  $z = z(x, t)$  and the function  $w(z)$  is a solution of an ODE. It was shown that there are reductions other than those obtained using the Lie group method for finding group-invariant solutions of PDEs.

All reductions of the Burgers equation

$$u_t + uu_x = u_{xx}, \tag{5}$$

the Korteweg–de Vries equation

$$u_t + uu_x = u_{xxx}, \tag{6}$$

and the modified Korteweg–de Vries equation

$$u_t + u^2u_x = u_{xxx} \tag{7}$$

were found in [5]. It was shown that reductions of these equations coincide with the symmetry reductions. It was also shown that for Equations (3) and (5)–(7), reductions (4) have the form

$$u = \alpha(x, t) + \beta(x, t)w(z). \tag{8}$$

In the works of [6,7], a method for constructing reductions of PDEs with two independent variables was proposed, based on the idea of invariance. In these articles, based on the proposed approach, all reductions of the equation of a steady-state plane boundary layer with a pressure gradient were found.

### 1.2. The Main Results

The main results of this article are the following:

- a method of constructing reductions, generalizing the previously proposed approach [6,7] to the case of three independent variables;
- the construction of all reductions of the unsteady axisymmetric boundary layer equation.

## 2. Reductions of the Burgers Equation

We demonstrate the effectiveness of the method presented in [6,7] by finding the reductions of the Burgers Equation (5).

Substituting (4) into Equation (5), we obtain the relation

$$U_w z_x^2 w'' + U_{ww} z_x^2 (w')^2 + (2U_{xw} z_x + U_w z_{xx} - U U_w z_x - U_w z_t) w' + U_{xx} - U U_x - U_t = 0. \tag{9}$$

We divide both sides of Relation (9) by the coefficient at the highest derivative by  $U_w z_x^2$  (normalization of the coefficients). The normalized coefficient at the term with  $(w')^2$  has the form

$$\frac{U_{ww}}{U_w} = \Gamma(z, w). \tag{10}$$

Integrating Relation (10), we obtain

$$U(x, t, w) = \beta(x, t) \Gamma_1(z, w) + \alpha(x, t).$$

The transformation  $w \rightarrow F(z, w)$  maps ODE into ODE. Then, the reductions of the Burgers equation can be sought in the form presented in (8).

Substituting (8) into Equation (5), we obtain the relation

$$\beta z_x^2 w'' - \beta^2 z_x w w' + [\beta z_{xx} + (2\beta_x - \alpha\beta) z_x - \beta z_t] w' + \beta \beta_x w^2 + (\beta_{xx} - \beta \alpha_x - \alpha \beta_x - \beta_t) w + \alpha_{xx} - \alpha \alpha_x - \alpha_t = 0. \tag{11}$$

Relation (11) is an ODE if the normalized coefficients depend on  $\alpha(x, t), \beta(x, t), z(x, t)$  and their derivatives are functions of  $z$ . From this condition, we obtain the overdetermined system of equations

$$\begin{aligned} \frac{\beta}{z_x} &= \Gamma_1(z), & \frac{\beta_x}{z_x^2} &= \Gamma_2(z), \\ \frac{\beta z_{xx} + (2\beta_x - \alpha\beta) z_x - \beta z_t}{\beta z_x^2} &= \Gamma_3(z), \\ \frac{\beta_{xx} - \beta \alpha_x - \alpha \beta_x - \beta_t}{\beta z_x^2} &= \Gamma_4(z), \\ \frac{\alpha_{xx} - \alpha \alpha_x - \alpha_t}{\beta z_x^2} &= \Gamma_5(z) \end{aligned} \tag{12}$$

for finding the unknown functions  $\alpha(x, t), \beta(x, t), z(x, t)$  and  $\Gamma_1(z), \dots, \Gamma_5(z)$ .

Let us state the main considerations underlying the method for constructing reductions for PDEs with two independent variables [6,7] as applied to the system of Equation (12):

1. Each equation of the system of Equation (12) is equivalent to the condition that the Jacobian of its left-hand side and the function  $z(x, t)$  is equal to zero. As a result, we can derive an overdetermined system of equations for determining the unknown functions  $\alpha(x, t), \beta(x, t)$  and  $z(x, t)$  ( $\mathcal{A}$ -system).  $\mathcal{A}$ -system is not presented here because of its cumbersomeness.
2. Consider the auxiliary functions  $\mu_1 = \mu_1(x, t), \mu_2 = \mu_2(x, t)$  and  $\mu_3 = \mu_3(x, t)$  defined by the relations

$$\begin{aligned} \mu_1 z_x + z_t &= 0, \\ \mu_1 \beta_x + \beta_y + \mu_2 \beta &= 0, \\ \mu_1 \alpha_x + \alpha_t + \mu_2 \alpha + \mu_3 &= 0. \end{aligned} \tag{13}$$

Finding the auxiliary functions, we find the reductions. As was noted in [5], the following transformations map a reduction of the form found in (8) into a reduction:

$$\begin{aligned} z &\rightarrow F_1(z), \\ \beta &\rightarrow \frac{\beta}{F_2(z)}, \end{aligned} \tag{14}$$

$$\alpha \rightarrow \alpha + \frac{\beta}{F_3(z)},$$

where  $F_1(z)$ ,  $F_2(z)$  and  $F_3(z)$  are arbitrary functions. These transformations are associated with the arbitrariness of  $w(z)$  in finding an ODE. It can be shown that the auxiliary functions  $\mu_1$ ,  $\mu_2$  and  $\mu_3$  are invariant under the transformations in (14). It can also be shown that the  $\mathcal{A}$ -system admits transformations (14).

3. Finding the derivatives  $\alpha_x$ ,  $\beta_x$  and  $z_x$  from the relations in (13) and substituting them into the  $\mathcal{A}$ -system, we obtain the following overdetermined system of equations for determining the auxiliary functions:

$$\begin{aligned} \mu_{1x} - \mu_2 &= 0, & \mu_{2x} &= 0, \\ 2\mu_1\mu_{1x} + \mu_{1t} + \mu_3 &= 0, \\ 2\mu_3\mu_{1x} + \mu_{3t} &= 0. \end{aligned} \tag{15}$$

Thus, the  $\mathcal{A}$ -system is reduced to the simpler system of Equation (15). The  $\mathcal{A}$ -system can be written only in terms of the invariants  $\mu_1$ ,  $\mu_2$  and  $\mu_3$  of transformations (14) because it is invariant under these transformations.

4. The overdetermined system of Equation (15) is easy to solve. The auxiliary functions  $\mu_1$ ,  $\mu_2$  and  $\mu_3$  are used to construct reductions.

**Remark 1.** According to the approach described in [5], the overdetermined system of Equation (12) is solved directly. Reductions are found using modulo transformations (14).

### 3. Reductions of the Steady-State Plane Boundary Layer Equation

#### 3.1. Basic Equation

We also demonstrate the effectiveness of the method for constructing reductions of PDEs with two independent variables based on the idea of invariance by finding the reductions of steady-state plane boundary layer equation [6,7].

We consider the equation

$$u_{yyy} - u_y u_{xy} + u_x u_{yy} - P(x) = 0, \tag{16}$$

which describes the steady motion of a viscous incompressible fluid in a laminar plane boundary layer with a pressure gradient [8]. Equation (16) is written in dimensionless variables. Here,  $u$  is the stream function,  $P(x) = \partial p / \partial x$  is a given function and  $p$  is the pressure. Self-similar solutions of Equation (16) were discussed in the classical monographs [1,8–11]. Symmetry reductions of Equation (16) can be obtained using the results of [2]. Reductions of Form (2) were found and studied in [4]. In [12], the method of non-classical symmetries [13] and its generalization were used to obtain new reductions of Form (8) for Equation (16). For invariant and noninvariant exact solutions, see also [10,11,14–20]. An extensive list of exact solutions to the boundary layer equation on a flat plane as well as related hydrodynamic equations can be found in the handbook [19].

