Construction of exact solutions in implicit form for PDEs: New functional separable solutions of non-linear reaction-diffusion equations with variable coefficients *

Andrei D. Polyanin[a,b,c]

[a] *Ishlinsky Institute for Problems in Mechanics, Russian Academy of Sciences, 101 Vernadsky Avenue, bldg 1, 119526 Moscow, Russia*
[b] *Bauman Moscow State Technical University, 5 Second Baumanskaya Street, 105005 Moscow, Russia*
[c] *National Research Nuclear University MEPhI, 31 Kashirskoe Shosse, 115409 Moscow, Russia*

The paper deals with different classes of non-linear reaction-diffusion equations with variable coefficients

$$c(x)u_t = [a(x)f(u)u_x]_x + b(x)g(u),$$

that admit exact solutions. The direct method for constructing functional separable solutions to these and more complex non-linear equations of mathematical physics is described. The method is based on the representation of solutions in implicit form

$$\int h(u)\,du = \xi(x)\omega(t) + \eta(x),$$

where the functions $h(u)$, $\xi(x)$, $\eta(x)$, and $\omega(t)$ are determined further by analyzing the resulting functional-differential equations. Examples of specific reaction-diffusion type equations and their exact solutions are given. The main attention is paid to non-linear equations of a fairly general form, which contain several arbitrary functions dependent on the unknown u and/or the spatial variable x (it is important to note that exact solutions of non-linear PDEs, that contain arbitrary functions and therefore have significant generality, are of great practical interest for testing various numerical and approximate analytical methods for solving corresponding initial-boundary value problems). Many new generalized traveling-wave solutions and functional separable solutions are described.

Keywords: method for solving non-linear PDEs, non-linear reaction-diffusion equations, equations with variable coefficients, exact solutions in implicit form, generalized traveling-wave solutions, functional separable solutions

This is a preprint of an article that will be published in *International Journal of Non-Linear Mechanics*, 2019, https://www.sciencedirect.com/science/article/pii/S0020746219300678

1 Introduction

The paper deals with diffusion-type equations equations in which x and t are independent variables, and $u = u(x, t)$ is the unknown function.

Transformations, symmetries, and exact solutions of various classes of non-linear reaction-diffusion-convection equations that do not depend explicitly on the variables x and t, have been considered in many studies (see, for example, [1–20] and the literature cited therein). To construct exact solutions, the most frequently used methods were those based on classical and nonclassical symmetry reductions [1–3, 5, 7, 10, 15, 16, 18–20], on generalized and functional separation of variables [6, 9, 11, 12, 15, 17, 18], and on differential constraints [6, 11, 14, 15, 17, 18].

A number of studies (e.g., see [8, 11, 13, 18, 21–25]) have been devoted to non-linear reaction-diffusion equations with variable coefficients dependent on the spatial variable x (from now on sometimes called autonomous coefficients). Table 1 lists the forms of exact solutions to some equations of this type with one or two arbitrary functions.

Таблица 1. Non-linear reaction-diffusion equations with variable coefficients and exact solutions

No.	Equation	Form of solution or remark	References
1	$u_t = [a(x)u^k u_x]_x + b(x)u^{k+1}$	$u = \varphi(x)\psi(t)$	[8, 13, 18]
2	$u_t = [a(x)e^{\lambda u} u_x]_x + b(x)e^{\lambda u}$	$u = \varphi(x) + \psi(t)$	[8, 13, 18]
3	$u_t = [a(x)u_x]_x + cu \ln u + ku$	$u = \varphi(x)\psi(t)$	[13, 18]
4	$u_t = (u^{-4/3} u_x)_x + b(x)u^{-1/3}$	Reduces to $v_t = (v^{-4/3} v_z)_z$	[8, 13, 18]
5	$u_t = [x^k f(u)u_x]_x + x^{k-2} g(u)$	$u = U(z),\ \ z = xt^{1/(k-2)},\ \ k \neq 2$	Remark 1
6	$u_t = [x^2 f(u)u_x]_x + g(u)$	$u = U(z),\ \ z = \lambda t + \ln x$	Remark 1
7	$u_t = [e^{\lambda x} f(u)u_x]_x + e^{\lambda x} g(u)$	$u = U(z),\ \ z = \lambda x + \ln t,\ \lambda \neq 0$	Remark 1
8	$u_t = [a(x)u_x]_x + [x^2/a(x)]g(u)$	$u = U(z),\ \ z = t + \int [x/a(x)]\, dx$	[25]
9	$u_t = u_{xx} + \tanh^2(kx)g(u)$	$u = U(z),\ \ z = t + k^{-2} \ln \cosh(kx)$	[25]

Here, $a(x)$, $b(x)$, $f(u)$, and $g(u)$ are arbitrary functions, c, k, and λ are free parameters.

Remark 1. Equations and solutions given in Table 1 in rows 5–7 generalize reaction-diffusion equations with power and exponential nonlinearities and their invariant solutions, which were considered in [21–23].

Symmetries, transformations, and some exact solutions of non-linear convection-diffusion equations with variable autonomous coefficients were considered in [26–31].

Other related and more complex non-linear diffusion-type equations were considered in [18, 32–36]. Exact solutions to a number of systems of coupled equations of the reaction-diffusion type are described in [18, 37] (these books give an extensive list of publications on this topic).

It is also noteworthy that lately much attention has been paid to studying hereditary systems, which are modeled by non-linear diffusion-type equations with delay. Exact solutions of such equations were obtained in [38–48].

The present paper deals with new exact solutions (in implicit form) admitted by non-linear reaction-diffusion equations of a fairly general form that depend on one or more arbitrary functions. It is important to note that exact solutions of mathematical physics equations, which contain arbitrary functions and therefore have a significant generality, are of great practical interest for evaluating the accuracy of various numerical and approximate analytical methods for solving corresponding initial-boundary value problems.

2 Non-linear reaction-diffusion type equations. Direct method for constructing functional separable solutions in implicit form

2.1 Preliminary remarks

The method proposed below for constructing implicitly given exact functional separable solutions is based on the generalization of traveling-wave solutions of some non-linear partial differential equations. Prior to describing the method, let us first give two simple examples that illustrate the existence of solutions defined implicitly.

Example 1. Let us look at the non-linear heat equation

$$u_t = [f(u)u_x]_x, \tag{1}$$

which contains an arbitrary function $f(u)$. This equation does not depend explicitly on x and t and admits the traveling-wave solution

$$u = u(z), \qquad z = \lambda t + \kappa x, \tag{2}$$

where κ and λ are arbitrary constants. Substituting (2) in (1), we obtain an ODE of the form $\lambda u'_z = \kappa^2[f(u)u'_z]'_z$. Integrating, we find its solution in implicit form

$$\kappa^2 \int \frac{f(u)\,du}{\lambda u + C_1} = \lambda t + \kappa x + C_2, \tag{3}$$

where C_1 and C_2 are arbitrary constants. On the right-hand side of (3), z has been replaced by the original variables using (2).

One can see that even for very simple functions such as $f(u) = u$, $f(u) = e^u$, $f(u) = \sin u$, $f(u) = \cos u$ in equation (1), its solution (3) cannot be represented explicitly in terms of elementary functions. Therefore, looking for exact solutions to more-complex diffusion-type equations in explicit form seems to be of little prospect.

Example 2. The non-linear wave equation

$$u_{tt} = [f(u)u_x]_x, \tag{4}$$

where $f(u)$ is an arbitrary function, also admits a traveling-wave solution $u = u(z)$, $z = \lambda t + \kappa x$, which can be represented in the implicit form

$$\int [\kappa^2 f(u) - \lambda^2]\,du = C_1(\lambda t + \kappa x) + C_2. \tag{5}$$

Examples 1 and 2 show that non-linear equations (1) and (4) have traveling-wave solutions that can be represented in implicit form. It is important that in the general case of arbitrary $f(u)$, these solutions cannot be represented explicitly. More complex examples of non-linear PDEs that have solutions in implicit form can be found, for example, in [18].

Section 2.3 will describe a method for constructing exact solutions to non-linear equations of mathematical physics, based on the generalization of solutions (3) and (5).

2.2 Class of non-linear reaction-diffusion equations in question

We will consider one-dimensional non-linear equations of the reaction-diffusion type with variable coefficients

$$c(x)u_t = [a(x)f(u)u_x]_x + b(x)g(u). \tag{6}$$

Note that if $a(x) = c(x) = x^n$, equation (6) describes reaction-diffusion processes with radial symmetry in two-dimensional (with $n = 1$) and three-dimensional (with $n = 2$) cases (x is the radial coordinate).

The present paper will focus on equations (6) of a fairly general form, which depend on one or two arbitrary functions.

