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Article

Symmetries, Reductions and Exact Solutions of Nonstationary Monge–Ampère Type Equations

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Abstract: A family of strongly nonlinear nonstationary equations of mathematical physics with three independent variables is investigated, which contain an arbitrary degree of the first derivative with respect to time and a quadratic combination of second derivatives with respect to spatial variables of the Monge–Ampère type. Individual PDEs of this family are encountered, for example, in electron magnetohydrodynamics and differential geometry. The symmetries of the considered parabolic Monge–Ampère equations are investigated by group analysis methods. Formulas are obtained that make it possible to construct multiparameter families of solutions based on simpler solutions. Two-dimensional and one-dimensional symmetry and non-symmetry reductions are considered, which lead to the original equation to simpler partial differential equations with two independent variables or ordinary differential equations or systems of such equations. Self-similar and other invariant solutions are described. A number of new exact solutions are constructed by methods of generalized and functional separation of variables, many of which are expressed in elementary functions or in quadratures. To obtain exact solutions, the principle of the structural analogy of solutions was also used, as well as various combinations of all the above-mentioned methods. In addition, some solutions are constructed by auxiliary intermediate-point or contact transformations. The obtained exact solutions can be used as test problems intended to check the adequacy and assess the accuracy of numerical and approximate analytical methods for solving problems described by highly nonlinear equations of mathematical physics.

Keywords: parabolic Monge–Ampère equations; strongly nonlinear partial differential equations; group analysis; one- and two-dimensional reductions; exact solutions; self-similar solutions; solutions with additive, multiplicative and generalized separation of variables; magnetohydrodynamics

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Contents

1. Introduction

1°. Stationary equations of the Monge–Ampère type with two independent variables containing quadratic nonlinearity with respect to the highest derivatives of the form $w_{xx}w_{yy} - w_{xy}^2$ have been considered in many papers (see, for example, [1–8]), where their qualitative features and applications are described. Exact solutions of such and related strongly nonlinear PDEs are given in [5,7–11].

2°. Nonstationary Monge–Ampère equations of the form

$$w_t(w_{xx}w_{yy} - w_{xy}^2) = \sigma, \tag{1}$$

where $\sigma < 0$ is a constant or a function of spatial variables; and generalizations of this equation to the case of many variables, when the expression in parentheses is replaced by $\det[w_{x_i x_j}]$, were considered in the works (see, for example, [12–17]) in which geometric applications were discussed and questions of existence and uniqueness of solutions for various internal and external initial-boundary value problems were investigated. Exact solutions of the nonlinear PDE (1) have not been considered so far.

In this paper, we will analyze the generalized nonstationary equation of magnetohydrodynamics with Monge–Ampère type nonlinearity in spatial variables and power nonlinearity with respect to the time derivative

$$w_{xx}w_{yy} - w_{xy}^2 = \sigma(w_t)^m, \qquad \sigma \neq 0, \tag{2}$$

where m and σ are free constants (unless otherwise specified). Equation (2) at $m = -1$ passes into Equation (1).

Equation (2) in the special case $m = 1$ extends to the equation of electron magnetohydrodynamics [18–20]. Simple exact solutions of this equation with additive and multiplicative separation of variables are described in [21,22]. In [23,24], a group analysis of Equation (2) for $m = 1$ is carried out, where some of its invariant solutions are described, and a number of non-invariant solutions with generalized separation of variables are constructed.

The issues of existence and uniqueness of solutions to Equation (2) are discussed in [25–28].

Equation (2) is strongly nonlinear (quadratic with respect to the highest derivatives). It belongs to the parabolic Monge–Ampère equations and has properties unusual for quasilinear equations, which are linear with respect to the highest derivatives. In particular, even for the simplest stationary case with $m = 0$, the qualitative features of Equation (2) depend on the sign of the constant σ, since for $\sigma > 0$, this equation is an equation of elliptic type, and for $\sigma < 0$, it is an equation of hyperbolic type [2,8]. Moreover, unlike the overwhelming majority of other equations of mathematical physics, which do not depend explicitly on the independent variables, Equation (2) has no solutions of the traveling wave type (this fact for the special case $m = 1$ was noted in [24]).

Table 1 presents some simple multiparameter exact solutions of the nonstationary Monge–Ampère Equation (2) with $m = \sigma = 1$, which are expressed in terms of elementary functions (according to [23,24]).

Table 1. Multiparameter exact solutions of the nonstationary Monge–Ampère Equation (2) with $m = \sigma = 1$. Notation: $A, C_1, \ldots, C_6, \mu$ are arbitrary constants.

No.	Exact Solutions of the NonStationary Monge–Ampère Equation (2) with $m = \sigma = 1$
1	$w = C_1y^2 + C_2xy + C_3x^2 + (4C_1C_3 - C_2^2)t$
2	$w = C_1y^4 + (24C_1C_2t + C_3)y^2 + C_2x^2 + 48C_1C_2^2t^2 + 4C_2C_3t$
3	$w = -At + \dfrac{1}{x + C_1}\Big(C_2y^2 + C_3y + \dfrac{C_3^2}{4C_2}\Big) - \dfrac{A}{12C_2}(x^3 + 3C_1x^2)$
4	$w = -At \pm \dfrac{2\sqrt{A}}{3C_1C_2}(C_1x - C_2^2y^2 + C_3)^{3/2}$
5	$w = -Axt + C_1y^2 + C_2xy - \dfrac{A}{12C_1}x^3 + \dfrac{C_2^2}{4C_1}x^2$
6	$w = -Axt + C_1\dfrac{y^2}{x} + C_2y - \dfrac{A}{24C_1}x^4 + C_3x + C_4$
7	$w = \dfrac{1}{2}C_1y^2 + C_2xy + \dfrac{1}{2}C_3x^2 + (C_1C_3 - C_2^2)t + C_4\exp(C_1\mu^2t \pm \mu x)$
8	$w = \dfrac{1}{2}C_1y^2 + C_2xy + \dfrac{1}{2}C_3x^2 + (C_1C_3 - C_2^2)t + C_4\exp(-C_1\mu^2t)\sin(\mu x)$
9	$w = -\dfrac{(y + C_1x^2 + C_2)^3}{36C_1(t + C_3)} + C_4x + C_5y + C_6$
10	$w = -\dfrac{1}{12(4C_1C_3 - C_2^2)(t + C_4)}\Big[C_1x^2 + C_2xy + C_3y^2\Big]^2$

3°. In this paper, the main attention is paid to the construction of exact solutions of Equation (2). Here and below, we understand the term "exact solution" in the same sense as in [24,29].

Exact solutions of nonlinear partial differential equations are most often constructed using methods of group analysis [4,7,30], methods of generalized and functional separation of variables [8,29,31,32], the method of differential relations [8,29,33,34] and some other analytical methods (see, for example, [8,29,35–39]).

In this paper, to find exact solutions to the generalized equation of magnetohydrodynamics (2), various modifications of the method of generalized separation of variables [8,29,31] and exact solutions of simpler than the original intermediate reduced equations with a smaller number of independent variables given in [8,9] are mainly used. In addition, to construct exact solutions of the nonlinear PDE (2), the principle of structural analogy of solutions was also used, which is formulated as follows: exact solutions of simpler PDEs can serve as a basis for constructing solutions of more complex related PDEs [36,37]. Namely, to construct a series of exact solutions of Equation (2) with $m \neq 1$, we used the structure of known exact solutions of the simpler equation with $m = 1$ [24]. It should be noted that we pay special attention to constructing simple exact solutions that are expressed through elementary functions or quadratures.