**Remark 2.** For some exact solutions and transformations of the unsteady plane boundary layer equations [8,11], see [2,19,21–28].

**Remark 3.** The studies [17,18,29–32] present exact solutions and transformations for the steady-state and unsteady equations of non-Newtonian fluids.

#### 3.2. Construction of Reductions

We look for reductions of Equation (16) in the form

$$u = U(x, y, w(z)), \tag{17}$$

where  $z = z(x, y)$  and the function  $w(z)$  is a solution of an ODE. Substituting (17) into Equation (16), we obtain the relation

$$\begin{aligned}
 &U_w z_y^3 w''' + 3U_{ww} z_y^3 w' w'' + z_y (3U_{yw} z_y + U_x U_w z_y - U_y U_w z_x + 3U_{wz} z_y) w'' \\
 &+ U_{ww} z_y^3 (w')^3 + (3U_{yw} z_y^2 - U_w U_x z_y^2 + U_w U_{yw} z_x z_y + U_x U_{ww} z_y^2 \\
 &- U_y U_{ww} z_x z_y + U_w^2 z_x z_y - U_w^2 z_y z_x y + 3U_{ww} z_y z_y) (w')^2 + (3U_{yy} z_y \\
 &- U_w U_{xy} z_y + U_w U_{yy} z_x - U_y U_x z_y - U_y U_{yw} z_x + 2U_x U_{yw} z_y + U_x U_{wz} z_y \\
 &- U_y U_w z_x y + 3U_{yw} z_y y + U_w z_y y) w' + U_{yyy} - U_y U_{xy} + U_x U_{yy} - P(x) = 0.
 \end{aligned}
 \tag{18}$$

Both sides of Relation (18) are divided by the coefficient at the highest derivative by  $U_w z_y^3$ . Consider the normalized coefficient at the term with  $w' w''$ . It has the form

$$\frac{3U_{ww}}{U_w} = \Gamma(z, w).
 \tag{19}$$

**Remark 4.** While deriving Relation (19), we assumed that  $z_y \neq 0$ . The case  $z_y = 0$  corresponds to a degenerate reduction. It is of no interest, and its detailed consideration is omitted.

The transformation  $w \rightarrow F(z, w)$  maps ODE into ODE. Then, the reductions of Equation (16) can be sought in the form

$$u = \alpha(x, y) + \beta(x, y)w(z),
 \tag{20}$$

where  $z = z(x, y)$ . Substituting (20) into Equation (16), we obtain the relation

$$\begin{aligned}
 &\beta z_y^3 w''' + \beta z_y (\beta_x z_y - \beta_y z_x) w w'' + z_y (3\beta z_{yy} + 3\beta_y z_y + \alpha_x \beta z_y - \alpha_y \beta z_x) w'' \\
 &+ \beta (\beta z_x z_{yy} - \beta z_y z_{xy} + \beta_y z_x z_y - z_y^2 \beta_x) (w')^2 \\
 &+ (\beta_x \beta z_{yy} + \beta z_x \beta_{yy} - \beta_y^2 z_x + \beta_y \beta_x z_y - \beta_y \beta z_{xy} - \beta z_y \beta_{xy}) w w' \\
 &+ (\beta z_{yyy} - \alpha_y \beta_x z_y - \alpha_y \beta_y z_x - \alpha_y \beta z_{xy} - \beta z_y \alpha_{xy} + 2\alpha_x \beta_y z_y \\
 &+ \alpha_x \beta z_{yy} + \beta z_x \alpha_{yy} + 3\beta_y z_{yy} + 3\beta_y z_y) w' \\
 &+ (\beta_x \beta_{yy} - \beta_y \beta_{xy}) w^2 + (\beta_{yyy} + \alpha_x \beta_{yy} - \beta_y \alpha_{xy} - \alpha_y \beta_{xy} + \beta_x \alpha_{yy}) w \\
 &+ \alpha_{yyy} - \alpha_y \alpha_{xy} + \alpha_x \alpha_{yy} - P(x) = 0.
 \end{aligned}
 \tag{21}$$

Relation (21) is an ODE if the normalized coefficients depend on  $\alpha(x, y)$ ,  $\beta(x, y)$  and  $z(x, y)$  and their derivatives are functions of  $z$ . From this condition, we obtain the over-determined system of equations

$$\begin{aligned}
 &\frac{\beta_x z_y - \beta_y z_x}{z_y^2} = \Gamma_1(z), \\
 &\frac{3\beta z_{yy} + 3\beta_y z_y + \alpha_x \beta z_y - \alpha_y \beta z_x}{\beta z_y^2} = \Gamma_2(z), \\
 &\frac{\beta (\beta z_x z_{yy} - \beta z_y z_{xy} + \beta_y z_x z_y - z_y^2 \beta_x)}{\beta z_y^3} = \Gamma_3(z), \\
 &\frac{\beta_x \beta z_{yy} + \beta z_x \beta_{yy} - \beta_y^2 z_x + \beta_y \beta_x z_y - \beta_y \beta z_{xy} - \beta z_y \beta_{xy}}{\beta z_y^3} = \Gamma_4(z), \\
 &\frac{\beta z_{yyy} - \alpha_y \beta_x z_y - \alpha_y \beta_y z_x - \alpha_y \beta z_{xy} - \beta z_y \alpha_{xy}}{\beta z_y^3} \\
 &+ \frac{2\alpha_x \beta_y z_y + \alpha_x \beta z_{yy} + \beta z_x \alpha_{yy} + 3\beta_y z_{yy} + 3\beta_y z_y}{\beta z_y^3} = \Gamma_5(z),
 \end{aligned}
 \tag{22}$$

$$\begin{aligned} \frac{\beta_x \beta_{yy} - \beta_y \beta_{xy}}{\beta z_y^3} &= \Gamma_6(z), \\ \frac{\beta_{yyy} + \alpha_x \beta_{yy} - \beta_y \alpha_{xy} - \alpha_y \beta_{xy} + \beta_x \alpha_{yy}}{\beta z_y^3} &= \Gamma_7(z), \\ \frac{\alpha_{yyy} - \alpha_y \alpha_{xy} + \alpha_x \alpha_{yy} - P(x)}{\beta z_y^3} &= \Gamma_8(z). \end{aligned}$$

Each of the equations in Equation (22) is equivalent to the condition that the Jacobian of its left-hand side and  $z(x, y)$  is equal to zero. As a result, we can derive an  $\mathcal{A}$ -system for determining the unknown functions  $\alpha(x, y)$ ,  $\beta(x, y)$  and  $z(x, y)$  (the system is not presented because of its cumbersomeness).