2.3 Direct method for constructing functional separable solutions in implicit form. An outline

Exact solutions of equation (6) or other non-linear partial differential equation,

$$G(x, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots) = 0, \tag{7}$$

are sought in the implicit form

$$\int h(u)\, du = \xi(x)\omega(t) + \eta(x), \tag{8}$$

where the functions $h(u)$, $\xi(x)$, $\eta(x)$, and $\omega(t)$ are determined in the subsequent analysis. The representation of the solution in the form (8) is based on a substantial generalization of solution (3), which is carried out as follows:

$$\frac{\kappa^2 f(u)}{\lambda u + C_1} \implies h(u), \qquad \lambda \implies \xi(x), \qquad t \implies \omega(t), \qquad \kappa x + C_2 \implies \eta(x).$$

The arguments of the functions $a = a(x)$, $b = b(x)$, $c = c(x)$, $f = f(u)$, $g = g(u)$, $h = h(u)$, $\xi = \xi(x)$, $\eta = \eta(x)$, and $\omega = \omega(t)$, which appear in equation (6) and solution (8), will often be omitted.

We describe the procedure for constructing exact solutions in implicit form. First, using (8), one calculates the partial derivatives u_x, u_t, u_{xx}, …, which are expressed in terms of functions h, ξ, η, ω and their derivatives. Then, these partial derivatives must be substituted into equation (7) followed by eliminating the variable t with the help of (8). As a result (with a suitable choice of ω), one arrives at a bilinear functional-differential equation

$$\sum_{j=1}^{N} \varphi_j[x]\psi_j[u] = 0,$$
$$\varphi_j[x] \equiv \varphi_j(x, \xi, \eta, \xi'_x, \eta'_x, \xi''_{xx}, \eta''_{xx} \dots), \quad \psi_j[u] \equiv \psi_j(u, h, h'_u, h''_{uu}, \dots). \tag{9}$$

Here, $\varphi_j[x]$ and $\psi_j[u]$ are differential forms (in some cases, functional coefficients) that depend respectively on only x and u. The following statement is true.

Proposition (first formulated by Birkhoff [49]). Functional-differential equations of the form (9) can have solutions only if the forms $\psi_j[u]$ ($j = 1, \dots, N$) are connected by linear relations (see, for example, [11, 15, 18]):

$$\sum_{j=1}^{m_i} k_{ij}\psi_j[u] = 0, \qquad i = 1, \dots, n, \tag{10}$$

where k_{ij} are some constants, $1 \le m_i \le N-1$, and $1 \le n \le N-1$. It is necessary to consider also degenerate cases when, in addition to the linear relations, some individual differential forms $\psi_j[u]$ vanish.

A similar proposition is also true for the forms $\varphi_j[x]$.

The above proposition will be used in Sections 3 and 4 to construct exact solutions of some functional-differential equations of the form (9), which arise in seeking solutions of the corresponding non-linear reaction-diffusion type equations (6).

Remark 2. The study [11] considered a non-linear reaction-diffusion equation of the form (6) with $a(x) = c(x) = 1$ and used the implicit representation (8) with $\xi(x) = 1$ to look for its exact solutions.

Remark 3. Constructing a solution in implicit form with an integral term on the left-hand side of (8) often leads to lower-order differential equations for the function h than when exact solutions are sought in explicit form. In addition, the implicit form of representation of solutions usually leads to simpler explicit representations of the functions f and g through h (when searching for exact solutions in explicit form, the functions f and g are often expressed in terms of u in parametric form [18]). Note also that, in the generic case, different linear relations of the form (10) correspond to different solutions of the PDE under consideration.

Note that solutions of the form (8) usually cannot be obtained by applying the classical Lie group analysis of PDEs [7, 50].

2.4 Derivation of functional-differential equation

We are looking for an exact solution of the equations of reaction-diffusion (6) in implicit form (8). Differentiating (8) with respect to t and x, we get

$$hu_t = \xi\omega'_t \quad\Longrightarrow\quad u_t = \frac{\xi\omega'_t}{h}; \quad hu_x = \xi'_x\omega + \eta'_x \quad\Longrightarrow\quad u_x = \frac{\xi'_x\omega + \eta'_x}{h};$$
$$(afu_x)_x = \Big[(a\xi'_x\omega + a\eta'_x)\frac{f}{h}\Big]_x = [(a\xi'_x)'_x\omega + (a\eta'_x)'_x]\frac{f}{h} + a(\xi'_x\omega + \eta'_x)^2\frac{1}{h}\Big(\frac{f}{h}\Big)'_u.$$

Substituting these expressions into (6) yields the functional-differential equation

$$\omega'_t = \Phi_1(x,u)\omega^2 + \Phi_2(x,u)\omega + \Phi_3(x,u), \tag{11}$$

where the functions Φ_n are not explicitly dependent on t and are defined by the formulas

$$\begin{aligned}
\Phi_1(x,u) &= \frac{a(\xi'_x)^2}{c\xi}\Big(\frac{f}{h}\Big)'_u,\\
\Phi_2(x,u) &= \frac{1}{c\xi}\Big[(a\xi'_x)'_x f + 2a\xi'_x\eta'_x\Big(\frac{f}{h}\Big)'_u\Big],\\
\Phi_3(x,u) &= \frac{1}{c\xi}\Big[(a\eta'_x)'_x f + a(\eta'_x)^2\Big(\frac{f}{h}\Big)'_u + bgh\Big].
\end{aligned} \tag{12}$$

Equation (11)–(12) depends on three variables t, x, u, which are connected by one additional relation (8); it contains unknown functions (and their derivatives) that have different arguments. This equation is more complex than equations of the form (9).

The functional-differential equation (11)–(12) is significantly simplified in two cases: (i) $\xi'_x = 0$ and (ii) $(f/h)'_u = 0$. These cases are discussed further in Sections 3 and 4.

3 Exact solutions in the case $\xi(x) = 1$

3.1 Generalized traveling-wave solutions for $\omega(t) = kt$

For $\xi'_x = 0$, without loss of generality, we can set $\xi = 1$. On substituting $\xi = 1$ into (12), we get $\Phi_1(x, u) = \Phi_2(x, u) = 0$. As a result, equation (11) is reduced to the form

$$\omega'_t = \frac{1}{c}\Big[(a\eta'_x)'_x f + a(\eta'_x)^2\Big(\frac{f}{h}\Big)'_u + bgh\Big]. \tag{13}$$

The functional-differential equation (13) is an equation with separated variables (the left-hand side depends only on t, while the right-hand side depends on x and u). Therefore, we can set $\omega'_t = k = \text{const}$, which gives $\omega(t) = kt$. The situation under consideration corresponds to generalized traveling-wave solutions given in implicit form

$$\int h(u)\,du = kt + \int \theta(x)\,dx. \tag{14}$$

Here, the integrands $h(u)$ and $\theta(x) = \eta'_x(x)$ will be determined in the subsequent analysis from the functional-differential equation

$$(a\theta)'_x f + a\theta^2\Big(\frac{f}{h}\Big)'_u + bgh - kc = 0, \tag{15}$$

which is obtained by substituting the functions $\omega(t) = kt$ and $\theta(x) = \eta'_x(x)$ in (13). Equation (15) is a bilinear functional-differential equation of the form (9) with $N = 4$.

Solution 1. First, let us look at the degenerate case where the differential form $(f/h)'_u$ in (15) is zero. In this case, equation (15) has solutions under the following conditions

$$h = f, \quad g = A + \frac{B}{f}, \quad (a\theta)'_x + Ab = 0, \quad Bb - kc = 0, \tag{16}$$

where A and B are arbitrary constants. From relations (16) with $b(x) = c(x) = 1$ it follows that the equation

$$u_t = [a(x)f(u)u_x]_x + A + \frac{k}{f(u)}, \tag{17}$$

which contains two arbitrary functions $a(x)$ and $f(u)$, has the generalized traveling-wave exact solution

$$\int f(u)\,du = kt - A\int \frac{x\,dx}{a(x)} + C_1 \int \frac{dx}{a(x)} + C_2, \tag{18}$$

where C_1 and C_2 are arbitrary constants. In the special case $a(x) = 1$, equation (17) and its solution (18) become the equation and solution obtained in [6].

Remark 4. It is easy to verify that the equation

$$u_t = [a(x)f(u)u_x]_x + b(x) + \frac{k}{f(u)},$$

which contains three arbitrary functions $a(x)$, $b(x)$, and $f(u)$ and generalizes the equation (17), has the exact solution in implicit form

$$\int f(u)\,du = kt - \int \frac{1}{a(x)}\Bigl(\int b(x)\,dx\Bigr)dx + C_1 \int \frac{dx}{a(x)} + C_2.$$

Solution 2. Equation (15) holds if we set

$$f = A, \quad g = \frac{1}{h}\Bigl(\frac{f}{h}\Bigr)'_u, \quad A(a\theta)'_x - kc = 0, \quad b = -a\theta^2, \tag{19}$$

where A is an arbitrary constant.