2. Symmetries of the Nonstationary Monge–Ampère Equation

2.1. Symmetries of the Equation in the Basic Case for $m \neq 2$

We seek the symmetry operators of Equation (2) in the form

$$X = \xi^1(x,y,t,w)\frac{\partial}{\partial x} + \xi^2(x,y,t,w)\frac{\partial}{\partial y} + \xi^3(x,y,t,w)\frac{\partial}{\partial t} + \eta(x,y,t,w)\frac{\partial}{\partial w}\,. \tag{3}$$

Applying the invariance criterion [4] to the nonlinear PDE (2) for $m \neq 2$, we obtain the following overdetermined linear homogeneous system of defining equations:

$$\begin{aligned}
&\xi^1_t = 0\,, \quad \xi^1_w = 0\,, \quad \xi^2_t = 0\,, \quad \xi^2_w = 0\,,\\
&\xi^3_x = 0\,, \quad \xi^3_y = 0\,, \quad \xi^3_w = 0\,,\\
&\eta_t = 0\,, \quad \eta_w + \frac{2\xi^1_x + 2\xi^2_y - m\xi^3_t}{m-2} = 0\,,\\
&\xi^1_{xx} = 0\,, \quad \xi^1_{xy} = 0\,, \quad \xi^1_{yy} = 0\,,\\
&\xi^2_{xx} = 0\,, \quad \xi^2_{xy} = 0\,, \quad \xi^2_{yy} = 0\,,\\
&\xi^3_{tt} = 0\,, \quad \eta_{xx} = 0\,, \quad \eta_{xy} = 0\,, \quad \eta_{yy} = 0\,.
\end{aligned} \tag{4}$$

It is easy to show that the solution of system (4) is given by the formulas

$$\begin{aligned}
\xi^1 &= \alpha_1 x + \alpha_2 y + \alpha_3\,,\\
\xi^2 &= \alpha_4 x + \alpha_5 y + \alpha_6\,,\\
\xi^3 &= \alpha_7 t + \alpha_8\,,\\
\eta &= \frac{mc_7 - 2(\alpha_1 + \alpha_5)}{m-2} w + \alpha_9 x + \alpha_{10} y + \alpha_{11},
\end{aligned}$$

where α_j $(j = 1, \dots, 11)$ are arbitrary constants.

Proposition 1. *The basis of the Lie algebra of symmetry operators of Equation (2) for $m \neq 2$ has the form*

$$\begin{aligned}
&X_1 = \frac{\partial}{\partial x}\,, \quad X_2 = \frac{\partial}{\partial y}\,, \quad X_3 = \frac{\partial}{\partial t}\,, \quad X_4 = \frac{\partial}{\partial w}\,,\\
&X_5 = y\frac{\partial}{\partial x}\,, \quad X_6 = x\frac{\partial}{\partial y}\,, \quad X_7 = x\frac{\partial}{\partial w}\,, \quad X_8 = y\frac{\partial}{\partial w}\,,\\
&X_9 = x\frac{\partial}{\partial x} - \frac{2w}{m-2}\frac{\partial}{\partial w}\,, \quad X_{10} = y\frac{\partial}{\partial y} - \frac{2w}{m-2}\frac{\partial}{\partial w}\,,\\
&X_{11} = t\frac{\partial}{\partial t} + \frac{mw}{m-2}\frac{\partial}{\partial w}\,.
\end{aligned}$$

From Proposition 1, the following preposition follows.

Proposition 2. *For $m \neq 2$, the transformation*

$$\begin{aligned}
&\bar{x} = a_1 x + b_1 y + c_1, \quad \bar{y} = a_2 x + b_2 y + c_2, \quad \bar{t} = pt + q, \quad p \neq 0,\\
&\bar{w} = kw + a_3 x + b_3 y + c_3,\\
&k = p\,|p|^{\frac{2}{m-2}}\,|a_1 b_2 - a_2 b_1|^{-\frac{2}{m-2}}, \quad a_1 b_2 - a_2 b_1 \neq 0,
\end{aligned} \tag{5}$$

where a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, p and q are arbitrary constants, transforms Equation (2) into itself.

The eleven-parameter transformation (5) allows one to construct more complex exact solutions of Equation (2) using simpler particular solutions. Namely, if $w = \Phi(x, y, t)$ is a solution of Equation (2) for $m \neq 2$, then the function

$$w = \frac{1}{k}[\Phi(a_1x + b_1y + c_1, a_2x + b_2y + c_2, pt + q)) - (a_3x + b_3y + c_3)] \tag{6}$$

is also a solution of this equation.

2.2. Symmetries of the Equation in the Special Case for $m = 2$

The symmetry operators of Equation (2) for $m = 2$, as before, are searched in Formula (3). Using the invariance criterion [4], in this case, we obtain an overdetermined system of determining equations:

$$\begin{array}{lllll}
\xi^1_t = 0\,, & \xi^1_w = 0\,, & \xi^2_t = 0\,, & \xi^2_w = 0\,, & \\
\xi^3_x = 0\,, & \xi^3_y = 0\,, & \xi^3_w = 0\,, & \xi^1_x + \xi^2_y - \xi^3_t = 0\,, & \\
\eta_t = 0\,, & \xi^1_{xx} = 0\,, & \xi^2_{xx} = 0\,, & \xi^2_{xy} = 0\,, & \xi^2_{yy} = 0\,, \\
\xi^3_{tt} = 0\,, & \eta_{xx} = 0\,, & \eta_{xy} = 0\,, & \eta_{yy} = 0\,, & \\
\eta_{xw} = 0\,, & \eta_{xw} = 0\,, & \eta_{ww} = 0\,. & &
\end{array}$$

The solution of this system has the form

$$\begin{aligned}
\xi^1 &= \alpha_1 x + \alpha_2 y + \alpha_3\,, \\
\xi^2 &= \alpha_4 x + \alpha_5 y + \alpha_6\,, \\
\xi^3 &= (\alpha_1 + \alpha_5)t + \alpha_7\,, \\
\eta &= \alpha_8 x + \alpha_9 y + \alpha_{10} w + \alpha_{11},
\end{aligned}$$

where α_j $(j = 1, \ldots, 11)$ are arbitrary constants.

Proposition 3. *The basis of the Lie algebra of symmetry operators of Equation (2) for $m = 2$ has the form*

$$\begin{array}{llll}
X_1 = \dfrac{\partial}{\partial x}\,, & X_2 = \dfrac{\partial}{\partial y}\,, & X_3 = \dfrac{\partial}{\partial t}\,, & X_4 = \dfrac{\partial}{\partial w}\,, \\
X_5 = y\dfrac{\partial}{\partial x}\,, & X_6 = x\dfrac{\partial}{\partial y}\,, & X_7 = x\dfrac{\partial}{\partial w}\,, & X_8 = y\dfrac{\partial}{\partial w}\,, \\
X_9 = w\dfrac{\partial}{\partial w}\,, & X_{10} = x\dfrac{\partial}{\partial x} + t\dfrac{\partial}{\partial t}\,, & X_{11} = y\dfrac{\partial}{\partial y} + t\dfrac{\partial}{\partial t}\,. &
\end{array}$$

From Proposition 3, the following proposition follows.

Proposition 4. *For $m = 2$, the transformation*

$$\begin{aligned}
&\bar{x} = a_1x + b_1y + c_1, \quad \bar{y} = a_2x + b_2y + c_2, \\
&\bar{t} = pt + q, \quad p = a_1b_2 - a_2b_1, \quad a_1b_2 - a_2b_1 \neq 0, \\
&\bar{w} = kw + a_3x + b_3y + c_3, \quad k \neq 0,
\end{aligned} \tag{7}$$

where a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, k and q are arbitrary constants, transforms Equation (2) into itself.

The eleven-parameter transformation (7) allows for using simpler partial solutions of Equation (2) to construct its more complex exact solutions. Namely, if $w = \Phi(x, y, t)$ is the solution of Equation (2) for $m = 2$, then the function

$$w = \frac{1}{k}[\Phi(a_1x + b_1y + c_1, a_2x + b_2y + c_2, pt + q) - (a_3x + b_3y + c_3)]$$

is also the solution of this equation.

3. Two-Dimensional Symmetry Reductions

The regular procedure for constructing two-dimensional symmetric reductions of partial differential equations is described in [4,30]. In this paper, we restrict ourselves to the most informative examples of constructing two-dimensional reductions of the parabolic Monge–Ampère Equation (2) based on the use of the symmetries described above.

1°. Passing into Equation (2) to variables of the traveling wave type,

$$w = W(\xi, \eta), \quad \xi = x + a_1t, \quad \eta = y + a_2t, \tag{8}$$

where a_1 and a_2 are arbitrary constants, we arrive at a two-dimensional equation of the Monge–Ampère type:

$$W_{\xi\xi}W_{\eta\eta} - W_{\xi\eta}^2 = \sigma(a_1W_\xi + a_2W_\eta)^m.$$

The solution of Formula (8) is invariant with respect to a one-parameter group of transformations defined by the symmetry operator

$$Y = a_1X_1 + a_2X_2 - X_3 = a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} - \frac{\partial}{\partial t}.$$

2°. For $m \neq 2$, passing into Equation (2) to variables of the self-similar type where α and β are arbitrary constants, we obtain a two-dimensional equation of the Monge–Ampère type with variable coefficients at lower derivatives.