Consider the auxiliary functions  $\mu_1 = \mu_1(x, y)$ ,  $\mu_2 = \mu_2(x, y)$  and  $\mu_3 = \mu_3(x, y)$  defined by the relations

$$\begin{aligned} z_x + \mu_1(x, y)z_y &= 0, \\ \beta_x + \mu_1(x, y)\beta_y + \mu_2(x, y)\beta &= 0, \\ \alpha_x + \mu_1(x, y)\alpha_y + \mu_2(x, y)\alpha + \mu_3(x, y) &= 0. \end{aligned} \tag{23}$$

Finding the partial derivatives  $\alpha_x$ ,  $\beta_x$  and  $z_x$  from (23) and substituting them into the  $\mathcal{A}$ -system, we obtain the following overdetermined system of equations for finding the auxiliary functions:

$$\begin{aligned} \mu_{1y}\mu_2 - \mu_{1xy} - \mu_{1y}^2 - \mu_1\mu_{1yy} &= 0, \\ \mu_2^2 - \mu_{2x} - \mu_2\mu_{1y} - \mu_1\mu_{2y} &= 0, \\ \mu_2\mu_3 - 3\mu_{1yy} - 3\mu_{2y} - \mu_{3x} - \mu_1\mu_{3y} - \mu_{1y}\mu_3 &= 0, \\ \mu_{2y}(2\mu_{1y} - \mu_2) &= 0, \\ \mu_2\mu_{2yy} - \mu_{2y}^2 &= 0, \\ 4\mu_{1yyy} + 6\mu_{2yy} + \mu_2\mu_{3y} - 2\mu_{2y}\mu_3 + 2\mu_{1y}\mu_{3y} &= 0, \\ \mu_{2yyy} + 2\mu_{2y}\mu_{3y} - \mu_{2yy}\mu_3 - \mu_2\mu_{3yy} &= 0, \\ \mu_{3yyy} + \mu_{3y}^2 - \mu_3\mu_{3yy} + 3P(x)\mu_{1y} + P(x)\mu_2 + P'(x) &= 0. \end{aligned} \tag{24}$$

The first seven equations of the overdetermined system of Equation (24) are easy to solve in explicit form. To solve them, it is enough to consider three cases: (1)  $\mu_{2y} = 0$ ,  $\mu_2 = \mu_2(x) \neq 0$ ; (2)  $\mu_2 = 0$ ; (3)  $2\mu_{1y} - \mu_2 = 0$ . The solution of the system of Equation (24) for functions  $\mu_1$ ,  $\mu_2$  and  $\mu_3$  is not presented. The system of Equation (24) has a solution only for the following functions  $P(x)$ :

$$\begin{aligned} P(x) &= \lambda(x + \tau)^\nu, \quad \lambda \neq 0, \quad \nu \neq 0, \\ P(x) &= \sigma e^{\gamma x}, \quad \sigma \neq 0, \quad \gamma \neq 0, \\ P(x) &= \sigma_0, \\ P(x) &= \lambda_1(x + \tau_1)^{-5/3} + \lambda_2(x + \tau_1)^{-1/3}. \end{aligned} \tag{25}$$

Here,  $\lambda, \tau, \nu, \sigma, \gamma, \sigma_0, \lambda_1, \lambda_2, \tau_1$  and  $\tau_2$  are arbitrary constants.

### 3.3. The Existence of Non-Symmetry Reductions

Let us find the symmetries of Equation (16). A symmetry operator is sought in the form [2]

$$X = \zeta^1(x, y, u) \frac{\partial}{\partial x} + \zeta^2(x, y, u) \frac{\partial}{\partial y} + \eta(x, y, u) \frac{\partial}{\partial u}.$$

The system of equations for determining the components of the symmetry operator is given by

$$\begin{aligned} \zeta_y^1 &= 0, & \zeta_u^1 &= 0, & \zeta_u^2 &= 0, & \zeta_{xy}^2 &= 0, \\ \eta_x &= 0, & \eta_y &= 0, & \zeta_x^1 - \zeta_y^2 - \eta_u &= 0, \\ \zeta^1 P'(x) + (3\zeta_y^2 - \eta_u)P(x) &= 0. \end{aligned}$$

The symmetries of Equation (16) depend on the form of the specified function  $P(x)$ . Let us present the result of the group classification:

- for an arbitrary function  $P(x)$ , the basis of symmetry operators is given by

$$X_1 = b(x) \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial u},$$

where  $b(x)$  is an arbitrary function;

- for  $P(x) = \lambda(x + \tau)^v$ , where  $\lambda \neq 0$  and  $v \neq 0$ , the basis of symmetry operators is given by

$$X_1, \quad X_2, \quad Y_1 = 4(x + \tau) \frac{\partial}{\partial x} + (1 - v)y \frac{\partial}{\partial y} + (v + 3)u \frac{\partial}{\partial u};$$

- for  $P(x) = \sigma e^{\gamma x}$ , where  $\sigma \neq 0$  and  $\gamma \neq 0$ , the basis of symmetry operators is given by

$$X_1, \quad X_2, \quad Y_2 = 4 \frac{\partial}{\partial x} - \gamma y \frac{\partial}{\partial y} + \gamma u \frac{\partial}{\partial u};$$

- for  $P(x) = \sigma_0$ , where  $\sigma_0 \neq 0$ , the basis of symmetry operators is given by

$$X_1, \quad X_2, \quad Z_1 = \frac{\partial}{\partial x}, \quad Z_2 = 4x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 3u \frac{\partial}{\partial u};$$

- for  $P(x) = 0$ , the basis of symmetry operators is given by

$$X_1, \quad X_2, \quad Z_1, \quad Z_3 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad Z_4 = x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}.$$

**Remark 5.** The results of the group classification of Equation (16) correspond to those of the system of equations describing a laminar steady-state plane boundary layer with a pressure gradient [2].

From the group classification, it follows that the last of the types of the function  $P(x)$  presented in Formula (25) leads to reductions other than those obtained using symmetries.

In conclusion, we note that the proposed method for finding reductions associated with the construction of a  $\mathcal{A}$ -system and the introduction of invariant auxiliary functions is general and applicable to PDEs with two independent variables. The proposed method is a modification of the Clarkson–Kruskal direct method and expands the possibilities for its application.

#### 4. Reductions of the Unsteady Axisymmetric Boundary Layer Equation

##### 4.1. Basic Equation

Let us consider an equation describing the unsteady flow of a viscous incompressible fluid in a laminar boundary layer on the surface of an axisymmetric body of rotation [8,10]

$$u_{yt} - u_{yyy} + u_y u_{xy} - \left( u_x + \frac{r'_0(x)}{r_0(x)} u \right) u_{yy} - f(x, t) = 0. \tag{26}$$

Equation (26) is written in dimensionless variables. Here  $u(x, y, t)$  is a stream function and a given function  $f(x, t) = -\partial p / \partial x$  is a pressure gradient. Function  $r_0(x)$  describes the shape of the streamlined body. For the simplicity of further calculations, it is convenient to use Stepanov–Mangler variables [8,10,11] by using the following formulas:

$$\bar{x} = \int_0^x r_0^2(\tau) d\tau, \quad \bar{y} = r_0(x)y, \quad \bar{u} = r_0(x)u. \tag{27}$$

This reversible transformation of variables maps the equation of a steady-state axisymmetric boundary layer into the equation of a steady-state plane boundary layer [8,10,11].

Applying the transformation of the variables in (27) to Equation (26), we obtain

$$\frac{1}{r_0^2(x)} u_{\bar{y}t} - u_{\bar{y}\bar{y}\bar{y}} + u_{\bar{y}} u_{\bar{x}\bar{y}} - u_{\bar{x}} u_{\bar{y}\bar{y}} - \frac{f(x, t)}{r_0^2(x)} = 0.$$

Expressing  $x$  by  $\bar{x}$  and omitting the bars, we have

$$r(x)u_{yt} - u_{yyy} + u_y u_{xy} - u_x u_{yy} - F(x, t) = 0, \tag{28}$$

where  $r(\bar{x}) = 1/r_0^2(x(\bar{x}))$ ,  $F(\bar{x}, t) = f(x(\bar{x}), t)/r_0^2(x(\bar{x}))$ . We consider Equation (28) as the basic one. In the following, all two-dimensional and one-dimensional reductions of this equation are found.