Using the formulas (19) with $c(x) = 1$ and $A = k = 1$, one can obtain the non-linear reaction-diffusion equation

$$u_t = [a(x)u_x]_x - \frac{x^2}{a(x)}g(u), \tag{20}$$

where $a(x)$ is an arbitrary function. The function $g(u)$ is expressed through the arbitrary function $h = h(u)$ as

$$g(u) = -h^{-3}h'_u. \tag{21}$$

Equation (20) under condition (21) admits the exact solution

$$\int h(u)\,du = t + \int \frac{x\,dx}{a(x)} + C_1. \tag{22}$$

Solving (21) for h, we get two functions

$$h(u) = \pm\Big(2\int g(u)\,du + C_2\Big)^{-1/2}.$$

By eliminating h from (22) with the help of the obtained expressions, we rewrite the solutions of equation (20) as

$$\pm\int\Big(2\int g(u)\,du + C_2\Big)^{-1/2} du = t + \int\frac{x\,dx}{a(x)} + C_1. \tag{23}$$

Here, $a(x)$ and $g(u)$ are arbitrary functions, and C_1 and C_2 are arbitrary constants.

Example 3. Exact solutions of the equation

$$u_t = (x^n u_x)_x - x^{2-n}g(u)$$

are determined by the formulas (23) with $a(x) = x^n$.

Note that equation (20) and its solutions were derived from other considerations in [25].

Solution 3. Equation (15) can be satisfied by setting

$$\Big(\frac{f}{h}\Big)'_u = A, \quad g = -\frac{f}{h}, \quad Aa\theta^2 = kc, \quad (a\theta)'_x = b, \tag{24}$$

where A is an arbitrary constant. Taking $c(x) = 1$, we obtain the non-linear reaction-diffusion equation

$$u_t = [a(x)f(u)u_x]_x \mp \frac{1}{2}\sqrt{\frac{k}{A}}\frac{a'_x(x)}{\sqrt{a(x)}}(Au + B), \tag{25}$$

where the function $f(u)$ is expressed through the arbitrary function $h = h(u)$ as follows:

$$f(u) = (Au + B)h(u). \tag{26}$$

Equations (25)–(26) admit exact solutions

$$\int h(u)\,du = kt \pm \sqrt{\frac{k}{A}}\int\frac{dx}{\sqrt{a(x)}} + C. \tag{27}$$

Formulas (26) and (27) include arbitrary constants A, B, and C.

Example 4. By setting $A = 1$, $B = 0$, and $k = 4$ in (25)–(27), we get the equations

$$u_t = [a(x)f(u)u_x]_x \mp \frac{a'_x(x)}{\sqrt{a(x)}}u, \tag{28}$$

which contains two arbitrary functions $a(x)$ and $f(u)$. With the relation $h(u) = f(u)/u$, exact solutions of these equations can be written in the implicit form

$$\int \frac{f(u)}{u}\,du = 4t \pm 2\int \frac{dx}{\sqrt{a(x)}} + C. \tag{29}$$

By setting $a(x) = \frac{1}{4}\alpha^2\beta^{-2}x^{2\beta}$ and $a(x) = \frac{1}{4}\alpha^2\beta^{-2}e^{2\beta x}$ in (28), we get the equations

$$\begin{aligned} u_t &= [x^{2\beta} f_1(u)u_x]_x + \alpha x^{\beta-1}u, \qquad f_1(u) = \tfrac{1}{4}\alpha^2\beta^{-2}f(u),\\ u_t &= [e^{2\beta x} f_1(u)u_x]_x + \alpha e^{\beta x}u, \qquad f_1(u) = \tfrac{1}{4}\alpha^2\beta^{-2}f(u), \end{aligned}$$

the exact solutions of which are found by the formula (29).

Solution 4. Equation (15) holds if we set

$$Af = \left(\frac{f}{h}\right)'_u, \quad g = \frac{k}{h}, \quad (a\theta)'_x + Aa\theta^2 = 0, \quad b = c, \tag{30}$$

where A is an arbitrary constant. With $c(x) = 1$ in (30), we obtain the non-linear reaction-diffusion equation

$$u_t = [a(x)f(u)u_x]_x + g(u), \tag{31}$$

where $a(x)$ is an arbitrary function, and the functions $f(u)$ and $g(u)$ are expressed through the arbitrary function $h = h(u)$ as

$$f(u) = Bh\exp\left(A\int h\,du\right), \quad g(u) = \frac{k}{h}. \tag{32}$$

Equation (31)–(32) has the exact solution

$$\int h(u)\,du = kt + \frac{1}{A}\ln\left(C_1\int \frac{dx}{a(x)} + C_2\right). \tag{33}$$

Formulas (32) and (33) include arbitrary functions $a(x)$ and $h(u)$ and arbitrary constants A, B, C_1, C_2, and k.

Example 5. By setting $h = 1/u$, $A = m+1$, and $B = 1$ in (32) and (33), we obtain the reaction-diffusion equation with power non-linearity

$$u_t = [a(x)u^m u_x]_x + ku, \qquad m \neq -1,$$

which has the exact solution

$$u = e^{kt}\left(C_1\int \frac{dx}{a(x)} + C_2\right)^{\frac{1}{m+1}},$$

where C_1 and C_2 are arbitrary constants.

Remark 5. The equation

$$u_t = [a(x)e^{\lambda u}u_x]_x + k$$

has the exact solution

$$u = kt + \frac{1}{\lambda}\ln\Bigl(C_1\int\frac{dx}{a(x)} + C_2\Bigr).$$

Solution 5. Equation (15) holds if we set

$$g = A_1\frac{f}{h} + A_2\frac{1}{h}, \quad \Bigl(\frac{f}{h}\Bigr)'_u = A_3 f + A_4, \tag{34}$$

$$(a\theta)'_x + A_3 a\theta^2 + A_1 b = 0, \quad A_4 a\theta^2 + A_2 b - kc = 0, \tag{35}$$

where A_1, ..., A_4 are arbitrary constants. From equations (34), we obtain the following representations of the functions g and f in terms of h:

$$\begin{aligned} &f = hE\Bigl(A_4\int\frac{du}{E} + C_1\Bigr), \quad E = \exp\Bigl(A_3\int h\,du\Bigr),\\ &g = A_1C_1E + A_1A_4E\int\frac{du}{E} + A_2\frac{1}{h}, \end{aligned} \tag{36}$$

where C_1 is an arbitrary constant.

Equations (35) for given functions $a = a(x)$ and $c = c(x)$ allow us to find two other functions, $b(x)$ and $\theta(x)$. Eliminating $b(x)$ from equations (35), we obtain a first-order ODE with quadratic non-linearity in $\theta(x)$,

$$A_2 a\theta'_x + (A_2A_3 - A_1A_4)a\theta^2 + A_2 a'_x\theta + A_1 kc = 0, \tag{37}$$

which is a Riccati equation [51]. With the substitution

$$\theta = \lambda\frac{\zeta'_x}{\zeta}, \qquad \lambda = \frac{A_2}{A_2A_3 - A_1A_4} \qquad (A_2A_3 - A_1A_4 \neq 0), \tag{38}$$

it is reduced to the linear second-order ODE

$$A_2\lambda(a\zeta'_x)'_x + A_1kc\zeta = 0. \tag{39}$$

Exact solutions of equation (39) for some functions $a = a(x)$ and $c = c(x)$ can be found in [51].

Using the last relation in (35), we can express the functional coefficient b through θ:

$$b = -\frac{1}{A_2}(A_4 a\theta^2 - kc). \tag{40}$$

Example 6. For $a(x) = c(x) = 1$, the general solution of equation (39) is given by

$$\zeta = \begin{cases} C_2 \cos(mx) + C_3 \sin(mx) & \text{if} \quad A_1 k(A_2 A_3 - A_1 A_4) > 0, \\ C_2 \cosh(mx) + C_3 \sinh(mx) & \text{if} \quad A_1 k(A_2 A_3 - A_1 A_4) < 0, \end{cases} \tag{41}$$

where C_2 and C_3 are arbitrary constants, and $m = \sqrt{|A_1 k|/|A_2 \lambda|}$. In particular, substituting $A_1 = A_2 = A_4 = 1$, $A_3 = 2$, $C_2 = 1$, $C_3 = 0$, and $k = -1$ in formulas (38), (40) and (41), we obtain $m = \lambda = 1$, $\zeta = \cosh x$, $\theta = \tanh x$, and $b = -(1 + \tanh^2 x)$.

Solution 6. Equation (15) also has solutions under the following conditions:

$$(a\theta)'_x = Ac, \quad a\theta^2 = Bc, \quad b = c, \quad Af + B\Big(\frac{f}{h}\Big)'_u + gh - k = 0, \tag{42}$$

where A and B are arbitrary constants.

1. Substituting $a(x) = b(x) = c(x) = 1$, $\theta(x) = \kappa$, $A = 0$, $B = \kappa^2$, and $k = \lambda$ into (42), we obtain a traveling-wave solution of the form (2), which is omitted here.