For $m \neq 2$, passing into Equation (2) to variables of the self-similar type,

$$w = t^{\frac{2\alpha+2\beta+m}{m-2}}W(\xi, \eta), \quad \xi = xt^\alpha, \quad \eta = yt^\beta, \tag{9}$$

where α and β are arbitrary constants, we obtain a two-dimensional equation of the Monge–Ampère type with variable coefficients at the lower derivatives:

$$W_{\xi\xi}W_{\eta\eta} - W_{\xi\eta}^2 = \sigma\Big(\alpha\xi W_\xi + \beta\eta W_\eta + \frac{2\alpha + 2\beta + m}{m-2}W\Big)^m. \tag{10}$$

The solution of Formula (10) is invariant with respect to a one-parameter group of transformations defined by the symmetry operator

$$Y = \alpha X_9 + \beta X_{10} - X_{11} = \alpha x\frac{\partial}{\partial x} + \beta y\frac{\partial}{\partial y} - t\frac{\partial}{\partial t} - \frac{(2\alpha + 2\beta + m)w}{m-2}\frac{\partial}{\partial w}.$$

Remark 1. *Substituting $\alpha = \beta = 0$ into (9), we arrive at the multiplicative separable solution*

$$w = t^{\frac{m}{m-2}}W(x, y).$$

Remark 2. *An equivalent form of solution representation can be obtained from (9) by taking, instead of the second argument, a combination of both arguments $\zeta = \xi^{-\beta}\eta^{\alpha} = x^{-\beta}y^{\alpha}$, which leads to a two-dimensional solution of the form*

$$w = t^{\frac{2\alpha+2\beta+m}{m-2}} W(\xi, \zeta), \quad \xi = xt^{\alpha}, \quad \zeta = y^{\alpha}x^{-\beta}. \tag{11}$$

3°. For $m \neq 2$, passing into Equation (2) to variables of the limiting self-similar type

$$w = \exp\Big[\frac{2(\alpha+\beta)}{m-2}t\Big] W(\xi, \eta), \quad \xi = x\exp(\alpha t), \quad \eta = y\exp(\beta t), \tag{12}$$

where α and β are arbitrary constants, we obtain another two-dimensional Monge–Ampère type equation with variable coefficients for lower derivatives:

$$W_{\xi\xi}W_{\eta\eta} - W_{\xi\eta}^2 = \sigma\Big(\alpha\xi W_{\xi} + \beta\eta W_{\eta} + \frac{2(\alpha+\beta)}{m-2}W\Big)^m. \tag{13}$$

The solution of Formula (13) is invariant with respect to a one-parameter group of transformations defined by the symmetry operator:

$$Y = \alpha X_9 + \beta X_{10} - X_3 = \alpha x\frac{\partial}{\partial x} + \beta y\frac{\partial}{\partial y} - \frac{\partial}{\partial t} - \frac{2(\alpha+\beta)w}{m-2}\frac{\partial}{\partial w}.$$

Remark 3. *An equivalent form of solution representation can be obtained from Equation (12) by taking, instead of the second argument, a combination of both arguments, $\zeta = \xi^{-\beta}\eta^{\alpha} = x^{-\beta}y^{\alpha}$, which leads to a two-dimensional solution of the form*

$$w = \exp\Big[\frac{2(\alpha+\beta)}{m-2}t\Big] W(\xi, \zeta), \quad \xi = x\exp(\alpha t), \quad \zeta = y^{\alpha}x^{-\beta}. \tag{14}$$

4°. For $m \neq 2$, passing into Equation (2) to invariant variables

$$w = t^{\frac{m}{m-2}} W(\xi, \eta), \quad \xi = x + \lambda_1 \ln t, \quad \eta = y + \lambda_2 \ln t, \tag{15}$$

where λ_1 and λ_2 are arbitrary constants, we obtain another two-dimensional equation of the Monge–Ampère type with constant coefficients:

$$W_{\xi\xi}W_{\eta\eta} - W_{\xi\eta}^2 = \sigma\Big(\lambda W_{\xi} + \lambda_2 W_{\eta} + \frac{m}{m-2}W\Big)^m.$$

The solution of Formula (15) is invariant with respect to a one-parameter group of transformations defined by the symmetry operator

$$Y = \lambda_1 X_1 + \lambda_2 X_2 - X_{11} = \lambda_1\frac{\partial}{\partial x} + \lambda_2\frac{\partial}{\partial y} - t\frac{\partial}{\partial t} - \frac{mw}{m-2}\frac{\partial}{\partial w}.$$

5°. Equation (2) for $m \neq 2$ using invariant variables

$$w = x^{\frac{2}{2-m}} W(\xi, \eta), \quad \xi = t + \alpha \ln x, \quad \eta = y + \beta \ln x, \tag{16}$$

where α and β are arbitrary constants, is reduced to a two-dimensional PDE, which is omitted here.

The solution of Formula (16) is invariant with respect to a one-parameter group of transformations defined by the symmetry operator

$$Y = -\beta X_2 - \alpha X_3 + X_9 = x\frac{\partial}{\partial x} - \beta\frac{\partial}{\partial y} - \alpha\frac{\partial}{\partial t} - \frac{2w}{m-2}\frac{\partial}{\partial w}.$$

Remark 4. *The values of $\alpha = \beta = 0$ correspond to the multiplicative separable solution:*

$$w = x^{\frac{2}{2-m}}\, W(y,t).$$

6°. Equation (2) for $m \neq 2$ using invariant variables

$$w = \exp\Bigl(\frac{\alpha m - 2\beta}{2-m}x\Bigr)W(\xi,\eta), \quad \xi = t\exp(\alpha x), \quad \eta = y\exp(\beta x), \tag{17}$$

is reduced to a two-dimensional PDE, which is omitted here.

The solution of Formula (17) is invariant with respect to a one-parameter group of transformations defined by the symmetry operator

$$Y = X_1 - \beta X_{10} - \alpha X_{11} = \frac{\partial}{\partial x} - \beta y\frac{\partial}{\partial y} - \alpha t\frac{\partial}{\partial t} - \frac{(\alpha m - 2\beta)w}{m-2}\frac{\partial}{\partial w}.$$

7°. Equation (2) for $m \neq 2$ using invariant variables

$$w = \exp\Bigl(\frac{2\alpha}{m-2}x\Bigr)W(\xi,\eta), \quad \xi = t + \beta x, \quad \eta = y\exp(\alpha x), \tag{18}$$

is reduced to a two-dimensional PDE, which is omitted here.

The solution of Formula (18) is invariant with respect to a one-parameter group of transformations defined by the symmetry operator

$$Y = X_1 - \beta X_3 - \alpha X_{10} = \frac{\partial}{\partial x} - \alpha y\frac{\partial}{\partial y} - \beta\frac{\partial}{\partial t} + \frac{2\alpha w}{m-2}\frac{\partial}{\partial w}.$$

8°. For $m = 2$, there are multiplicative separable solutions of the form

$$w = e^{\lambda t}W(x,y), \tag{19}$$

where λ is an arbitrary constant, and the function $W = W(x,y)$ is described by the two-dimensional equation

$$W_{xx}W_{yy} - W_{xy}^2 = \sigma\lambda^2 W^2.$$

The solution of Formula (19) is invariant with respect to a one-parameter group of transformations defined by the symmetry operator

$$Y = X_3 + \lambda X_9 = \frac{\partial}{\partial t} + \lambda w\frac{\partial}{\partial w}.$$

9°. For $m = 2$, there are other multiplicative separable solutions

$$w = e^{\gamma x}W(y,t), \tag{20}$$

where λ is an arbitrary constant, and the function $W = W(x,y)$ is described by the two-dimensional equation

$$WW_{yy} - W_y^2 = \sigma\gamma^{-2}W_t^2.$$

The solution of Formula (20) is invariant with respect to a one-parameter group of transformations defined by the symmetry operator

$$Y = X_1 + \gamma X_9 = \frac{\partial}{\partial x} + \gamma w \frac{\partial}{\partial w}\,.$$

Remark 5. *More complicated two-dimensional reductions of Equation (2) can be obtained by replacing in (8), (9), (11), (12), (14), (15), (16)–(20) the spatial variables by their arbitrary linear combinations according to the rule $x \Longrightarrow a_1x + b_1y$ and $y \Longrightarrow a_2x + b_2y$.*

4. One-Dimensional Symmetry Reductions and Exact Solutions

The regular procedure for constructing one-dimensional reductions of partial differential equations is described in [4]. In this paper, we restrict ourselves to characteristic examples of constructing one-dimensional reductions and invariant exact solutions by using symmetries of the Monge–Ampère parabolic Equation (2).