**Remark 6.** Some exact solutions to unsteady axisymmetric boundary layer equations are found in [24,33,34].

**Remark 7.** Some exact solutions of axisymmetric boundary layer equations for non-Newtonian fluids are found in [24].

4.2. Reductions to Simpler PDEs (Two-Dimensional Reductions)

Let us find all two-dimensional reductions of Equation (28), i.e., reductions to a PDE with two independent variables. We are looking for two-dimensional reductions in the following form:

$$u = U(x, y, t, w(s(x, y, t), q(x, y, t))). \tag{29}$$

Substituting Expression (29) into Equation (28), we have

$$-U_w s_y^3 w_{sss} - U_w q_y^3 w_{qqq} - 3U_{ww} s_y^3 w_s w_{ss} - 3U_{ww} q_y^3 w_q w_{qq} + \dots = 0. \tag{30}$$

Here, for simplicity, only four terms are written out.

Next, we divide both parts of Equation (30) by a coefficient with a derivative of  $w_{sss}$ , i.e., by  $-U_w s_y^3$  (the procedure for normalizing coefficients).

The condition that the resulting equation is a PDE for the function  $w(s, q)$  is the dependence of each of the normalized coefficients for derivatives of the function  $w(s, q)$  only on the variables  $s, q$  and  $w$ . Consider the normalized coefficient for the term  $w_s w_{ss}$ . This term has the following form:

$$\frac{3U_{ww}}{U_w} = \Gamma(s, q, w). \tag{31}$$

Integrating (31) twice by  $w$  gives the following:

$$U(x, y, w) = \beta(x, y, t)\tilde{\Gamma}(s, q, w) + \alpha(x, y, t).$$



Since an arbitrary function of the variables  $s, q$  and  $w$  can be taken as a function of  $w(s, q)$ , then two-dimensional reductions of Equation (28) can be searched in the form of a linear function of  $w(s, q)$ , so we have

$$u = \beta(x, y, t)w(s, q) + \alpha(x, y, t). \tag{32}$$

**Remark 8.** When deriving Expression (31), it was assumed that  $s_y \neq 0$ . If  $s_y = 0$ , then with  $q_y \neq 0$  from Expression (30), similarly, we obtain Form (32). Case  $s_y^2 + q_y^2 = 0$  corresponds to degenerate reduction that does not contain partial derivatives of the third and second orders. This case is not of particular interest, and its detailed consideration is not given here.

Let us show that one of the independent variables of the function  $w(s, q)$  can, without loss of generality, be used independently of the variable  $y$ . Indeed, considering in Ratio (30) the normalized coefficient for the partial derivative  $w_{qqq}$ , we obtain  $q_y/s_y = G(s, q)$ . We can present the function  $G(s, q)$  as  $G(s, q) = -\Gamma_s(s, q)/\Gamma_q(s, q)$ . From which it follows that

$$\Gamma_s s_y + \Gamma_q q_y = 0. \tag{33}$$

From Expression (33), it follows that  $\Gamma(s, q) = h(x, t)$ . Let us take, instead of the independent variable  $s$ , a new independent variable  $\bar{s} = \Gamma(s, q)$ , i.e., let us move on to the new independent variables  $\bar{s}, q$ . Then, in Expression (32), we assume  $s = s(x, t)$ .

Let us now show that it is possible, without loss of generality, to state  $\beta = \beta(x, t)$ . Substituting Expression (32) into Equation (28), we have

$$-\beta q_y^3 w_{qqq} + \beta^2 s_x q_y^2 (w_q w_{sq} - w_s w_{qq}) + \beta \beta_y s_x q_y w w_{sq} + \dots = 0.$$

Here, for simplicity of recording, only three terms are written out. Considering the ratio of coefficients for derivatives of the function  $w$  in the third and second terms, we obtain  $\beta_y/\beta = q_y \Gamma(s, q)$ . Whence it follows that  $\ln \beta = \tilde{\Gamma}(s, q) + g(x, t)$ . Since the function  $\beta(x, y, t)$  is defined up to an arbitrary multiplier depending on the variables  $s$  and  $q$ , then we can state  $\beta = \beta(x, t)$ .

Thus, we have the result that two-dimensional reductions of Equation (28) can be searched in the following form:

$$u = \beta(x, t)w(s(x, t), q(x, y, t)) + \alpha(x, y, t). \tag{34}$$

Substituting Expression (34) into Equation (28), we have

$$\begin{aligned} &-\beta q_y^3 w_{qqq} + \beta^2 s_x q_y^2 (w_q w_{sq} - w_s w_{qq}) - \beta \beta_x q_y^2 w w_{qq} + \beta q_y [\alpha_y s_x + r(x) s_t] w_{sq} \\ &\quad - \beta q_y [3q_{yy} - \alpha_y q_x + \alpha_x q_y - r(x) q_t] w_{qq} - \beta^2 s_x q_y w_s w_q \\ &\quad + \beta (\beta q_y q_{xy} - \beta q_x q_{yy} + \beta_x q_y^2) w_q^2 - \beta \beta_x q_{yy} w w_q - \beta \alpha_{yy} s_x w_s \\ &\quad - [\beta (q_{yyy} - \alpha_y q_{xy} + \alpha_x q_{yy} + \alpha_{yy} q_x) - (\beta \alpha_{xy} + \beta_x \alpha_y) q_y \\ &\quad \quad \quad - r(x) (\beta q_{yt} + \beta_t q_y)] w_q \\ &\quad - \beta_x \alpha_{yy} w - \alpha_{yyy} + \alpha_y \alpha_{xy} - \alpha_x \alpha_{yy} + r(x) \alpha_{yt} - F(x, t) = 0. \end{aligned} \tag{35}$$

From the condition that Relation (35) is a PDE, we obtain the following overdetermined system of differential equations:

$$\begin{aligned} \frac{\beta s_x}{q_y} = \Gamma_1(s, q), \quad \frac{\beta_x}{q_y} = \Gamma_2(s, q), \quad \frac{\alpha_y s_x + r(x) s_t}{q_y^2} = \Gamma_3(s, q), \\ \frac{3q_{yy} - \alpha_y q_x + \alpha_x q_y - r(x) q_t}{q_y^2} = \Gamma_4(s, q), \quad \frac{\beta s_x q_{yy}}{q_y^3} = \Gamma_5(s, q), \end{aligned}$$

$$\begin{aligned} \frac{\beta q_y q_{xy} - \beta q_x q_{yy} + \beta_x q_y^2}{q_y^3} &= \Gamma_6(s, q), & \frac{\beta_x q_{yy}}{q_y^3} &= \Gamma_7(s, q), & \frac{\alpha_{yy} s_x}{q_y^3} &= \Gamma_8(s, q), & (36) \\ \frac{\beta(q_{yyy} - \alpha_y q_{xy} + \alpha_x q_{yy} + \alpha_{yy} q_x) - (\beta \alpha_{xy} + \beta_x \alpha_y) q_y - r(x)(\beta q_{yt} + \beta_t q_y)}{\beta q_y^3} &= \Gamma_9(s, q), \\ \frac{\beta_x \alpha_{yy}}{\beta q_y^3} &= \Gamma_{10}(s, q), & \frac{\alpha_{yyy} - \alpha_y \alpha_{xy} + \alpha_x \alpha_{yy} - r(x) \alpha_{yt} + F(x, t)}{\beta q_y^3} &= \Gamma_{11}(s, q). \end{aligned}$$