2. By setting $c(x) = 1$ and $A = B = 1$ in the first three equations (42), we find

$$a(x) = x^2, \quad b(x) = 1, \quad \theta(x) = 1/x. \tag{43}$$

As a result, we get the equation

$$u_t = [x^2 f(u) u_x]_x + g(u), \tag{44}$$

where

$$g(u) = \frac{k}{h(u)} - \frac{f(u)}{h(u)} - \frac{1}{h(u)} \frac{d}{du}\left[\frac{f(u)}{h(u)}\right], \tag{45}$$

which admits the exact solution in implicit form

$$\int h(u)\, du = kt + \ln x. \tag{46}$$

Note that equation (44)–(45) includes two arbitrary functions, $f = f(u)$ and $h = h(u)$.

Remark 6. The invariant solution (46) of equation (44) can be found in the usual form $u = U(z)$, where $z = kt + \ln x$ (in this case, relation (45) between the functions g and h is not used). The function $U(z)$ is determined from the ODE

$$kU'_z = [f(U)U'_z]'_z + f(U)U'_z + g(U).$$

Remark 7. The non-linear reaction-diffusion equation with delay (τ being the delay time)

$$u_t = [x^2 f(u, w) u_x]_x + g(u, w), \qquad w = u(x, t - \tau),$$

which is more general than (44), also has an exact solution of the form $u = U(z)$, where $z = kt + \ln x$.

3.2 Functional separable solutions for $\omega(t) = ke^{\lambda t}$

Let us return to equation (13). Section 3.1 investigated the simplest case of linear dependence, $\omega(t) = kt$, which immediately led to a functional-differential equation with two variables of type (9).

The function $\omega(t)$ enters formula (8) in a linear fashion. If we choose $\omega(t) = ke^{\lambda t}$ (k is an arbitrary constant), then the solution will take the form

$$H(u) = ke^{\lambda t} + \eta(x), \qquad H(u) = \int h(u)\,du, \tag{47}$$

and the function $e^{\lambda t}$ can be eliminated from equation (13) using (47). As a result, we arrive at a functional-differential equation of the form (9) with $N = 5$:

$$\lambda\eta - \lambda H + \frac{(a\eta'_x)'_x}{c} f + \frac{a(\eta'_x)^2}{c}\Big(\frac{f}{h}\Big)'_u + \frac{b}{c} gh = 0. \tag{48}$$

Remark 8. Equation (48) can be derived from other considerations. Indeed, rewriting (8) as

$$\xi\omega/(H - \eta) = 1, \tag{49}$$

we multiply the right-hand side of equation (16) by $\xi\omega/(H - \eta)$ and, after simple manipulations, taking into account that $\xi = 1$, we obtain

$$\frac{\omega'_t}{\omega} = \frac{1}{c(H-\eta)}\Big[(a\eta'_x)'_x f + a(\eta'_x)^2\Big(\frac{f}{h}\Big)'_u + bgh\Big]. \tag{50}$$

In equation (50), the variables are separated: the left-hand side depends only on t, and the right-hand side depends on x and u. Equating both sides (50) with the constant λ, we obtain two equations. The left-hand side of (50) gives the equation $\omega'_t/\omega = \lambda$, whose solution is $\omega = ke^{\lambda t}$. The right-hand side of (50) leads to equation (48).

Solution 7. Equation (48) holds if we set

$$\begin{gathered} f = C_1uh + C_2h, \quad g = \lambda\frac{H}{h} - C_1C_3u - C_2C_3, \\ b = c, \quad (a\eta'_x)'_x = C_3c, \quad C_1a(\eta'_x)^2 + \lambda c\eta = 0, \end{gathered} \tag{51}$$

where C_1, C_2, and C_3 are arbitrary constants. Relations (51) include two arbitrary functions h and c, and the functions f, g, a, b, and η are expressed through them.

The general solution of the system consisting of the last two equations in (51) has the form

$$\begin{aligned} a(x) &= \frac{C_5}{c(x)}\Big[C_3\int c(x)\,dx + C_4\Big]^{2+\lambda/(C_1C_3)}, \\ \eta(x) &= -\frac{C_1}{C_5\lambda}\Big[C_3\int c(x)\,dx + C_4\Big]^{-\lambda/(C_1C_3)}, \end{aligned} \tag{52}$$

where C_4 and C_5 are arbitrary constants.

Example 7. By setting $c(x) = 1$, $C_1 = C_3 = C_5 = 1$, $C_2 = C_4 = 0$, and $\lambda = n-2$ in (52), we obtain $a(x) = x^n$, $b(x) = 1$, and $\eta(x) = x^{2-n}/(2-n)$. Taking into account the first two relations in (51), we find that the reaction-diffusion equation

$$\begin{gathered} u_t = [x^n f(u) u_x]_x + g(u), \\ f(u) = u h(u), \quad g(u) = \frac{(n-2)}{h(u)} \int h(u)\,du - u, \end{gathered} \tag{53}$$

where $h(u)$ is an arbitrary function and $n \neq 2$ is an arbitrary constant, admits the functional separable solution in implicit form

$$\int h(u)\,du = k e^{(n-2)t} + \frac{x^{2-n}}{2-n}; \tag{54}$$

k is an arbitrary constant.

In view of the ratio $h = f/u$, equation (53) can be rewritten in explicit form as

$$u_t = [x^n f(u) u_x]_x - u + \frac{(n-2)u}{f(u)} \int \frac{f(u)}{u}\,du.$$

Its solution is written as follows:

$$\int \frac{f(u)}{u}\,du = k e^{(n-2)t} + \frac{x^{2-n}}{2-n}.$$

Solution 8. Equation (48) can be satisfied in another way, by setting

$$\begin{gathered} f = 1, \quad g = \lambda \frac{H}{h} + C_1 \frac{h'_u}{h^3}, \\ b = c, \quad (a\eta'_x)'_x + \lambda c \eta = 0, \quad a(\eta'_x)^2 - C_1 c = 0, \end{gathered} \tag{55}$$

where C_1 is an arbitrary constant. Relations (55) include two arbitrary functions h and c, with the functions g, a, b, and η expressed through them.

The general solution of the system consisting of the last two equations in (55) has the form

$$\begin{gathered} a(x) = \frac{C_4^2}{C_1 c(x)} \exp\left(-\frac{\lambda}{C_1}\eta^2\right), \\ \int \exp\left(-\frac{\lambda}{2C_1}\eta^2\right) d\eta = \frac{C_1}{C_4} \int c(x)\,dx + C_5, \end{gathered} \tag{56}$$

where C_4, C_5 are arbitrary constants (this solution can be represented in terms of the inverse error function).

Remark 9. If in the last two equations (55), the function $\eta=\eta(x)$ is considered to be given, then the solution can be written as

$$a(x)=\frac{C_4}{\eta'_x(x)}\exp\Bigl(-\frac{\lambda}{2C_1}\eta^2\Bigr),\qquad c(x)=\frac{C_4}{C_1}\eta'_x(x)\exp\Bigl(-\frac{\lambda}{2C_1}\eta^2\Bigr).$$

3.3 Functional separable solutions for $\omega(t)=k\ln t$

Substituting $\xi=1$ and $\omega(t)=k\ln t$ in (8), we look for solutions in the form

$$\int h(u)\,du=k\ln t+\eta(x). \tag{57}$$

Eliminating t from (13), with $\omega=k\ln t$, and (57), we obtain the functional-differential equation

$$(a\eta'_x)'_x f+a(\eta'_x)^2\Bigl(\frac{f}{h}\Bigr)'_u+bgh-kce^{\eta/k}e^{-H/k}=0,\qquad H=\int h(u)\,du. \tag{58}$$

Remark 10. Equation (58) can be derived from other considerations. Indeed, rewriting formula (8) with $\xi=1$ as

$$e^{(H-\eta-\omega)/k}=1,$$

where k is some constant, multiplying the right-hand side of equation (13) by $e^{(H-\eta-\omega)/k}$, and rearranging. we obtain

$$e^{\omega/k}\omega'_t=\frac{e^{(H-\eta)/k}}{c}\Bigl[(a\eta'_x)'_x f+a(\eta'_x)^2\Bigl(\frac{f}{h}\Bigr)'_u+bgh\Bigr]. \tag{59}$$

In equation (59), the variables are separated: the left-hand side depends on t alone and the right-hand side depends on x and u. Equating both parts (59) with the constant λ, we obtain two equations. The left-hand side of (59) gives the equation $e^{\omega/k}\omega'_t=\lambda$, whose solution is $\omega=k\ln(t+t_0)+k\ln(\lambda/k)$. The right-hand side of (59) with $\lambda=k$ leads to equation (58).