1°. For $m \neq 2$, the simplest invariant solution of Equation (2) that allows for a scaling transformation is a solution in the form of a product of the corresponding powers of the independent variables

$$w = A(xy)^{-\frac{2}{m-2}} t^{\frac{m}{m-2}}, \qquad A = \left[\frac{4(m+2)(m-2)^{m-3}}{\sigma m^m}\right]^{\frac{1}{m-2}}. \tag{21}$$

This formula can be used for those values of parameters m and σ when A is a real number.

Below, we consider several invariant solutions that generalize solution (21) and can be obtained using simple methods described in [36,37].

Solution (21) is a special case of a wider family of invariant solutions of the form

$$w = x^{-\frac{2}{m-2}} t^{\frac{m}{m-2}} f(z), \qquad z = y + \beta \ln t, \tag{22}$$

where β is a free parameter, and the function $f = f(z)$ is described by ODE

$$\frac{2m}{(m-2)^2} f f''_{zz} - \frac{4}{(m-2)^2}(f'_z)^2 = \sigma\Big(\frac{m}{m-2} f + \beta f'_z\Big)^m.$$

The solution of Formula (22) is invariant with respect to a two-parameter group of transformations defined by the symmetry operators

$$Y_1 = X_9 = x\frac{\partial}{\partial x} - \frac{2w}{m-2}\frac{\partial}{\partial w},$$
$$Y_2 = -\beta X_2 + X_{11} = -\beta\frac{\partial}{\partial y} + t\frac{\partial}{\partial t} + \frac{mw}{m-2}\frac{\partial}{\partial w}.$$

Solution (21) is a special case of another, broader family of invariant solutions of the form

$$w = x^{-\frac{2}{m-2}} t^{\frac{m}{m-2}} g(\xi), \qquad \xi = y + \lambda \ln x, \tag{23}$$

where λ is a free parameter and the function $f = f(z)$ satisfies the ODE

$$\lambda g'_\xi g''_{\xi\xi} - \frac{2m}{(m-2)^2} g g''_{\xi\xi} + \frac{4}{(m-2)^2}(g'_\xi)^2 = -\sigma\Big(\frac{m}{m-2} g\Big)^m, \qquad m \neq 2.$$

The solution of Formula (23) is invariant with respect to a two-parameter group of transformations defined by the symmetry operators

$$Y_1 = X_{11} = t\frac{\partial}{\partial t} + \frac{mw}{m-2}\frac{\partial}{\partial w},$$
$$Y_2 = \lambda X_2 - X_9 = \lambda\frac{\partial}{\partial y} - x\frac{\partial}{\partial x} + \frac{2w}{m-2}\frac{\partial}{\partial w}.$$

Solution (21) is also a special case of another broader family of invariant solutions of the form

$$w = (xy)^{-\frac{2}{m-2}}\varphi(\eta), \quad \eta = t + \gamma\ln y, \tag{24}$$

where γ is a free parameter, and the function $\varphi = \varphi(\eta)$ satisfies the ODE

$$\frac{2m\gamma^2}{(2-m)^2}\varphi\varphi''_{\eta\eta} - \frac{4\gamma^2}{(2-m)^2}(\varphi'_\eta)^2 + \frac{2\gamma(m-6)}{(2-m)^3}\varphi\varphi'_\eta - \frac{4(m+2)}{(2-m)^3}\varphi^2 = \sigma(\varphi'_\eta)^m.$$

The solution of Formula (24) is invariant with respect to a two-parameter group of transformations defined by the symmetry operators

$$Y_1 = X_9 = x\frac{\partial}{\partial x} - \frac{2w}{m-2}\frac{\partial}{\partial w},$$
$$Y_2 = \gamma X_3 - X_{10} = y\frac{\partial}{\partial y} - \gamma\frac{\partial}{\partial t} - \frac{2w}{m-2}\frac{\partial}{\partial w}.$$

Remark 6. *In solutions (22)–(24), the spatial variables x and y can be swapped or Formula (6) can be used. For example, by applying Formula (6) with $a_1 = b_1 = a_2 = p = 1$, $b_2 = -1$, $a_4 = b_4 = c_1 = c_2 = c_4 = q = 0$ to the solution (21), we obtain a solution of a more complex form as follows:*

$$w = B(x^2 - y^2)^{\frac{2}{2-m}}t^{\frac{m}{m-2}}, \quad B = \left[\frac{16(m+2)(m-2)^{m-3}}{\sigma m^m}\right]^{\frac{1}{m-2}}.$$

2°. Equation (2) for $m \neq 2$ using invariant variables

$$w = t^{\frac{2\alpha+m}{m-2}}x^{\frac{2(\beta-1)}{m-2}}V(\zeta), \quad \zeta = t^\alpha x^\beta y \tag{25}$$

is reduced to a second-order ODE, which is not given here due to its bulkiness.

The solution of Formula (25) is invariant with respect to a two-parameter group of transformations defined by the symmetry operators

$$Y_1 = X_9 - \beta X_{10} = x\frac{\partial}{\partial x} - \beta y\frac{\partial}{\partial y} + \frac{2(\beta-1)w}{m-2}\frac{\partial}{\partial w},$$
$$Y_2 = X_{11} - \alpha X_{10} = t\frac{\partial}{\partial t} - \alpha y\frac{\partial}{\partial y} + \frac{(2\alpha+m)w}{m-2}\frac{\partial}{\partial w}.$$

3°. Equation (2) for $m \neq 2$ using invariant variables

$$w = \exp\left(\frac{2\alpha t}{m-2}\right)x^{\frac{2(\beta-1)}{m-2}}V(\zeta), \quad \zeta = \exp(\alpha t)x^\beta y \tag{26}$$

is reduced to a second-order ODE, which is not given here.

The solution of Formula (26) is invariant with respect to a two-parameter transformation group defined by symmetry operators

$$Y_1 = X_3 - \alpha X_{10} = \frac{\partial}{\partial t} - \alpha y \frac{\partial}{\partial y} + \frac{2\alpha w}{m-2}\frac{\partial}{\partial w},$$
$$Y_2 = X_9 - \beta X_{10} = x\frac{\partial}{\partial x} - \beta y \frac{\partial}{\partial y} + \frac{2(\beta-1)w}{m-2}\frac{\partial}{\partial w}.$$

4°. Equation (2) for $m \neq 2$ using invariant variables

$$w = t^{\frac{2\alpha+m}{m-2}} \exp\Big(\frac{2\beta x}{m-2}\Big) V(\zeta), \quad \zeta = t^{\alpha}\exp(\beta x) y \tag{27}$$

is reduced to a second-order ODE, which is not given here.

The solution of Formula (27) is invariant with respect to a two-parameter group of transformations defined by the symmetry operators

$$Y_1 = X_{11} - \alpha X_{10} = t\frac{\partial}{\partial t} - \alpha y \frac{\partial}{\partial y} + \frac{(2\alpha+m)w}{m-2}\frac{\partial}{\partial w},$$
$$Y_2 = X_1 - \beta X_{10} = \frac{\partial}{\partial x} - \beta y \frac{\partial}{\partial y} + \frac{2\beta w}{m-2}\frac{\partial}{\partial w}.$$

5°. Equation (2) for $m \neq 2$ using invariant variables

$$w = \exp\Big(\frac{2\alpha t + 2\beta x}{m-2}\Big) V(\zeta), \quad \zeta = \exp(\alpha t + \beta x) y \tag{28}$$

is reduced to a second-order ODE, which is not given here.