The system of Equation (36) allows the following transformations:

$$\begin{aligned} s &\rightarrow G_1(s), & q &\rightarrow G_2(s, q), & (37) \\ \beta &\rightarrow G_3(s)\beta, & \alpha &\rightarrow \alpha + G_4(s, q)\beta, \end{aligned}$$

where  $G_1(s)$ ,  $G_2(s, q)$ ,  $G_3(s)$  and  $G_4(s, q)$  are arbitrary functions. These transformations are associated with arbitrariness in finding a reduction of Form (34). Let us introduce the auxiliary functions  $\mu_1 = \mu_1(x, t)$ ,  $\mu_2 = \mu_2(x, y, t)$ ,  $\mu_3 = \mu_3(x, t)$  and  $\mu_4 = \mu_4(x, y, t)$ , determined from the following relations:

$$\begin{aligned} s_x - \mu_1 s_t &= 0, & q_x - \mu_2 q_y - \mu_1 q_t &= 0, & (38) \\ \beta_x - \mu_1 \beta_t - \mu_3 \beta &= 0, & \alpha_x - \mu_2 \alpha_y - \mu_1 \alpha_t - \mu_3 \alpha - \mu_4 &= 0. \end{aligned}$$

It can be shown that the introduced auxiliary functions are invariants of the transformations in (37).

From the first two relations in (38), it follows that  $Xs = 0$ ,  $Xq = 0$ , where the operator  $X$  has the form

$$X = \frac{\partial}{\partial x} - \mu_2 \frac{\partial}{\partial y} - \mu_1 \frac{\partial}{\partial t}.$$

By acting with the operator  $X$  on the left and right sides of Equation (36), one can obtain a system of equations that does not contain unknown functions  $\Gamma_1(s, q), \dots, \Gamma_{11}(s, q)$ . Further, in the resulting system of equations, the corresponding partial derivatives of the form  $s_x, q_x, \beta_x, \alpha_x, \dots$  can be excluded using the relations in (38). So, it is possible to obtain a system of equations containing only auxiliary functions. This system of equations has the form

$$\begin{aligned} \mu_{1x} - \mu_1 \mu_{2y} + \mu_1 \mu_3 &= 0, & \mu_1 \mu_{2yy} &= 0, & \mu_1 \mu_{4yy} &= 0, \\ \mu_{2xy} - \mu_2 \mu_{2yy} + \mu_{3x} - \mu_{2y}^2 + \mu_3^2 &= 0, & \mu_{3x} - \mu_3 \mu_{2y} + \mu_3^2 &= 0, \\ 2r(x) \mu_{2y} - \mu_1 \mu_{4y} - r(x) \mu_{1t} - r'(x) &= 0, & \mu_3 \mu_{2yy} &= 0, & \mu_3 \mu_{4yy} &= 0, \\ 3\mu_{2yy} + \mu_{4x} - \mu_4 \mu_{2y} - \mu_2 \mu_{4y} - r(x) \mu_{2t} + \mu_3 \mu_4 &= 0, & (39) \\ \mu_{2yyy} - \mu_{4xy} + \mu_4 \mu_{2yy} + \mu_2 \mu_{4yy} - r(x) \mu_{2yt} - 2\mu_3 \mu_{4y} - r(x) \mu_{3t} &= 0, \\ \mu_{4yyy} + \mu_4 \mu_{4yy} - r(x) \mu_{4yt} - \mu_{4y}^2 - 3F(x, t) \mu_{2y} + F_x(x, t) & & & & & \\ - \mu_1 F_t(x, t) - F(x, t) \mu_3 &= 0. \end{aligned}$$

The overdetermined system of Equation (39) is solved explicitly. To solve it, it is enough to consider four cases: (1)  $\mu_1 = \mu_3 = 0$ ; (2)  $\mu_1 = 0, \mu_3 \neq 0$ ; (3)  $\mu_1 \neq 0, \mu_3 = 0$ ; (4)  $\mu_1 \mu_3 \neq 0$ . The system of Equation (39) has a solution only for certain types of functions  $r(x)$  and  $F(x, t)$ . Let us write out these solutions.

- $r(x) = \alpha_0(x + \beta_0)^\gamma, F(x, t) = (x + \beta_0)^{2(\gamma+\delta)+1} G[(t + \varepsilon)(x + \beta_0)^\delta].$

In this case, we have

$$\begin{aligned} \mu_1 &= \frac{\delta(t + \varepsilon)}{x + \beta_0}, & \mu_2 &= \frac{y(\gamma + \delta)}{2(x + \beta_0)}, \\ \mu_3 &= \frac{\gamma + \delta + 2}{2(x + \beta_0)}, & \mu_4 &= 0. \end{aligned}$$

Then, from the relations in (38), we find

$$s(x, t) = (x + \beta_0)^\delta (t + \varepsilon), \quad q(x, y, t) = y(x + \beta_0)^{\frac{\gamma + \delta}{2}},$$

$$\beta(x, t) = (x + \beta_0)^{\frac{\gamma + \delta}{2} + 1}, \quad \alpha(x, y, t) = 0.$$

The corresponding two-dimensional reduction has the form:

$$w_{qqq} + \left(1 + \frac{\gamma + \delta}{2}\right) w w_{qq} - \delta s(w_q w_{sq} + w_s w_{qq}) - \alpha_0 w_{sq}$$

$$- (\gamma + \delta + 1) w_q^2 + G(s) = 0.$$

2.  $r(x) = \alpha_0 \exp(\beta_0 x)$ ,  $F(x, t) = \exp(2\gamma x) G[(t + \varepsilon) \exp(-x(\beta_0 - \gamma))]$ .

In this case, we have

$$\mu_1 = (\gamma - \beta_0)(t + \varepsilon), \quad \mu_2 = \frac{\gamma y}{2}, \quad \mu_3 = \frac{\gamma}{2}, \quad \mu_4 = 0.$$

Then, from the relations in (38), we find

$$s(x, t) = (t + \varepsilon) \exp[(\gamma - \beta_0)x], \quad q(x, y, t) = y \exp\left(\frac{\gamma x}{2}\right),$$

$$\beta(x, t) = \exp\left(\frac{\gamma x}{2}\right), \quad \alpha(x, y, t) = 0.$$

The corresponding two-dimensional reduction has the form:

$$w_{qqq} - (\gamma - \beta_0) s w_q w_{sq} - \alpha_0 w_{sq} + (\gamma - \beta_0) s w_s w_{qq}$$

$$+ \frac{\gamma}{2} w w_{qq} - \gamma w_q^2 + G(s) = 0.$$

3.  $r(x) = \alpha_0 \exp(\beta_0 x) + \gamma$ ,

$$F(x, t) = -\frac{\gamma(\alpha_0 \exp(\beta_0 x) + \gamma)}{\beta_0(t + \varepsilon)^2} + G[(t + \varepsilon) \exp(-\beta_0 x)].$$

In this case, we have

$$\mu_1 = -\beta_0(t + \varepsilon), \quad \mu_2 = 0, \quad \mu_3 = 0, \quad \mu_4 = -\frac{\gamma y}{t + \varepsilon}.$$

Then, from the relations in (38), we find

$$s(x, t) = (t + \varepsilon) \exp(-\beta_0 x), \quad q(x, y, t) = y,$$

$$\beta(x, t) = 1, \quad \alpha(x, y, t) = \frac{\gamma y}{\beta_0(t + \varepsilon)}.$$