Solution 9. First, let us look at the degenerate case where the differential form $(f/h)'_u$ vanishes. In this case, equation (58) has solutions under the following conditions

$$h=f,\quad g=A+\frac{B}{f}e^{-F/k},\quad (a\eta'_x)'_x+Ab=0,\quad Bb-kce^{\eta/k}=0, \tag{60}$$

where A and B are arbitrary constants, and $F=\int f(u)\,du$. From relations (60) with $B=k$, it follows that the equation

$$c(x)u_t=[a(x)f(u)u_x]_x+c(x)e^{\eta(x)/k}\Bigl[A+\frac{k}{f(u)}e^{-F(u)/k}\Bigr], \tag{61}$$

where $a(x)$, $c(x)$, and $f(u)$ are arbitrary functions, and the function $\eta = \eta(x)$ is the solution of the second-order non-linear ODE

$$[a(x)\eta'_x]'_x + Ac(x)e^{\eta/k} = 0, \tag{62}$$

has the generalized traveling-wave solution

$$\int f(u)\,du = k\ln t + \eta(x). \tag{63}$$

Example 8. For $a(x) = x^n$ $(n \neq 1, 2)$, $c(x) = 1$, and $A = -k(n-1)(n-2)$, equation (62) admits the exact solution $\eta = k(n-2)\ln x$. Therefore, the equation

$$u_t = [x^n f(u)u_x]_x + x^{n-2}\left[-k(n-1)(n-2) + \frac{k}{f(u)}e^{-F(u)/k}\right] \tag{64}$$

admits the exact solution in implicit form $\int f(u)\,du = k\ln t + k(n-2)\ln x$.

Example 9. For $a(x) = e^{\lambda x}$, $c(x) = 1$, and $A = -k\lambda^2$, equation (62) admits the exact solution $\eta = k\lambda x$. Therefore, the equation

$$u_t = [e^{\lambda x} f(u)u_x]_x + e^{\lambda x}\left[-k\lambda^2 + \frac{k}{f(u)}e^{-F(u)/k}\right] \tag{65}$$

admits the exact solution in implicit form $\int f(u)\,du = k\ln t + k\lambda x$.

Solution 10. Equation (58) can be satisfied if we take

$$Af = \left(\frac{f}{h}\right)'_u, \quad g = \frac{1}{h}e^{-H/k}, \quad (a\eta'_x)'_x + Aa(\eta'_x)^2 = 0, \quad b = kce^{\eta/k}, \tag{66}$$

where A is an arbitrary constant. Substituting $c = 1$ into (66), we obtain the non-linear reaction-diffusion equation

$$u_t = [a(x)f(u)u_x]_x + k\left(C_1\int\frac{dx}{a(x)} + C_2\right)^{1/(Ak)} g(u), \tag{67}$$

where $a(x)$ is an arbitrary function, and the functions $f(u)$ and $g(u)$ are expressed as follows through the arbitrary function $h = h(u)$:

$$f(u) = Bh\exp\left(A\int h\,du\right), \quad g(u) = \frac{1}{h}\exp\left(-\frac{1}{k}\int h\,du\right). \tag{68}$$

Equation (67)–(68) has the exact solution

$$\int h(u)\,du = k\ln t + \frac{1}{A}\ln\left(C_1\int\frac{dx}{a(x)} + C_2\right). \tag{69}$$

Formulas (68) and (69) include arbitrary functions $a(x)$ and $h(u)$ and arbitrary constants A, B, C_1, C_2, and k.

Example 10. If $a(x) = 1$, $A = 1/(kn)$, $C_1 = k^{-1/n}$, and $C_2 = 0$, equation (67) becomes

$$u_t = [f(u)u_x]_x + x^n g(u),$$

where the functions $f(u)$ and $g(u)$ are given by formulas (68). This equation has the exact solution $\int h(u)\,du = k(\ln t + n\ln x - \ln k)$.

Solution 11. Equation (58) holds if we set

$$\Bigl(\frac{f}{h}\Bigr)'_u - Ae^{-H/k} = 0, \quad g = -\frac{f}{h}, \quad Aa(\eta'_x)^2 = kce^{\eta/k}, \quad (a\eta'_x)'_x = b, \tag{70}$$

where A is an arbitrary constant. As a result, we obtain the non-linear reaction-diffusion equation

$$c(x)u_t = [a(x)f(u)u_x]_x + b(x)g(u), \tag{71}$$

with the functions $f(u)$ and $g(u)$ expressed in terms of the arbitrary function $h = h(u)$ as follows:

$$f(u) = h(u)\biggl[A\int e^{-H(u)/k}du + B\biggr], \quad g(u) = -A\int e^{-H(u)/k}du - B, \tag{72}$$

and the function $b(x)$ is determined by the formulas (two variants are possible)

$$b(x) = [a(x)\eta'_x]'_x, \qquad \eta = -2k\ln\biggl[C_1 \pm \frac{1}{2\sqrt{Ak}}\int\sqrt{\frac{c(x)}{a(x)}}dx\biggr]. \tag{73}$$

Equations (71)–(73) have the exact solutions

$$\int h(u)\,du = k\ln t - 2k\ln\biggl[C_1 \pm \frac{1}{2\sqrt{Ak}}\int\sqrt{\frac{c(x)}{a(x)}}dx\biggr]. \tag{74}$$

Formulas (72) and (74) include arbitrary constants A, B, and C.

Eliminating the function h from formulas (72), we obtain a simple relation between the functions f and g,

$$f = k\frac{gg''_{uu}}{g'_u}.$$

Solution 12. Equation (58) can be satisfied by setting

$$f = Ae^{-H/k}, \quad g = \frac{1}{h}\Bigl(\frac{f}{h}\Bigr)'_u, \quad A(a\eta'_x)'_x - kce^{\eta/k} = 0, \quad b = -a(\eta'_x)^2, \tag{75}$$

where A is an arbitrary constant.

From the first two equations in (75), we express g and h in terms of f to obtain

$$H = -k \ln \frac{f}{A}, \quad h = -k\frac{f'_u}{f}, \quad g = \frac{f}{k^2 f'_u}\left(\frac{f^2}{f'_u}\right)'_u. \tag{76}$$

Suppose that in the last two equations in (75), the variable functions $c = c(x)$ and $\eta = \eta(x)$ are given. Then the functional coefficients $a = a(x)$ and $b = b(x)$ are determined by the formulas

$$a(x) = \frac{k}{A\eta'_x}\left(\int ce^{\eta/k}dx + B\right), \qquad b(x) = -\frac{k\eta'_x}{A}\left(\int ce^{\eta/k}dx + B\right),$$

where B is an arbitrary constant.

Solution 13. Equation (58) holds if we set

$$\begin{gathered}(a\eta'_x)'_x = Ace^{\eta/k}, \quad a(\eta'_x)^2 = Bce^{\eta/k}, \quad b = ce^{\eta/k},\\ Af + B\left(\frac{f}{h}\right)'_u + gh - k\exp(-H/k) = 0,\end{gathered} \tag{77}$$

where A and B are arbitrary constants and $H = \int h(u)\,du$.

Substituting $c(x) = 1$, $A = 1/k$, and $B = 1$ into the first three equations in (77), we find that

$$a(x) = b(x) = e^{\lambda x}, \quad \eta(x) = x, \quad \lambda = \frac{1}{k}. \tag{78}$$

As a result, we obtain the equation

$$u_t = [e^{\lambda x} f(u) u_x]_x + e^{\lambda x} g(u), \tag{79}$$

where

$$g(u) = \frac{1}{\lambda h(u)} \exp\left[-\lambda \int h(u)\,du\right] - \lambda\frac{f(u)}{h(u)} - \frac{1}{h(u)}\frac{d}{du}\left[\frac{f(u)}{h(u)}\right], \tag{80}$$

which admits the exact solution in implicit form

$$\int h(u)\,du = x + \frac{1}{\lambda}\ln t. \tag{81}$$

Note that equation (79)–(80) includes two arbitrary functions, $f = f(u)$ and $h = h(u)$.

Remark 11. The invariant solution (81) of equation (79) can be found in the usual form $u = U(z)$, where $z = x + (1/\lambda)\ln t$ (in this case, the relation (80) between the functions g and h is not used). The function $U(z)$ is determined by the ODE

$$\frac{1}{\lambda}U'_z = [e^{\lambda z}f(U)U'_z]'_z + e^{\lambda z}g(U).$$

Solution 14. By setting $c(x) = 1$, $A = (1+k)/k$, and $B = 1$ in the first three equations of (77), we find that

$$a(x) = x^n, \quad b(x) = x^{n-2}, \quad \eta(x) = \ln x, \quad n = 2 + \frac{1}{k}. \tag{82}$$

This leads to the reaction-diffusion equation

$$u_t = [x^n f(u)u_x]_x + x^{n-2}g(u), \tag{83}$$

where $n \neq 2$ and

$$g(u) = \frac{1}{(n-2)h(u)}\exp\left[-(n-2)\int h(u)\,du\right] - (n-1)\frac{f(u)}{h(u)} - \frac{1}{h(u)}\frac{d}{du}\left[\frac{f(u)}{h(u)}\right], \tag{84}$$

which admits the exact solution in implicit form

$$\int h(u)\,du = \ln x + \frac{1}{n-2}\ln t. \tag{85}$$

Remark 12. The self-similar solution (85) of equation (83) can be found in the usual form $u = U(z)$, where $z = xt^{1/(n-2)}$ (in this case, relation (84) between the functions g and h is not used). The function $U(z)$ is determined by the ODE

$$\frac{1}{n-2}zU'_z = [z^n f(U)U'_z]'_z + z^{n-2}g(U).$$

Solution 15. Equation (58) holds if we set

$$\begin{gathered}\left(\frac{f}{h}\right)'_u = Af, \quad \exp(-H/k) = Bf, \quad gh = f,\\ (a\eta'_x)'_x + Aa(\eta'_x)^2 + b - Bkce^{\eta/k} = 0,\end{gathered} \tag{86}$$

where A and B are arbitrary constants and $H = \int h(u)\,du$.