The solution of Formula (28) is invariant with respect to a two-parameter group of transformations defined by the symmetry operators

$$Y_1 = X_3 - \alpha X_{10} = \frac{\partial}{\partial t} - \alpha y \frac{\partial}{\partial y} + \frac{2\alpha w}{m-2}\frac{\partial}{\partial w},$$
$$Y_2 = X_1 - \beta X_{10} = \frac{\partial}{\partial x} - \beta y \frac{\partial}{\partial y} + \frac{2\beta w}{m-2}\frac{\partial}{\partial w}.$$

6°. For $m = 2$, there is a solution of the form

$$w = \exp(a_1 x + b_1 y + c_1 t) W(\xi),$$
$$\xi = a_2 x + b_2 y + c_2 t, \quad a_1 b_2 - a_2 b_1 \neq 0,$$

where a_1, a_2, b_1, b_2, c_1, and c_2 are arbitrary constants, and the function $W = W(\xi)$ satisfies the autonomous ODE

$$WW''_{\xi\xi} - (W'_{\xi})^2 = \sigma (a_1 b_2 - a_2 b_1)^{-2} (c_1 W + c_2 W'_{\xi})^2. \tag{29}$$

The solution of Formula (29) is invariant with respect to a two-parameter group of transformations defined by the symmetry operators

$$Y_1 = \frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}\frac{\partial}{\partial x} - \frac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}\frac{\partial}{\partial y} + \frac{\partial}{\partial t},$$
$$Y_2 = \frac{b_2}{a_1 b_2 - a_2 b_1}\frac{\partial}{\partial x} - \frac{a_2}{a_1 b_2 - a_2 b_1}\frac{\partial}{\partial y} + w\frac{\partial}{\partial w}.$$

Equation (29) for $c_1 = 0$ is easily integrated and has a simple solution

$$W = (A\xi + B)^{-k}, \quad k = \frac{(a_1 b_2 - a_2 b_1)^2}{\sigma c_2^2},$$

where A and B are arbitrary constants.

For $c_2 = 0$, the general solution of Equation (29) is given by the formula

$$W = A\exp(\lambda\xi^2 + B\xi), \quad \lambda = \tfrac{1}{2}\sigma c_1^2(a_1b_2 - a_2b_1)^{-2},$$

where A and B are arbitrary constants.

Remark 7. *In the case $m = 2$, it is not difficult to construct other examples of two-dimensional reductions and exact solutions similar to those given above.*

5. Reductions with Additive Separation of Variables Leading to Stationary Monge–Ampère Type Equations

1°. Equation (2) has additive separable solutions of the form

$$w = At + u(x,y), \tag{30}$$

where A is an arbitrary constant, and the function u is described by the inhomogeneous Monge–Ampère equation with a constant right-hand side:

$$u_{xx}u_{yy} - u_{xy}^2 = \sigma A^m. \tag{31}$$

2°. It is easy to verify that Equation (2) admits an exact additive separable solution of Formula (30), which is expressed in elementary functions:

$$w = C_1x^2 + C_2xy + \frac{1}{4C_1}(\sigma A^m + C_2^2)y^2 + C_4x + C_5y + At + C_6,$$

where $A, C_1, \ldots, C_5$ ($C_1 \neq 0$) are arbitrary constants.

3°. Using the results of [8], for example, one can obtain the following exact solutions of Formula (30) of Equation (2):

$$w = At \pm \frac{\sqrt{-\sigma A^m}}{C_2}x(C_1x + C_2y) + \varphi(C_1x + C_2y) + C_3x + C_4y + C_5,$$
$$w = At + \frac{1}{x + C_1}\left(C_2y^2 + C_3y + \frac{C_3^2}{4C_2}\right) + \frac{\sigma A^m}{12C_2}(x^3 + 3C_1x^2) + C_4y + C_5x + C_6,$$
$$w = At \pm \frac{2\sqrt{-\sigma A^m}}{3C_1C_2}(C_1x - C_2^2y^2 + C_3)^{3/2} + C_4x + C_5y + C_6,$$

where $C_1, \ldots, C_6$ are arbitrary constants, and $\varphi = \varphi(z)$ is an arbitrary function.

Remark 8. *For $\sigma A^m < 0$, the general solution of the inhomogeneous Monge–Ampère Equation* (31) *can be represented in parametric form [3,8].*

4°. Equation (2) admits more complicated solutions than (30) with generalized separation of variables of the form

$$w = (ax + by + c)t + u(x,y),$$

where a, b, and c are arbitrary constants, and the function u is described by the inhomogeneous Monge–Ampère equation with a variable right-hand side:

$$u_{xx}u_{yy} - u_{xy}^2 = \sigma(ax + by + c)^m. \tag{32}$$

For $a = 1$, $b = c = 0$, Equation (32) has, for example, the following exact solutions with generalized separation of variables:

$$u = \pm\frac{2\sqrt{-\sigma}}{m+2}x^{\frac{m+2}{2}}y + \varphi(x), \quad m \neq 2;$$
$$u = C_1y^2 + C_2xy + \frac{C_2^2}{4C_1}x^2 + \frac{\sigma}{2C_1(m+1)(m+2)}x^{m+2}, \quad m \neq -1,\, -2;$$
$$u = \frac{1}{x}\left(C_1y^2 + C_2y + \frac{C_2^2}{4C_1}\right) + \frac{\sigma}{2C_1(m+2)(m+3)}x^{m+3}, \quad m \neq -2,\, -3,$$

where $\varphi(x)$ is an arbitrary function and C_1 and C_2 are arbitrary constants.

6. Reductions with Multiplicative Separation of Variables Leading to Stationary Monge–Ampère Type Equations

1°. Equation (2) for $m \neq 2$ has the multiplicative separable solution

$$w = (t + A)^{\frac{m}{m-2}}W(x, y),$$

where A is an arbitrary constant, and the function $W = W(x, y)$ is described by the stationary Monge–Ampère equation

$$W_{xx}W_{yy} - W_{xy}^2 = \sigma\left(\frac{m}{m-2}\right)^m W^m. \tag{33}$$

Equation (33), in turn, admits the multiplicative separable solution

$$W = x^{\frac{2}{2-m}}\theta(y),$$

where $\theta = \theta(y)$ satisfies the autonomous ODE

$$2m\theta\theta''_{yy} - 4(\theta'_y)^2 = \sigma(m-2)^2\left(\frac{m}{m-2}\right)^m\theta^m.$$

Substituting $Z(\theta) = (\theta'_y)^2$ reduces this equation to the first-order linear ODE

$$m\theta Z'_\theta - 4Z = \sigma(m-2)^2\left(\frac{m}{m-2}\right)^m\theta^m,$$

which is easily integrated.

2°. Equation (2) for $m = 2$ admits the multiplicative separable solution

$$w = e^{\lambda t}W(x, y), \tag{34}$$

where λ is an arbitrary constant, and the function $W = W(x, y)$ is described by the stationary Monge–Ampère equation

$$W_{xx}W_{yy} - W_{xy}^2 = \sigma\lambda^2W^2. \tag{35}$$

Equation (35) admits the multiplicative separable solution

$$W = C_1e^{\beta x}\theta(y), \quad \theta = \exp\left[\tfrac{1}{2}\sigma(\lambda/\beta)^2y^2 + C_2y\right],$$

where C_1 and C_2 are arbitrary constants.

7. Reductions with Generalized Separation of Variables Leading to a Two-Dimensional Nonstationary Equation

1°. Equation (2) allows for solutions with generalized separation of variables of the form

$$w = \tfrac{1}{2}y^2 + axy + \tfrac{1}{2}a^2x^2 + by + W(x,t),$$

where a and b are arbitrary constants, and the function $W = W(x,t)$ is described by a relatively simple nonlinear equation:

$$W_{xx} = \sigma W_t^m. \tag{36}$$

For the magnetohydrodynamic equation, which corresponds to $m = \sigma = 1$, the reduced Equation (36) is the linear heat equation.

Some exact solutions of Equation (36) are described below.

2°. Equation (36) has a simple solution with an additive separation of variables

$$W = At + \tfrac{1}{2}\sigma A^m x^2 + Bx + C,$$

where A, B, and C are arbitrary constants.

3°. Equation (36) has an exact solution in the form of a product of functions of different arguments $W = T(t)X(x)$, which includes a simple solution

$$W = A(t + C_1)^{\frac{m}{m-1}}(x + C_2)^{\frac{2}{1-m}}, \quad A = \left[\frac{2(m+1)(m-1)^{m-2}}{\sigma m^m}\right]^{\frac{1}{m-1}},$$

where C_1 and C_2 are arbitrary constants.

4°. Equation (36) has the traveling wave solution

$$W = W(z), \quad z = x + \lambda t,$$

where λ is an arbitrary constant, and the function $W(z)$ is described by the simple autonomous ODE

$$W''_{zz} = \sigma\lambda^m (W'_z)^m.$$

The general solution of this equation for $m \neq 1$, 2 is determined by the formula

$$W = \frac{1}{(2-m)\sigma\lambda^m}\left[(1-m)\sigma\lambda^m z + C_1\right]^{\frac{2-m}{1-m}} + C_2,$$

where C_1 and C_2 are arbitrary constants.