The corresponding two-dimensional reduction has the form:

$$w_{qqq} - \alpha_0 w_{sq} + \beta_0 s(w_q w_{sq} - w_s w_{qq}) + G(s) = 0.$$

4.  $r(x) = \alpha_0(x + \beta_0)^{-4/3} + \gamma(x + \beta_0)^{-2/3}$ ,

$$F(x, t) = \frac{-3\alpha_0\gamma(x + \beta_0)^{\delta - 1/3}}{(3\delta + 2)(t + \varepsilon)^2} - \frac{9\alpha_0^2(\delta + 1)}{(3\delta + 2)^2(x + \beta_0)^{5/3}(t + \varepsilon)^2} + \frac{G[(x + \beta_0)^{-\delta - 2/3}]}{(x + \beta_0)^{1/3}}.$$

In this case, we have

$$\mu_1 = -\frac{(t + \varepsilon)(3\delta + 2)}{3(x + \beta_0)}, \quad \mu_2 = -\frac{y}{3(x + \beta_0)},$$

$$\mu_3 = \frac{2}{3(x + \beta_0)}, \quad \mu_4 = -\frac{\alpha y(3\delta + 4)}{(3\delta + 2)(t + \varepsilon)(x + \beta_0)^{4/3}}.$$

Then, from the relations in (38), we find

$$s(x, t) = (t + \epsilon)(x + \beta_0)^{-\delta-2/3}, \quad q(x, y, t) = \frac{y}{(x + \beta_0)^{1/3}},$$

$$\beta(x, t) = (x + \beta_0)^{2/3}, \quad \alpha(x, y, t) = \frac{3\alpha y}{(3\delta + 2)(t + \epsilon)(x + \beta_0)^{1/3}}.$$

The corresponding two-dimensional reduction has the form:

$$3w_{qqq} - 3\gamma(3\delta + 2)w_{sq} + (3\delta + 2)s(w_q w_{sq} - w_s w_{qq}) + 2w w_{qq} - w_q^2 + 3G(s) = 0.$$

5.  $r(x) = \alpha_0(x + \beta_0)^{-4/3}$ ,  
 $F(x, t) = G[(x + \beta_0)^{1/3} \exp(\int dt/a(t))] \exp[\int dt/a(t)] + \frac{3\alpha_0^2(a'(t)-1)}{(x+\beta_0)^{5/3}a(t)^2}.$

In this case, we have

$$\mu_1 = \frac{a(t)}{3(x + \beta_0)}, \quad \mu_2 = -\frac{y}{3(x + \beta_0)},$$

$$\mu_3 = \frac{2}{3(x + \beta_0)}, \quad \mu_4 = \frac{\alpha_0 y(2 - a'(t))}{a(t)(x + \beta_0)^{4/3}}.$$

Then, from the relations in (38), we find

$$s(x, t) = (x + \beta_0)^{1/3} \exp\left(\int \frac{dt}{a(t)}\right), \quad q(x, y, t) = \frac{y}{(x + \beta_0)^{1/3}},$$

$$\beta(x, t) = (x + \beta_0)^{2/3}, \quad \alpha(x, y, t) = -\frac{3\alpha_0 y}{a(t)(x + \beta_0)^{1/3}}.$$

The corresponding two-dimensional reduction has the form:

$$3w_{qqq} + s(w_{qq}w_s - w_q w_{sq}) + 2w w_{qq} - w_q^2 + 3G(s) = 0.$$

### 4.3. Reductions to ODEs (One-Dimensional Reductions)

Let us find all one-dimensional reductions of Equation (28), i.e., its reduction to an ODE. We are looking for such one-dimensional reductions in the following form:

$$u = U(x, y, t, w(z(x, y, t))). \tag{40}$$

Substituting Expression (40) into Equation (28), we obtain the following:

$$-U_w z_y^3 w''' - 3U_{ww} z_y^3 w' w'' + \dots = 0. \tag{41}$$

Here, for simplicity, only two terms are written out. Similarly to the above, using the condition that the normalized Ratio (41) is an ODE, we obtain a linear form of one-dimensional reduction

$$u = \beta(x, y, t)w(z(x, y, t)) + \alpha(x, y, t). \tag{42}$$

**Remark 9.** When deriving Expression (42), it was assumed that  $z_y \neq 0$ . The case  $z_y = 0$  corresponds to a degenerate reduction. This case is not of particular interest, and its detailed consideration is not given here.

Substituting Expression (42) into Equation (28), we obtain the following equation:

$$-\beta z_y^3 w''' + \beta z_y (\beta_y z_x - \beta_x z_y) w w'' + w'^2(z) \beta (-z_x \beta z_{yy})$$

$$\begin{aligned}
 &+ z_y(\beta_x z_y - \beta_y z_x + \beta z_{xy}) + w w'(-\beta_x(2\beta_y z_y + \beta z_{yy}) - \beta z_x \beta_{yy}) \\
 &+ \beta_y(\beta_x z_y + \beta_y z_x + \beta z_{xy}) + \beta z_y \beta_{xy} + w^2(\beta_y \beta_{xy} - \beta_x \beta_{yy}) \\
 &+ w''(-3\beta z_y z_{yy} + r(x)\beta z_y z_t - \alpha_x \beta z_y^2 + \alpha_y \beta z_y z_x - 3\beta_y z_y^2) \\
 &+ w'(-\alpha_x(2\beta_y z_y + \beta z_{yy}) - \beta z_x \alpha_{yy} + \alpha_y(\beta_x z_y + \beta_y z_x + \beta z_{xy}) \\
 &- 3\beta_y z_{yy} + r(x)(\beta_t z_y + \beta_y z_t + \beta z_{ty}) - 3\beta_{yy} z_y + \beta z_y \alpha_{xy} - \beta z_{yyy}) \\
 &+ w(-\alpha_x \beta_{yy} + r(x)\beta_{ty} - \beta_x \alpha_{yy} - \beta_{yyy} + \alpha_y \beta_{xy} + \beta_y \alpha_{xy}) \\
 &+ r(x)\alpha_{ty} - \alpha_{yyy} + \alpha_y \alpha_{xy} - \alpha_x \alpha_{yy} - F(x, t) = 0.
 \end{aligned} \tag{43}$$

From the condition that Equation (43) is an ODE, we obtain the following overdetermined system of differential equations:

$$\begin{aligned}
 \frac{\beta_y z_x - \beta_x z_y}{z_y^2} &= \Gamma_1(z), \\
 \frac{-\beta_x z_y^2 + z_y \beta_y z_x - z_y \beta z_{xy} + z_x \beta z_{yy}}{z_y^3} &= \Gamma_2(z), \\
 \frac{-\beta_y \beta_x z_y + \beta_y^2 z_x + \beta_y \beta z_{xy} + \beta z_y \beta_{xy} - \beta_x \beta z_{yy} - \beta z_x \beta_{yy}}{\beta z_y^3} &= \Gamma_3(z), \\
 \frac{\beta_y \beta_{xy} - \beta_x \beta_{yy}}{\beta z_y^3} &= \Gamma_4(z), \\
 \frac{r(x)\beta z_t z_y + \alpha_y \beta z_y z_x - \alpha_x \beta z_y^2 - 3\beta_y z_y^2 - 3\beta z_y z_{yy}}{\beta z_y^3} &= \Gamma_5(z), \\
 \frac{r(x)(\beta_y z_t + \beta_t z_y + \beta z_{ty}) + \alpha_y(\beta_x z_y + \beta_y z_x + \beta z_{xy}) + \beta z_y \alpha_{xy}}{\beta z_y^3} \\
 + \frac{-\alpha_x(2\beta_y z_y + \beta z_{yy}) - \beta z_x \alpha_{yy} - 3\beta_{yy} z_y - 3\beta_y z_{yy} - \beta z_{yyy}}{\beta z_y^3} &= \Gamma_6(z), \\
 \frac{r(x)\beta_{ty} + \alpha_y \beta_{xy} + \beta_y \alpha_{xy} - \alpha_x \beta_{yy} - \beta_x \alpha_{yy} - \beta_{yyy}}{\beta z_y^3} &= \Gamma_7(z), \\
 \frac{r(x)\alpha_{ty} + \alpha_y \alpha_{xy} - \alpha_x \alpha_{yy} - \alpha_{yyy} - F(x, t)}{\beta z_y^3} &= \Gamma_8(z).
 \end{aligned} \tag{44}$$