By setting $A = -1/k = \lambda$ and $B = 1$ in the first three equations of (86), we get

$$f(u) = g(u) = e^{\lambda u}, \quad h(u) = 1. \tag{87}$$

Therefore, the equation

$$c(x)u_t = [a(x)e^{\lambda u}u_x]_x + b(x)e^{\lambda u} \tag{88}$$

admits the exact solution in explicit form

$$u = -\frac{1}{\lambda}\ln t + \eta(x), \tag{89}$$

where the function $\eta = \eta(x)$ is determined by the ODE

$$[a(x)e^{\lambda\eta}\eta'_x]'_x + b(x)e^{\lambda\eta} + \frac{1}{\lambda}c(x) = 0. \tag{90}$$

Equations (88) and (90) include three arbitrary functions $a(x)$, $b(x)$, and $c(x)$.

Solution 16. By setting $A = n+1$, $B = 1$, and $k = -1/n$ in the first three equations (86), we find

$$f(u) = u^n, \quad g(u) = u^{n+1}, \quad h(u) = 1/u. \tag{91}$$

As a result, we have the reaction-diffusion equation

$$c(x)u_t = [a(x)u^n u_x]_x + b(x)u^{n+1} \tag{92}$$

where $a(x)$, $b(x)$, and $c(x)$ are arbitrary functions, which admits an exact solution of the form $\ln u = -(1/n)\ln t + \eta(x)$. This solution can be represented explicitly as

$$u = t^{-1/n}\zeta(x), \qquad \zeta(x) = e^{\eta(x)}, \tag{93}$$

where the function ζ is described by the ODE

$$[a(x)\zeta^n\zeta'_x]'_x + b(x)\zeta^{n+1} + \frac{1}{n}c(x)\zeta = 0. \tag{94}$$

4 Exact solutions in the case $h = f$

4.1 Generalized traveling-wave solutions for $\omega(t) = t$

For $(f/h)'_u = 0$, without loss of generality, we can set $h = f$. Substituting $h = f$ in (11)–(12), we get the equation

$$\omega'_t = \frac{(a\xi'_x)'_x}{c\xi}f\omega + \frac{1}{c\xi}\big[(a\eta'_x)'_x f + bfg\big]. \tag{95}$$

Solution 17. In the degenerate case determined by the condition

$$(a\xi'_x)'_x = 0, \tag{96}$$

the variables in equation (95) are separated and we can set $\omega(t) = t$. As a result, we arrive at a functional-differential equation

$$(a\eta'_x)'_x f + bfg - c\xi = 0. \tag{97}$$

Integrating (96), we find the connection between the functions $a = a(x)$ and $\xi = \xi(x)$:

$$\xi = C_1 \int \frac{dx}{a(x)} + C_2, \tag{98}$$

where C_1 and C_2 are arbitrary constants. Equation (97) admits exact solutions as long as the conditions

$$g = k_1 + k_2 f^{-1}, \quad (a\eta'_x)'_x + k_1 b = 0, \quad k_2 b - c\xi = 0, \tag{99}$$

where k_1 and k_2 are arbitrary constants, are satisfied. From relations (98) and (99) it follows that the non-linear reaction-diffusion equation

$$c(x)u_t = [a(x)f(u)u_x]_x + c(x)\xi(x)\left[k + \frac{1}{f(u)}\right], \quad k = \frac{k_1}{k_2}, \tag{100}$$

where $a(x)$, $c(x)$, and $f(u)$ are arbitrary functions, and the function $\xi = \xi(x)$ is defined by formula (98), admits the generalized traveling-wave solution in implicit form

$$\begin{gathered}\int f(u)\,du = \xi(x)t + \eta(x),\\ \eta(x) = -k\int \frac{1}{a(x)}\left(\int c(x)\xi(x)\,dx\right)dx + C_3\int \frac{dx}{a(x)} + C_4;\end{gathered} \tag{101}$$

C_3 and C_4 are arbitrary constants.

Example 11. We set $a(x) = c(x) = 1$ in equation (100) and $C_1 = 1$, $C_2 = 0$ in formulas (98) and (101) to obtain the equation

$$u_t = [f(u)u_x]_x + x\left[k + \frac{1}{f(u)}\right], \tag{102}$$

which contains an arbitrary function $f(u)$ and an arbitrary constant k and has the generalized traveling-wave solution

$$\int f(u)\,du = xt - \frac{1}{6}kx^3 + C_3 x + C_4.$$

In the particular case of $f(u) = e^u$, equation (102) takes the form

$$u_t = (e^u u_x)_x + x(k + e^{-u}).$$

Its exact solution is expressed explicitly, $u = \ln(xt - \frac{1}{6}kx^3 + C_3x + C_4)$.

Solution 18. We substitute $\omega(t) = t$ in (8) and (95), and then eliminate t to obtain the functional-differential equation

$$\xi(a\eta'_x)'_x - \eta(a\xi'_x)'_x + b\xi g - c\xi^2 f^{-1} + (a\xi'_x)'_x F = 0, \qquad F = \int f(u)\,du. \tag{103}$$

In its derivation, the relation $h = f$ has been taken into account.

Equation (103) is a special case of equation (9) with $N = 5$. It admits solutions under the following conditions:

$$g = k_1 + k_2 f^{-1} + k_3 F, \qquad F = \int f(u)\,du, \tag{104}$$

$$\xi(a\eta'_x)'_x - \eta(a\xi'_x)'_x + k_1 b\xi = 0, \tag{105}$$

$$k_2 b - c\xi = 0, \tag{106}$$

$$(a\xi'_x)'_x + k_3 b\xi = 0, \tag{107}$$

where $f(u)$ is an arbitrary function and k_1, k_2, and k_3 are arbitrary constants. Assuming that the functions $a = a(x)$ and $c = c(x)$ are known, and eliminating b from equations (106) and (107), we arrive at an Emden–Fowler type equation for ξ:

$$(a\xi'_x)'_x + \frac{k_3}{k_2} c\xi^2 = 0. \tag{108}$$

Equation (105) is a linear nonhomogeneous ordinary differential equation with respect to η, which has a particular solution $\eta_p = -k_1/k_3$ (with this value substituted, equation (105) reduces to (107)). The shortened homogeneous equation (105) corresponding to $k_1 = 0$ has a particular solution $\eta_0 = \xi$. Hence, the order of this equation can be reduced [51]. Considering the above, we find the general solution of equation (105):

$$\eta = C_1\xi + C_2\xi \int \frac{dx}{a\xi^2} - \frac{k_1}{k_3}, \tag{109}$$

where C_1 and C_2 are arbitrary constants. The functional coefficient b is determined from equation (106).

To sum up, we have obtained the non-linear reaction-diffusion equation

$$c(x)u_t = [a(x)f(u)u_x]_x + c(x)\xi(x)\left[k_1 + \frac{1}{f(u)} + k_3 \int f(u)\,du\right], \tag{110}$$

where $a(x)$, $c(x)$, and $f(u)$ are arbitrary functions, and the function $\xi = \xi(x)$ satisfies equation (108) with $k_2 = 1$. Equation (110) has the exact solution in implicit form

$$\int f(u)\,du = \xi(x)t + \eta(x), \tag{111}$$

where the function $\eta(x)$ is defined by formula (109).

Note that equation (110) is a generalization of equation (100).

Example 12. For $a(x) = c(x) = k_2 = 1$, equation (108) has the exact solution $\xi = -(6/k_3)x^{-2}$. In this case, the function η in (109) takes the form $\eta = A_1x^{-2} + A_2x^3 - (k_1/k_3)$, where A_1 and A_2 are arbitrary constants that can be expressed in terms of C_1, C_2, and k_3.