Equation (36) also has a more general solution of the form

$$w = ax^2 + bx + ct + W(z), \quad z = x + \lambda t,$$

where a, b, c, and λ are arbitrary constants, and the function $W(z)$ is described by the autonomous ODE

$$W''_{zz} = \sigma\big(\lambda W'_z + c\big)^m - 2a.$$

5°. Equation (36) for $m \neq 1$ admits the self-similar solution

$$W = t^{\frac{m+2\beta}{m-1}} V(\zeta), \quad \zeta = xt^{\beta},$$

where the function $V(\xi)$ is described by the non-autonomous ODE

$$V''_{\xi\xi} = \sigma\Big(\frac{m+2\beta}{m-1}V + \beta\xi V'_{\xi}\Big)^m.$$

6°. Equation (36) for $m \neq 1$ has an invariant solution of the form

$$W = t^{\frac{m}{m-1}}\theta(z), \quad z = x + \beta\ln t, \tag{37}$$

where β is an arbitrary constant, and the function $\theta = \theta(z)$ is described by the autonomous ODE

$$\theta''_{zz} = \sigma\Big(\frac{m}{m-1}\theta + \beta\theta'_z\Big)^m.$$

The general solution of this equation for $\beta = 0$, which corresponds to the solution with a multiplicative separation of variables (37), can be represented in implicit form.

7°. Equation (36) for $m \neq 1$ has another invariant solution of the form

$$W = \exp\Big(\frac{2\beta}{m-1}t\Big)U(\xi), \quad \xi = x\exp(\beta t),$$

where β is an arbitrary constant, and the function $U = U(\xi)$ is described by the non-autonomous ODE

$$U''_{\xi\xi} = \sigma\Big(\frac{2\beta}{m-1}U + \beta\xi U'_{\xi}\Big)^m.$$

8°. Equation (36) for $m = -1$ using the Euler transformation [8]

$$w(x,t) + u(\xi,\tau) = x\xi, \quad x = u_\xi, \quad t = -\tau/\sigma,$$

reduces to the linear heat equation

$$u_\tau = u_{\xi\xi}.$$

8. Reduction to the Stationary Monge–Ampère Equation Using Traveling Wave Type Variables

1°. Equation (2) allows for the generalized separable solutions of combined type:

$$\begin{aligned} &w = C_1x^2 + C_2xy + C_3y^2 + C_4x + C_5y + C_6t + W(\xi,\eta),\\ &\xi = a_1x + b_1y + \lambda_1t, \quad \eta = a_2x + b_2y + \lambda_2t, \end{aligned} \tag{38}$$

where C_i, a_j, b_j, λ_j ($i = 1,\dots,6$; $j = 1, 2$) are arbitrary constants, ξ and η are new traveling wave variables, and the function $W = W(\xi,\eta)$ is described by the stationary Monge–Ampère type equation:

$$\begin{aligned} &(a_1b_2 - b_1a_2)^2(W_{\xi\xi}W_{\eta\eta} - W_{\xi\eta}^2)\\ &\quad + 2(a_1^2C_3 + b_1^2C_1 - a_1b_1C_2)W_{\xi\xi} + 2(a_2^2C_3 + b_2^2C_1 - a_2b_2C_2)W_{\eta\eta}\\ &\quad + 2[(2a_1a_2C_3 + 2b_1b_2C_1 - (a_1b_2 + a_2b_1)C_2]W_{\xi\eta} + 4C_1C_3 - C_2^2\\ &\quad = \sigma(C_6 + \lambda_1W_\xi + \lambda_2W_\eta)^m. \end{aligned} \tag{39}$$

2°. Consider the special case of (38) and (39), setting

$$a_1 = a, \quad b_1 = b, \quad \lambda_1 = \lambda, \quad a_2 = b_2 = 0, \quad \lambda_2 = 1, \quad \eta = t,$$

which corresponds to a solution of the form

$$w = C_1x^2 + C_2xy + C_3y^2 + C_4x + C_5y + C_6t + W(\xi, t), \quad \xi = ax + by + \lambda t, \tag{40}$$

where C_i, a, b, λ $(i = 1, \dots, 6)$ are arbitrary constants. In this case, the function $W = W(\xi, t)$ is described by the nonlinear equation:

$$2(a^2C_3 + b^2C_1 - abC_2)W_{\xi\xi} = \sigma(C_6 + W_t + \lambda W_\xi)^m - 4C_1C_3 + C_2^2. \tag{41}$$

3°. In particular, taking in (40) and (41), the function W with one argument ξ, we arrive at a nonlinear ODE of the autonomous form

$$2(a^2C_3 + b^2C_1 - abC_2)W''_{\xi\xi} = \sigma(C_6 + \lambda W'_\xi)^m - 4C_1C_3 + C_2^2. \tag{42}$$

Substituting $U(\xi) = W'_\xi$ reduces it to a first-order ODE with separable variables. Under the condition $4C_1C_3 - C_2^2 = 0$, $m \neq 1, 2$, the general solution of Equation (42) is written as follows:

$$W = \frac{1}{A(2-m)}\big[A(1-m)\xi + B_1\big]^{\frac{2-m}{1-m}} - \frac{C_6}{\lambda}\xi + B_2, \quad A = \frac{\sigma\lambda^m}{2(a^2C_3 + b^2C_1 - abC_2)},$$

where B_1 and B_2 are arbitrary constants.

9. Reduction Using a New Variable, Parabolic in Spatial Coordinates

1°. In the variables, one of which is time and the other is given by a parabolic function in spatial variables,

$$w = W(z, t), \quad z = y + ax^2,$$

where a is an arbitrary constant, Equation (2) is reduced to the two-dimensional PDE:

$$2aW_zW_{zz} = \sigma(W_t)^m. \tag{43}$$

Some exact solutions of Equation (43) are described below.

2°. The reduced Equation (43) admits additive separable solutions

$$W = C_1t \pm \frac{2}{3}\sqrt{\frac{\sigma C_1^m}{a}}(z + C_2)^{3/2} + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants.

3°. Equation (43) for $m \neq 2$ has a simple solution in the form of a product of power functions of different arguments:

$$W = A(t + C_1)^{\frac{m}{m-2}}(z + C_2)^{\frac{3}{2-m}}, \quad A = \left[-\frac{18a(m+1)(m-2)^{m-3}}{\sigma m^m}\right]^{\frac{1}{m-2}}.$$

where C_1 and C_2 are arbitrary constants.

4°. Equation (43) has traveling wave solutions:

$$W = W(\xi), \quad \xi = z + \lambda t \equiv y + ax^2 + \lambda t, \tag{44}$$

where λ is an arbitrary constant, and the function $W(z)$ is described by the autonomous ODE

$$2aW''_{\xi\xi} = \sigma\lambda^m(W'_\xi)^{m-1},$$

whose general solution for $m \neq 2$, 3 is determined by the formula

$$W = \frac{1}{\kappa(3-m)}\big[\kappa(2-m)\xi + C_1\big]^{\frac{3-m}{2-m}} + C_2, \quad \kappa = \frac{\sigma\lambda^m}{2a},$$

where C_1 and C_2 are arbitrary constants.

Remark 9. *More general than* (44)*, a solution of Equation* (43) *can be obtained if we look for a solution in the form*

$$W = Ct + U(\xi), \quad \xi = z + \lambda t \equiv y + ax^2 + \lambda t.$$

5°. Equation (43) for $m \neq 2$ admits self-similar solutions:

$$W = t^{\frac{m+3\beta}{m-2}} V(\zeta), \quad \zeta = zt^\beta,$$

where β is an arbitrary constant, and the function $V = V(\xi)$ is described by the non-autonomous ODE

$$2aV'_\zeta V''_{\zeta\zeta} = \sigma\Big(\frac{m+3\beta}{m-2}V + \beta\zeta V'_\zeta\Big)^m.$$

6°. Equation (43) for $m \neq 2$ has invariant solutions of the form

$$W = t^{\frac{m}{m-2}} f(\eta), \quad \eta = z + \lambda \ln t,$$

where λ is an arbitrary constant, and the function $f = f(\eta)$ is described by the autonomous ODE

$$2af'_\eta f''_{\eta\eta} = \sigma\Big(\frac{m}{m-2}f + \lambda f'_\eta\Big)^m.$$

7°. Equation (43) for $m \neq 2$ also admits other invariant solutions

$$W = \exp\Big(\frac{3\beta}{m-2}t\Big)g(\tau), \quad \tau = \exp(\beta t)z,$$

where β is an arbitrary constant, and the function $g = g(\tau)$ is described by the non-autonomous ODE

$$2ag'_\tau g''_{\tau\tau} = \sigma\Big(\frac{3\beta}{m-2}g + \beta\tau g'_\tau\Big)^m.$$

8°. Equation (43) for $m = 2$ has simple solutions of the exponential form

$$W = A\exp\big[kz \pm (2ak^3/\sigma)^{1/2}t\big],$$

where A and k are arbitrary constants. There are also more complex solutions of the form $W = e^{\lambda t}\varphi(z)$, where the function $\varphi = \varphi(z)$ is described by an autonomous ODE, the general solution of which can be represented in implicit form.