Similarly to the case of two-dimensional reduction, the system of Equation (44) admits the following transformations:

$$z \rightarrow G_1(z), \quad \beta \rightarrow \beta G_2(z), \quad \alpha \rightarrow \alpha + \beta G_3(z), \tag{45}$$

where  $G_1(z)$ ,  $G_2(z)$  and  $G_3(z)$  are arbitrary functions. These transformations are associated with arbitrariness in finding a reduction of Form (42). Consider auxiliary functions  $\mu_1 = \mu_1(x, y, t)$ ,  $\mu_2 = \mu_2(x, y, t)$ ,  $\mu_3 = \mu_3(x, y, t)$ ,  $\mu_4 = \mu_4(x, y, t)$ ,  $\mu_5 = \mu_5(x, y, t)$  and  $\mu_6 = \mu_6(x, y, t)$ , determined from the following relations:

$$\begin{aligned}
 z_x - z_y \mu_1 &= 0, & z_t - z_y \mu_2 &= 0, \\
 \beta_x - \beta \mu_3 - \beta_y \mu_1 &= 0, & \beta_t - \beta \mu_4 - \beta_y \mu_2 &= 0, \\
 \alpha_x - \mu_5 - \alpha_y \mu_1 - \alpha \mu_3 &= 0, & \alpha_t - \mu_6 - \alpha_y \mu_2 - \alpha \mu_4 &= 0.
 \end{aligned} \tag{46}$$

It can be shown that the introduced auxiliary functions are invariants of the transformations in (45).

Each of the equations of the system of Equation (44) can be rewritten as two equations  $\phi_x z_y - \phi_y z_x = 0$ ,  $\phi_t z_y - \phi_y z_t = 0$ , where  $\phi(x, y, t)$  is the left-hand side of the corresponding equation. From here, we can obtain a system of sixteen equations, which is not given

here due to its bulkiness. Expressing the derivatives of  $z_x, \alpha_x, \beta_x, z_t, \beta_t, \alpha_t \dots$  via the derivatives  $z_y, \beta_y, \alpha_y \dots$  using the relations in (46), it is possible to rewrite the resulting system only in terms of the invariants  $\mu_1 \dots, \mu_6$ . To the resulting sixteen equations for  $\mu_1 \dots, \mu_6$ , three more compatibility conditions should be added, obtained from the relations  $z_{xt} = z_{tx}, \beta_{xt} = \beta_{tx}$  and  $\alpha_{xt} = \alpha_{tx}$ . Thus, the following overdetermined system of equations consisting of nineteen equations for six unknown functions is subject to solution:

$$\begin{aligned}
 &\mu_{1y}\mu_3 + \mu_1\mu_{3y} - \mu_{3x} - \mu_3^2 = 0, & \mu_3\mu_{2y} - \mu_{3t} - \mu_3\mu_4 + \mu_2\mu_{3y} &= 0, \\
 &\mu_{1y}^2 - \mu_{1xy} + \mu_1\mu_{1yy} - \mu_{1y}\mu_3 = 0, & \mu_{1y}\mu_4 - \mu_{1y}\mu_{2y} + \mu_{1ty} - \mu_2\mu_{1yy} &= 0, \\
 &\mu_{3y}(2\mu_{1y} - \mu_3) = 0, & \mu_{4y}(2\mu_{1y} - \mu_3) &= 0, \\
 &\mu_{3y}^2 - \mu_3\mu_{3yy} = 0, & \mu_{3y}\mu_{4y} - \mu_3\mu_{4yy} &= 0, \\
 &2r(x)\mu_2\mu_{1y} - r'(x)\mu_2 - r(x)\mu_{1t} + \mu_3\mu_5 - \mu_5\mu_{1y} - \mu_1\mu_{5y} + 3\mu_{3y} + \mu_{5x} + 3\mu_{1yy} &= 0, \\
 &2r(x)\mu_2\mu_{2y} - r(x)\mu_{2t} - \mu_{2y}\mu_5 - \mu_2\mu_{5y} + 3\mu_4 + 3\mu_{2yy} + \mu_{5t} + \mu_3\mu_6 &= 0, \\
 &2r(x)\mu_{1y}\mu_4 - r(x)\mu_2\mu_{3y} + 2r(x)\mu_{1y}\mu_{2y} - r'(x)\mu_{2y} - r(x)\mu_{1ty} - 2\mu_3\mu_{5y} + \mu_{1yy}\mu_5 & \\
 &\quad + \mu_1\mu_{5yy} + \mu_{3y}\mu_5 - r(x)\mu_{3t} - r'(x)\mu_4 + \mu_{1yyy} - \mu_{5xy} + 3\mu_{3yy} &= 0, \\
 &r(x)\mu_2\mu_{4y} - 2r(x)\mu_{2y}\mu_4 - 2\mu_{2y}\mu_{5y} - \mu_2\mu_{5yy} - \mu_{2yy}\mu_5 + 2\mu_{1y}\mu_{6y} + 2\mu_3\mu_{6y} + \mu_{3y}\mu_6 & \\
 &\quad - 2\mu_{4y}\mu_5 - \mu_{2yyy} - 3\mu_{4yy} + \mu_{5ty} + r(x)\mu_{2ty} + r(x)\mu_{4t} - 2r(x)\mu_{2y}^2 &= 0, \\
 &\mu_3\mu_{5yy} - r(x)\mu_{3ty} - r'(x)\mu_{4y} + \mu_{3yy} + \mu_{3yy}\mu_5 - 2\mu_{3y}\mu_{5y} & \\
 &\quad - r(x)\mu_{3y}\mu_4 + 2r(x)\mu_{1y}\mu_{4y} &= 0, \\
 &r(x)\mu_4\mu_{4y} - 2r(x)\mu_{2y}\mu_{4y} + r(x)\mu_{4ty} - \mu_3\mu_{6yy} - \mu_{4yyy} & \\
 &\quad + \mu_{3y}\mu_{6y} - \mu_{4yy}\mu_5 + \mu_{4y}\mu_{5y} &= 0, \\
 &2r(x)\mu_{1y}\mu_{6y} - r(x)\mu_{3y}\mu_6 + \mu_{5yyy} - \mu_{5y}^2 + F_x(x, t) - r(x)\mu_{5ty} + \mu_{5yy}\mu_5 & \\
 &\quad - r'(x)\mu_{6y} - 3F(x, t)\mu_{1y} - F(x, t)\mu_3 &= 0, \\
 &r(x)\mu_{4y}\mu_6 - 2r(x)\mu_{2y}\mu_{6y} - F_t(x, t) - \mu_{6yyy} + r(x)\mu_{6ty} + 3F(x, t)\mu_{2y} & \\
 &\quad - \mu_{6yy}\mu_5 + F(x, t)\mu_4 + \mu_{6y}\mu_{5y} &= 0, \\
 &\mu_{1t} + \mu_1\mu_{2y} - \mu_{2x} - \mu_2\mu_{1y} = 0, & \mu_{4x} + \mu_2\mu_{3y} - \mu_{3t} - \mu_1\mu_{4y} &= 0, \\
 &\mu_{5t} + \mu_1\mu_{6y} + \mu_3\mu_6 - \mu_{6x} - \mu_2\mu_{5y} - \mu_4\mu_5 = 0.
 \end{aligned} \tag{47}$$

The overdetermined system of Equation (47) is solved explicitly. To solve it, it is enough to consider two cases: (1)  $\mu_3 = \mu_3(x, t), \mu_3 \neq 0$ ; (2)  $\mu_3 = 0$  (it can be shown that  $\mu_3 = \mu_3(x, t)$  and  $\mu_4 = \mu_4(x, t)$ ). The system of Equation (47) has a solution only for certain types of functions  $r(x)$  and  $F(x, t)$ . Let us write out these solutions.