Solution 19. It is easy to verify that the functional-differential equation (103) has solutions also if the relations

$$g = k_1 f^{-1}, \qquad F = k_2 f^{-1} + k_3, \tag{112}$$

where k_n are arbitrary constants, are satisfied. By setting $k_1 = 1$, $k_2 = 2$, and $k_3 = 0$ in (112), we obtain $f = u^{-1/2}$, $g = u^{1/2}$, and $F = 2u^{1/2}$. The corresponding non-linear reaction-diffusion type equation

$$c(x)u_t = [a(x)u^{-1/2}u_x]_x + b(x)u^{1/2}, \tag{113}$$

where $a(x)$, $b(x)$, and $c(x)$ are arbitrary functions, has an exact solution that can be represented explicitly as

$$u = \tfrac{1}{4}[\xi(x)t + \eta(x)]^2. \tag{114}$$

Here, the functions $\xi = \xi(x)$ and $\eta = \eta(x)$ are determined by solving the ordinary differential equations

$$\begin{aligned} 2(a\xi'_x)'_x + b\xi - c\xi^2 &= 0, \\ \xi(a\eta'_x)'_x - \eta(a\xi'_x)'_x &= 0. \end{aligned} \tag{115}$$

Let $\xi = \xi(x)$ be a solution of the first equation in (115). Then the general solution of the second equation (115) is determined by formula (109) with $k_1 = 0$.

Example 13. The first equation (115) can be satisfied if we take $\xi(x) = b(x) = k$ and $c(x) = 1$, where $k = \text{const}$. Therefore, the equation

$$u_t = [a(x)u^{-1/2}u_x]_x + ku^{1/2}, \tag{116}$$

which depends on an arbitrary function, $a(x)$, has the exact solution

$$u = \tfrac{1}{4}[kt + \eta(x)]^2, \qquad \eta(x) = C_1 + C_2 \int \frac{dx}{a(x)}. \tag{117}$$

Remark 13. The equation

$$c(x)u_t = [a(x)u^{-1/2}u_x]_x + b(x)u^{1/2} + p(x), \tag{118}$$

which is more general than (113), has an exact solution of the form (114). In the cases $b(x) = 0$ and $p(x)/b(x) = \text{const}$, equation (118) belongs to the class of equations (6) in question.

Remark 14. The non-linear delay PDE

$$\begin{aligned} c(x)u_t &= [a_1(x)u^{-1/2}u_x]_x + [a_2(x)w^{-1/2}w_x]_x + b_1(x)u^{1/2} + b_2(x)w^{1/2} + p(x), \\ w &= u(x, t-\tau), \end{aligned}$$

where τ is the delay time and $a_1(x)$, $a_2(x)$, $b_1(x)$, $b_2(x)$, $c(x)$, and $p(x)$ are arbitrary functions, also admits an exact solution of the form (114).

4.2 Functional separable solutions for $\omega(t) = e^{\lambda t}$

We substitute $\omega(t) = e^{\lambda t}$ in (8) and (95) and then eliminate t to obtain the functional-differential equation

$$\frac{\lambda}{f} = \frac{(a\xi'_x)'_x}{c\xi} + \left[\frac{(a\eta'_x)'_x}{c} + \frac{b}{c}g\right]\frac{1}{F - \eta}, \qquad F = \int f\, du. \tag{119}$$

Equation (119) holds if we set

$$(a\xi'_x)'_x = -k_1 c\xi, \quad \eta = -k_2, \quad b = c, \quad g = \Big(\frac{\lambda}{f} + k_1\Big)(F + k_2), \tag{120}$$

where k_1 and k_2 are arbitrary constants.

Solution 20. It follows from the relations (120) that the non-linear reaction-diffusion equation

$$c(x)u_t = [a(x)f(u)u_x]_x + c(x)\left[\frac{\lambda}{f(u)} + k_1\right]\left[\int f(u)\, du + k_2\right], \tag{121}$$

where $a(x)$, $c(x)$, and $f(u)$ are arbitrary functions and k_1, k_2, and λ are arbitrary constants, admits the functional separable solution in implicit form

$$\int f(u)\, du = \xi(x)e^{\lambda t} - k_2. \tag{122}$$

The function $\xi = \xi(x)$ is found by solving the linear ordinary differential equation $[a(x)\xi'_x]'_x + k_1 c(x)\xi = 0$.

In the degenerate case of $k_1 = 0$, the function $\xi = \xi(x)$ is defined by formula (98).

4.3 Functional separable solutions for $\omega(t) = t^{\beta}$

We substitute $\omega(t) = t^{1/(1-n)}$ in (8) and (95) and then eliminate t to obtain the functional-differential equation

$$\frac{1}{(n-1)f} = \frac{(a\xi'_x)'_x}{c\xi^{2-n}}\frac{1}{(F-\eta)^{n-1}} + \left[\frac{(a\eta'_x)'_x}{c\xi^{1-n}} + \frac{b}{c\xi^{1-n}}g\right]\frac{1}{(F-\eta)^n}, \tag{123}$$

where $F = \int f\,du$. Equation (123) can be satisfied if we take

$$(a\xi'_x)'_x = -k_1 c\xi^{2-n}, \quad \eta = -k_2, \quad b = c\xi^{1-n}, \quad g = k_1(F+k_2) + \frac{(F+k_2)^n}{(n-1)f}, \tag{124}$$

where k_1 and k_2 are arbitrary constants.

Solution 21. From relations (124) it follows that the non-linear reaction-diffusion equation

$$c(x)u_t = [a(x)f(u)u_x]_x + c(x)\xi^{1-n}\left\{k_1[F(u)+k_2] + \frac{[F(u)+k_2]^n}{(n-1)f(u)}\right\}, \tag{125}$$

where $a(x)$, $c(x)$, and $f(u)$ are arbitrary functions, k_1, k_2, n, and λ are arbitrary constants, and $F(u) = \int f(u)\,du$, admits the functional separable solution in implicit form

$$\int f(u)\,du = \xi(x)t^{1/(1-n)} - k_2. \tag{126}$$

The function $\xi = \xi(x)$ in (125) and (126) is described by the non-linear ordinary differential equation

$$[a(x)\xi'_x]'_x + k_1 c(x)\xi^{2-n} = 0. \tag{127}$$

Note that for $n = 2$, the general solution of equation (127) is

$$\xi = -k_1\int\frac{1}{a(x)}\left(\int c(x)\,dx\right)dx + C_1\int\frac{dx}{a(x)} + C_2, \tag{128}$$

where C_1 and C_2 are arbitrary constants.

Example 14. Substituting $a(x) = c(x) = 1$, $k_1 = 0$, $C_1 = 1$, and $C_2 = 0$ in (125)–(127), we get the equation

$$u_t = [f(u)u_x]_x + x^{1-n}\frac{[F(u)+k_2]^n}{(n-1)f(u)}, \qquad F(u) = \int f(u)\,du, \tag{129}$$

which allows the exact solution in implicit form $\int f(u)\,du = xt^{1/(1-n)} - k_2$. This solution is non-invariant and it is of a self-similar type; when substituted into equation (129), it causes the term $[f(u)u_x]_x$ to vanish.

5 Relationship of the method of functional separation of variables with other methods. Some comments

The direct method of constructing functional separable solutions in implicit form, based on formula (8), is closely related to (i) the method of differential constraints (which is based on the compatibility theory of PDEs [52]) and (ii) the nonclassical method of symmetry reduction (which is based on the invariant surface condition [53]). To show this, we differentiate formula (8) with respect to t and x. As a result, we obtain two differential relations

$$u_t = \xi(x)\bar{\omega}(t)p(u), \tag{130}$$

$$u_x = [\bar{\xi}(x)\omega(t) + \bar{\eta}(x)]p(u), \tag{131}$$

where $\bar{\xi}(x) = \xi'_x(x)$, $\bar{\eta}(x) = \eta'_x(x)$, $\bar{\omega}(t) = \omega'_t(t)$, and $p(u) = 1/h(u)$.

(i) Either relation (130) or (131) can be treated as a first-order differential constraint, which can be used to find exact solutions of equation (6) through a compatibility analysis of the overdetermined pair of equations (6) and (130) (or (6) and (131)) with the single unknown u. The differential constraints (130) and (131) are equivalent to relation (8); at the initial stage, all functions included on the right-hand sides of (130) and (131) are considered arbitrary, and the specific form of these functions is determined in the subsequent analysis.

From the first-order differential constraints (130) and (131), one can obtain, by multiplying by suitable functions and differentiating, second-order differential constraints. For example, if we divide both sides of (130) by $\xi(x)p(u)$ and then differentiate by x, we obtain a second-order differential constraint with a mixed derivative. Differential constraints of the second and higher orders can also be used to construct exact solutions to equation (6) (see, for example, [18, 54–56]). In the general case, any PDE (or ODE, in a degenerate case) that depends on the same variables as the original equation can be treated as a differential constraint.

For a description of the method of differential constraints, its relationship with other methods, as well as a number of specific examples of its application, see, for example, [6, 14, 15, 18, 52, 54–59].

The construction of exact solutions by the method of differential constraints is based on a compatibility analysis of PDEs and is carried out in several steps briefly described below.

1. Two PDEs (the original PDE and a differential constraint) are differentiated (sufficiently many times) with respect to x and t, and then the highest-order

derivatives are eliminated from the differential relations obtained and PDEs considered. As a result, one arrives at an equation involving powers of lower-order derivatives, for example, u_x.