10. Reduction Using a New Quadratic Variable in Spatial Coordinates

1°. In the variables, one of which is time and the other is quadratic with respect to spatial variables,

$$w = W(z,t), \quad z = ax^2 + bxy + cy^2 + kx + sy, \tag{45}$$

where a, b, c, k, and s are arbitrary constants, Equation (2) is reduced to the two-dimensional nonstationary PDE:

$$\begin{aligned}&2(Az+B)W_z W_{zz} + AW_z^2 = \sigma (W_t)^m;\\ &A = 4ac - b^2, \quad B = as^2 + ck^2 - bks.\end{aligned} \tag{46}$$

Note that depending on the coefficients of the quadratic terms a, b, and c in (45), the curve $z = \text{const}$ can be an ellipse (for $A = 4ac - b^2 > 0$), a hyperbola (for $A < 0$), or a parabola (for $A = 0$).

Let us consider some classes of exact solutions that Equation (46) admits.

2°. The reduced Equation (46) admits additive separable solutions:

$$W = Ct + \zeta(z),$$

where C is an arbitrary constant, and the function $\zeta = \zeta(z)$ is described by the nonlinear ODE

$$2(Az+B)\zeta'_z \zeta''_{zz} + A(\zeta'_z)^2 = \sigma C^m,$$

which is easily integrated, since it admits a reduction in order and is simultaneously linearized using the substitution $u(z) = (\zeta'_z)^2$. As a result, we obtain

$$\begin{aligned}&\zeta = \pm \frac{C_1}{2\sqrt{\sigma A^3 C^m}} \ln\left[\frac{A^2 C_1}{2} + \sigma A^3 C^m \left(z + \frac{B}{A}\right) + s\sqrt{\sigma A^3 C^m}\right] + C_2\,,\\ &s = \sqrt{\sigma A^3 C^m \left(z + \frac{B}{A}\right)^2 + A^2 C_1 \left(z + \frac{B}{A}\right)}\,,\end{aligned}$$

where C_1 and C_2 are arbitrary constants. The first formula above was transformed by changing C_2.

Remark 10. *A more general result can be obtained if we seek a solution to Equation (46) in the form*

$$W = Ct + U(\xi), \quad \xi = z - \lambda t,$$

where λ is an arbitrary constant.

3°. For $4ac - b^2 \neq 0$, $m \neq 2$, the reduced Equation (46) admits solutions in the form of a product of functions of different arguments:

$$W = t^{\frac{m}{m-2}} f(z),$$

where the function $f = f(z)$ is described by the non-autonomous ODE

$$2(Az+B)f'_z f''_{zz} + A(f'_z)^2 = \sigma \left(\frac{m}{m-2}\right)^m f^m,$$

which has the simple particular solution:

$$f = k(Az+B)^{\frac{2}{2-m}}, \quad k = \left[\frac{4A^3(m+2)}{\sigma(2-m)^3}\left(\frac{m-2}{m}\right)^m\right]^{\frac{1}{m-2}}.$$

4°. For $4ac - b^2 \neq 0$, $m \neq 2$ the reduced Equation (46) admits solutions of the quasi-self-similar form

$$W = t^{\frac{m+2\beta}{m-2}} V(\eta), \quad \eta = (Az+B)t^\beta,$$

where β is an arbitrary constant, and the function $V = V(\eta)$ satisfies the nonlinear ordinary differential equation:

$$2\eta V'_\eta V''_{\eta\eta} + (V'_\eta)^2 = \sigma A^{-3}\Big(\frac{m+2\beta}{m-2}V + \beta\eta V'_\eta\Big)^m.$$

5°. When $4ac - b^2 \neq 0$, the transformation

$$t = t, \quad z = \frac{\sqrt{|A|}}{2}\rho^2 - \frac{B}{A}, \quad W = U(\rho, t),$$

leads Equation (46) to the canonical form:

$$\operatorname{sign} A\, \rho^{-1} U_\rho U_{\rho\rho} = \sigma (U_t)^m. \tag{47}$$

11. Reductions and Exact Solutions in Polar Coordinates

In polar coordinates r, φ, where $x = r\cos\varphi$ and $y = r\sin\varphi$, the original Equation (2) takes the form

$$r^{-2} w_{rr}(w_{\varphi\varphi} + r w_r) - [(r^{-1} w_\varphi)_r]^2 = \sigma (w_t)^m. \tag{48}$$

Remark 11. *In elliptical coordinates r, φ, where $x = ar\cos\varphi$, $y = br\sin\varphi$ (a and b are positive constants), Equation (2) is written as follows:*

$$r^{-2} w_{rr}(w_{\varphi\varphi} + r w_r) - [(r^{-1} w_\varphi)_r]^2 = (ab)^2 \sigma (w_t)^m. \tag{49}$$

It can be seen that Equation (49) differs from Equation (48) only by overestimating the coefficient σ.

1°. Equation (48), written in polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$, allows for radially symmetric solutions independent of the angular variable, which are described by a two-dimensional equation:

$$r^{-1} w_r w_{rr} = \sigma (w_t)^m, \tag{50}$$

which, up to a redesignation of the independent variable, coincides with Equation (47) for $A > 0$. Three exact solutions of Equation (50) are obtained using the results given in paragraphs 2°–4° Section 9.

2°. Equation (50) has an exact solution with the additive separation of variables

$$w = At + \zeta(r)\,,$$

where

$$\zeta = \pm\left[\frac{rs}{2} - \frac{C_1}{2\sqrt{\sigma A^m}}\ln\Big(r\sqrt{\sigma A^m} + s\Big)\right] + C_2\,,$$

$$s = \sqrt{\sigma A^m r^2 - C_1}\,.$$

Here, A, C_1, and C_2 are arbitrary constants.

3°. Equation (48) for $m \neq 2$ admits the self-similar solution

$$w = t^{\frac{m+4\gamma}{m-2}} F(z), \quad z = r t^\gamma,$$

where γ is an arbitrary constant, and the function $F = F(z)$ is described by the ODE

$$z^{-1} F'_z F''_{zz} = \sigma\Big(\frac{m+4\gamma}{m-2}F + \gamma z F'_z\Big)^m.$$

4°. Equation (49) for $m \neq 2$ also has exact solutions with separation of variables of the form

$$w = r^{\frac{4}{2-m}} v(\varphi, t), \tag{51}$$

where the function $v = v(\varphi, t)$ is described by the two-dimensional PDE

$$\frac{4(2+m)}{(2-m)^2} v\Big(v_{\varphi\varphi} + \frac{4}{2-m} v\Big) - \Big(\frac{2+m}{2-m}\Big)^2 v_\varphi^2 = \sigma v_t^m. \tag{52}$$

5°. Since Equation (52) does not depend explicitly on the independent variables, it has the traveling wave solution

$$v = v(Z), \quad Z = \varphi + \lambda t,$$

where λ is an arbitrary constant, and the function $v = v(Z)$ is described by the autonomous ODE

$$\frac{4(2+m)}{(2-m)^2} v\Big(v''_{ZZ} + \frac{4}{2-m} v\Big) - \Big(\frac{2+m}{2-m}\Big)^2 (v'_Z)^2 = \sigma \lambda^m (v'_Z)^m.$$

6°. Equation (52) admits the multiplicative separable solution of the form

$$v = (t + C)^{\frac{m}{m-2}} V(\varphi),$$

where C is an arbitrary constant, and the function $V = V(\varphi)$ is described by the autonomous ODE

$$\frac{4(2+m)}{(2-m)^2} V\Big(V''_{\varphi\varphi} + \frac{4}{2-m} V\Big) - \Big(\frac{2+m}{2-m}\Big)^2 (V'_\varphi)^2 = \sigma\Big(\frac{m}{m-2}\Big)^m V^m.$$

There is also a more complex solution of the form $v = (t + C_1)^{\frac{m}{m-2}} V(\zeta)$, where $\zeta = \varphi + C_2 \ln(t + C_1)$.