1.  $r(x) = \alpha_0(x + \beta_0), F(x, t) = \gamma(x + at + b) + \alpha(\alpha x + \beta).$

In this case, we have

$$\begin{aligned}
 &\mu_1 = 0, & \mu_2 &= 0, \\
 &\mu_3 = \frac{1}{x + at + b}, & \mu_4 &= \frac{a}{x + at + b}, \\
 &\mu_5 = -\frac{ay(\alpha(at + b) - \beta)}{x + at + b}, & \mu_6 &= \frac{a^2y(\alpha * x + \beta)}{x + at + b}.
 \end{aligned}$$

Then, from the relations in (46), we find

$$\begin{aligned}
 &z(x, y, t) = y, & \beta(x, t) &= (x + at + b), \\
 &\alpha(x, y, t) = -ay(\alpha x + \beta).
 \end{aligned}$$

The corresponding one-dimensional reduction has the form:

$$\alpha\alpha(w' - zw'') - w'^2 + w''' + ww'' + \gamma = 0.$$

2.  $r(x)$  is an arbitrary function,  $F(x, t) = \alpha(x + \beta)^{\gamma-3}, \gamma \neq 0$ .

In this case, we have

$$\begin{aligned} \mu_1 &= \frac{y(\gamma - 4)}{4(x + \beta)}, & \mu_2 &= 0, \\ \mu_3 &= \frac{\gamma}{4(x + \beta)}, & \mu_4 &= 0, \\ \mu_5 &= 0, & \mu_6 &= 0. \end{aligned}$$

Then, from the relations in (46), we find

$$\begin{aligned} z(x, y, t) &= y(x + \beta)^{\gamma/4-1}, & \beta(x, t) &= (x + \beta)^{\gamma/4}, \\ \alpha(x, y, t) &= 0. \end{aligned}$$

The corresponding one-dimensional reduction has the form:

$$4w'^2 + 4w''' + 4\alpha + \gamma ww'' - 2\gamma w'^2 = 0.$$

3.  $r(x) = \alpha_0(x + \beta_0)^\gamma, F(x, t) = \delta(x + \beta)^{2\gamma+1}/t^2$ .

In this case, we have

$$\begin{aligned} \mu_1 &= \frac{\gamma y}{2(x + \beta_0)}, & \mu_2 &= -\frac{y}{2t}, \\ \mu_3 &= \frac{\gamma + 2}{2(x + \beta_0)}, & \mu_4 &= -\frac{1}{2t}, \\ \mu_5 &= 0, & \mu_6 &= 0. \end{aligned}$$

Then, from the relations in (46), we find

$$\begin{aligned} z(x, y, t) &= y^2(x + \beta_0)^\gamma/t, & \beta(x, y, t) &= (x + \beta_0)/y, \\ \alpha(x, y, t) &= 0. \end{aligned}$$

The corresponding one-dimensional reduction has the form:

$$\begin{aligned} z(2 + 3\gamma)ww' + 2z^2(2 + \gamma)ww'' - 4z^2(\gamma + 1)w'^2 + zw'(\alpha_0 z + 6) \\ + 2\alpha_0 z^3 w'' - 6w + w^2 + 8w'''z^3 = 0. \end{aligned}$$

4.  $r(x) = \alpha_0 \exp(\beta_0 x), F(x, t) = \gamma(x + \delta)^{-3+\epsilon}$ .

In this case, we obtain

$$\begin{aligned} \mu_1 &= \frac{(\epsilon - 4)y}{4(x + \delta)}, & \mu_2 &= 0, \\ \mu_3 &= \frac{\epsilon}{4(x + \delta)}, & \mu_4 &= 0, \\ \mu_5 &= 0, & \mu_6 &= 0. \end{aligned}$$

Then, from the relations in (46), we find

$$\begin{aligned} z(x, y, t) &= y(x + \delta)^{\gamma/4-1}, & \beta(x, y, t) &= (x + \delta)^{\epsilon/4}, \\ \alpha(x, y, t) &= 0. \end{aligned}$$

The corresponding one-dimensional reduction has the form:

$$2(2 - \epsilon)w'^2 + \epsilon ww'' + 4w''' + 4\gamma = 0.$$

$$5. \quad r(x) = \alpha_0(x + \beta_0)^{-4/3} + \gamma(x + \beta_0)^\delta, \\ F(x, t) = t^{-2}(x + \beta_0)^{2\delta+1}[\varepsilon - 3\alpha_0(2 + 3\delta)^{-2}(3\alpha_0(\delta + 1)(x + \beta_0)^{-8/3-2\delta} \\ + \gamma(2 + 3\delta)(x + \beta_0)^{-4/3-\delta}].$$

In this case, we obtain

$$\mu_1 = -\frac{\delta y}{2(x + \beta_0)}, \quad \mu_2 = -\frac{y}{2t}, \\ \mu_3 = \frac{\delta + 2}{2(x + \beta_0)}, \quad \mu_4 = -\frac{1}{2t}, \\ \mu_5 = -\frac{\alpha(4 + 3\delta)y}{(2 + 3\delta)t(x + \beta_0)^{4/3}}, \quad \mu_6 = 0.$$

Then, from the relations in (46), we find

$$z(x, y, t) = \frac{t(x + \beta_0)^{-\delta}}{y^2}, \quad \beta(x, y, t) = \frac{(x + \beta_0)}{y}, \\ \alpha(x, y, t) = \frac{3\alpha_0 y}{(x + \beta_0)^{1/3}t(2 + 3\delta)}.$$

The corresponding one-dimensional reduction has the form:

$$2z^4(\delta + 2)ww'' + z^3(\delta + 6)ww' - 4z^4(\delta + 1)w'^2 + 3z^2(\gamma - 18z)w' \\ + 2z^3(\gamma - 24z)w'' + z^2w^2 - 8z^5w''' - 6wz^2 + \varepsilon = 0.$$

### 5. Conclusions

In this work, a method for constructing reductions of PDEs with two independent variables was considered. The method is based on the idea of invariance. An equation describing a steady-state laminar flat boundary layer with a pressure gradient was considered, and all reductions of this equation were obtained.

Further development of this method for finding reductions, based on the idea of invariance, was proposed for a PDE with three independent variables. The proposed method is a modification of the Clarkson–Kruskal direct method and expands the possibilities for its application. An equation describing a laminar unsteady axisymmetric boundary layer with a pressure gradient was analyzed. All reductions of this equation to ODEs (one-dimensional reductions) and simpler PDEs (two-dimensional reductions) were obtained.

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