12. Constructing Exact Solutions Using a Special Point Transformation

The special point transformation

$$x = \frac{\xi}{1 + \alpha\xi + \beta\eta}, \quad y = \frac{\eta}{1 + \alpha\xi + \beta\eta}, \quad w = \frac{u}{1 + \alpha\xi + \beta\eta}, \tag{53}$$

where α and β are free parameters, leads the nonlinear PDE (2) to the form

$$u_{\xi\xi} u_{\eta\eta} - u_{\xi\eta}^2 = \sigma(1 + \alpha\xi + \beta\eta)^{-m-4} (u_t)^m. \tag{54}$$

Note that transformation (53) was used in [5,8] to study stationary Monge–Ampère equations of the form $w_{xx} w_{yy} - w_{xy}^2 = f(x, y)$.

Setting $\beta = 0$ in (53) and (54), we arrive at the equation

$$u_{\xi\xi} u_{\eta\eta} - u_{\xi\eta}^2 = f(\xi)(u_t)^m. \tag{55}$$

where $f(\xi) = \sigma(1 + \alpha\xi)^{-m-4}$.

Let us now describe some exact solutions of Equation (55) for the general case, considering the function $f(\xi)$ to be arbitrary.

1°. Equation (55) admits generalized separable solutions

$$u = (a\xi + b)t + Z(\xi, \eta),$$

where a and b are arbitrary constants, and the function $Z = Z(\xi, \eta)$ is described by the stationary Monge–Ampère equation

$$Z_{\xi\xi}Z_{\eta\eta} - Z_{\xi\eta}^2 = f(\xi)(a\xi + b)^m. \tag{56}$$

PDEs of this type were considered in [8]. Equation (56) has the following exact generalized separable solutions:

$$\begin{aligned}
Z &= \pm\eta \int \sqrt{-f(\xi)(a\xi + b)^m}\, d\xi + \varphi(\xi),\\
Z &= C_1\eta^2 + C_2\xi\eta + \frac{C_2^2}{4C_1}\xi^2 + \frac{1}{2C_1}\int_0^\xi (\xi - \zeta)f(\zeta)(a\zeta + b)^m\, d\zeta + C_3\xi + C_4\eta,\\
Z &= \frac{1}{\xi + C_1}\left(C_2\eta^2 + C_3\eta + \frac{C_3^2}{4C_2}\right) + \frac{1}{2C_2}\int_0^\xi (\xi - \zeta)(\zeta + C_1)f(\zeta)(a\zeta + b)^m\, d\zeta,
\end{aligned}$$

where $\varphi(\xi)$ is an arbitrary function and $C_1, \ldots, C_4$ are arbitrary constants.

2°. Equation (55) for $m \neq 2$ admits two-dimensional solutions of the form

$$u = t^{\frac{m+2k}{m-2}} U(\xi, \theta), \quad \theta = \eta t^k,$$

where k is a free parameter, and the function $U = U(\xi, \theta)$ is described by the PDE

$$U_{\xi\xi}U_{\theta\theta} - U_{\xi\theta}^2 = f(\xi)\Big(\frac{m + 2k}{m - 2}U + k\theta U_\theta\Big)^m.$$

3°. Equation (55) for $m \neq 2$ admits other two-dimensional solutions of the form

$$u = \exp\Big(\frac{2\gamma t}{m - 2}\Big)U(\xi, \theta), \quad \theta = \exp(\gamma t)\eta,$$

where γ is a free parameter, and the function $U = U(\xi, \theta)$ is described by the PDE

$$U_{\xi\xi}U_{\theta\theta} - U_{\xi\theta}^2 = f(\xi)\Big(\frac{2\gamma}{m - 2}U + \gamma\theta U_\theta\Big)^m.$$

4°. Moreover, Equation (55) for $m \neq 2$ also admits two-dimensional solutions of the form

$$u = \exp\Big(\frac{m\gamma\eta}{m - 2}\Big)U(\xi, \theta), \quad \theta = t\exp(\gamma\eta),$$

where γ is a free parameter, and the function $U = U(\xi, \theta)$ is described by a PDE that is not given here due to its cumbersomeness.

5°. Equation (55) for $m = 2$ has multiplicative separable solutions of the form

$$u = e^{\lambda t}U(\xi, \eta),$$

where λ is a free parameter, and the function $U = U(\xi, \eta)$ is described by the two-dimensional PDE

$$U_{\xi\xi}U_{\eta\eta} - U_{\xi\eta}^2 = \lambda^2 f(\xi)U^2.$$

6°. Equation (55) for $m = 2$ has other multiplicative separable solutions,

$$u = e^{\gamma\eta}U(\xi, t),$$

where γ is a free parameter, and the function $U = U(\xi, t)$ is described by the two-dimensional equation

$$UU_{\xi\xi} - U_{\xi}^2 = \gamma^{-2} f(\xi) U_t^2.$$

7°. Equation (55) for $m \neq 2$ admits the one-dimensional solution

$$u = t^{\frac{m}{m-2}} \eta^{\frac{2}{2-m}} \varphi(\xi),$$

where the function $\varphi = \varphi(\xi)$ is described by the ODE

$$\frac{2m}{(m-2)^2} \varphi \varphi''_{\xi\xi} - \frac{4}{(m-2)^2} (\varphi'_{\xi})^2 = \left(\frac{m}{m-2}\right)^m f(\xi) \varphi^m.$$

8°. Equation (55) for $m = 2$ admits the one-dimensional solution

$$u = \exp(\alpha t + \beta \eta) \varphi(\xi),$$

where α and β are free parameters, and the function $\varphi = \varphi(\xi)$ is described by the ODE

$$\varphi \varphi''_{\xi\xi} - (\varphi'_{\xi})^2 = (\alpha/\beta)^2 f(\xi) \varphi^2.$$

13. Using the Euler–Legendre Contact Transformation

For further analysis of the original Equation (2), we use the Euler–Legendre contact transformation, which is defined by the following formulas [40]:

Direct transformation:

$$t = T, \quad x = W_X, \quad y = W_Y, \quad w = XW_X + YW_Y - W. \tag{57}$$

Inverse transformation:

$$T = t, \quad X = w_x, \quad Y = u_y, \quad W = xw_x + yw_y - w, \tag{58}$$

where $w = w(t, x, y)$ and $W = W(T, X, Y)$, and the time derivatives are related by the relation

$$w_t = -W_T. \tag{59}$$

Using (57) and (58), we find the second derivatives

$$w_{xx} = JW_{YY}, \quad w_{xy} = w_{yx} = -JW_{XY}, \quad w_{yy} = JW_{XX}, \tag{60}$$

where

$$J = w_{xx}w_{yy} - w_{xy}^2, \quad \frac{1}{J} = W_{XX}W_{YY} - W_{XY}^2. \tag{61}$$

Replacing the old derivatives in Equation (2) with new ones according to Formulas (59)–(61) and additionally making the substitution $W = -U$, we arrive at an equation of a similar type with a different exponent for the first derivative

$$U_{XX}U_{YY} - U_{XY}^2 = \sigma^{-1}(U_T)^{-m}. \tag{62}$$

If $W = W(T, X, Y)$ is a solution of Equation (62), then Formula (57) defines the corresponding solution of Equation (2) in parametric form.

From a comparison of Equations (2) and (62), in particular, it follows that to construct exact solutions of Equation (2) for $m = -1$, one can use the exact solutions of the simpler Equation (2) for $m = 1$ obtained in [24].

14. Brief Conclusions

The generalized equation of electron magnetohydrodynamics with nonlinearity of the Monge–Ampère type

$$w_{xx}w_{yy} - w_{xy}^2 = \sigma(w_t)^m,$$

which is also encountered in differential geometry, is investigated. Two-dimensional and one-dimensional reductions are considered, leading to simpler partial differential equations with two independent variables (including stationary equations of the Monge–Ampère type) or ordinary differential equations. Some self-similar and other invariant solutions are described. A number of more complex solutions with generalized separation of variables are obtained, many of which are expressed in elementary functions or quadratures.

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