2. By equating the coefficients of all degrees of the derivative u_x with zero in this equation, one obtains compatibility conditions connecting the functional coefficients of the PDEs.

3. The compatibility conditions make up a non-linear system of ODEs for determining functional coefficients. In this step, it is necessary to find a solution to this system in a closed form.

4. The obtained coefficients are substituted into the differential constraint, which must then be integrated to find a form (or forms) of the unknown function u (in this step, intermediate solutions are obtained that contain undetermined functions).

5. The final form of the unknown function is determined from the original PDE.

In the last three steps of the method of differential constraints, one has to solve various equations (systems of equations). If no solution can be constructed in at least one of these steps, then one fails to construct an exact solution of the original equation.

Note that for constructing exact solutions, one may use several differential constraints (see, for example, [15, 18, 58]). In the general case, when investigating the compatibility of several PDEs with a single unknown function, one has to use the methods of analysis of overdetermined systems of PDEs based on the Cartan algorithm or on the Janet–Spenser–Kuranishi algorithm (the description of these algorithms and other information on the theory of overdetermined systems of PDEs can be found, for example, in [58, 60, 61]).

(ii) The first-order differential constraints (130) and (131) are special cases of an invariant surface condition [53], which characterizes the nonclassical method of symmetry reduction (in general, an invariant surface condition is a quasilinear first-order PDE of general form). This method, just like the method of different constraints, is also based on the compatibility analysis of two PDEs; specific examples of its use can be found, for example, in [11,15,18,53,62–68]. For first-order differential constraints, the results of applying the method of differential constraint and the nonclassical method of symmetry reduction coincide (provided that the differential constraint coincides with the invariant surface condition [15, 18]).

In practice, for the vast majority of nonlinear PDEs, it is preferable to use the direct method functional separation of variables, since this method require less steps where it is necessary to solve intermediate differential equations (in addition, intermediate equations are simpler) than the method of differential constraints or the nonclassical method of symmetry reduction. In all cases, a complicating factor is the presence of arbitrary functions, if included in the equation in question (it is precisely such equations that are discussed in this article). In addition, the method of differential constraints and the nonclassical method of symmetry reduction are very difficult to use to construct exact solutions of non-linear third- and higher-order equations, as this would lead to very cumbersome calculations and complex equations (often, the original equations look simpler). A good illustration of the complexity of using the nonclassical method of symmetry reduction is the article [69], where the linear heat equation and the Fokker–Planck equation were considered as examples.

The method of differential constraints and the nonclassical method of symmetry reduction are very general. The most important and informal problem arising in the application of these methods is as follows: how to find a suitable differential constraint or invariant surface condition for a given equation. If, for example, a differential constraint (or an invariant surface condition) contains insufficient functional arbitrariness, then no solutions can be obtained; if it is too general, then the compatibility analysis of the nonlinear PDEs under consideration will be so complex that one will not be able to find any exact solutions. The successful solution of the above problem lies beyond the formal description of these methods and, in each specific case, is usually determined by the experience and intuition of the researcher.

Note that the implicit representation for functional separable solutions (8) used in the present paper resulted from the natural generalization of solutions to two fairly simple model problems for non-linear heat and wave equations (see Sections 2.1 and 2.3). The technical use of (8) leads to relatively simple calculations and gives many exact solutions to the non-linear equation (6). Moreover, the currently available preliminary results (which have not yet been fully completed) show that the implicit representation of solution (8) can also be successfully used to construct exact solutions to non-linear convective-diffusion equations, non-linear Klein–Gordon equations with variable coefficients, and some higher-order PDEs.

Examples of the construction of exact functional separable solutions in an explicit form for non-linear equations of the hydrodynamic type of the third and fourth orders can be found in [70–73].

Let us now briefly discuss the direct method by Clarkson and Kruskal [74] (see also [18, 54, 67, 68, 75–77]), which is based on looking for exact solutions in the form $u = U(x, t, w(z))$ with $z = z(x, t)$. The functions $U(x, t, w)$ and $z(x, t)$ should be chosen so as to obtain ultimately a single ordinary differential equation for $w(z)$. The requirement that the function $w(z)$ must satisfy a single ODE greatly limits the capabilities of this method and does not allow it to be effectively used to construct exact solutions of equation (6).

6 Conclusions

The direct method for constructing functional separable solutions to non-linear equations of mathematical physics has been described. The solutions are sought in the form of an implicit relation containing several free functions (these functions are determined in the subsequent analysis). Different classes of non-linear reaction-diffusion equations with variable coefficients, which admit exact solutions, have been considered. Special attention has been paid to non-linear reaction-diffusion equations of general form, which depend on one or several arbitrary functions. Many new generalized traveling-wave solutions and functional separable solutions have been obtained.

Acknowledgments

The work was supported by the Russian State Assignment (State Registration Number AAAA-A17-117021310385-6) and was partially supported by the Russian Foundation for Basic Research (project No. 18-29-10025).

I would like to thank Dr. Alexei Zhurov for his assistance in improving the language of the paper.

References

1. L.V. Ovsiannikov, Group properties of nonlinear heat equations, Doklady Acad. Nauk USSR 125(3) (1959) 492–495 (in Russian).

2. V.A. Dorodnitsyn, On invariant solutions of the equation of non-linear heat conduction with a source, USSR Comput. Math. & Math. Phys. 22(6) (1982) 115–122.

3. V.A. Galaktionov, V.A. Dorodnitsyn, G.G. Yelenin, S.P. Kurdyumov, A.A. Samarskii, A quasilinear equation of heat conduction with a source: peaking, localization, symmetry, exact solutions, asymptotic behavior, structures, J. Soviet Math. 41(5) (1988) 1222–1292.

4. N.A. Kudryashov, On exact solutions of families of Fisher equations. Theor. Math. Phys. 94(2) (1993) 211–218.

5. P.A. Clarkson, E.L. Mansfield, Symmetry reductions and exact solutions of a class of nonlinear heat equations, Phys. D 70 (1994) 250–288.

6. V.A. Galaktionov, Quasilinear heat equations with first-order sign-invariants and new explicit solutions, Nonlinear Anal. Theor. Meth. Appl. 23 (1994) 1595–621.

7. N.H. Ibragimov (ed.), CRC Handbook of Lie Group Analysis of Differential Equations. Symmetries, Exact solutions and Conservation Laws, vol. 1, CRC Press, Boca Raton, 1994.

8. V.F. Zaitsev, A.D. Polyanin, Handbook of Partial Differential Equations: Exact Solutions, International Program of Education, Moscow, 1996 (in Russian).

9. P.W. Doyle, P.J. Vassiliou, Separation of variables for the 1-dimensional non-linear diffusion equation, Int. J. Non-Linear Mech. 33(2) (1998) 315–326.

10. S. Hood, On direct, implicit reductions of a nonlinear diffusion equation with an arbitrary function - generalizations of Clarkson's and Kruskal's method, IMA J. Appl. Math. 64(3) (2000) 223–244.

11. E. Pucci, G. Saccomandi, Evolution equations, invariant surface conditions and functional separation of variables, Physica D 139 (2000) 28–47.

12. P.G. Estevez, C. Qu, S. Zhang, Separation of variables of a generalized porous medium equation with nonlinear source, J. Math. Anal. Appl. 275 (2002) 44–59.

13. A.D. Polyanin, V.F. Zaitsev, Handbook of Nonlinear Equations of Mathematical Physics, Fizmatlit, Moscow, 2002 (in Russian).

14. O.V. Kaptsov, I.V. Verevkin, Differential constraints and exact solutions of nonlinear diffusion equations, J. Phys. A: Math. Gen. 36 (2003) 1401–1414.

15. A.D. Polyanin, V.F. Zaitsev, A.I. Zhurov, Solution methods for nonlinear equations of mathematical physics and mechanics, Fizmatlit, Moscow, 2005 (in Russian).

16. R.M. Cherniha, O. Pliukhin, New conditional symmetries and exact solutions of nonlinear reaction-diffusion-convection equations, J. Physics A: Math. Theor. 40(33) (2007) 10049–10070.

17. V.A. Galaktionov, S.R. Svirshchevskii, Exact solutions and invariant subspaces of nonlinear partial differential equations in mechanics and physics, Chapman & Hall/CRC Press, Boca Raton, 2006.

18. A.D. Polyanin, V.F. Zaitsev, Handbook of Nonlinear Partial Differential Equations, 2nd ed., CRC Press, Boca Raton, 2012 (see also 1st ed., 2004).

19. R.M. Cherniha, O. Pliukhin, New conditional symmetries and exact solutions of reaction-diffusion-convection equations with exponential nonlinearities, J. Math. Anal. Appl. 403 (2013) 23–37.

20. R. Cherniha, M. Serov, O. Pliukhin, Nonlinear Reaction-Diffusion-Convection Equations: Lie and Conditional Symmetry, Exact Solutions and Their Applications, Chapman & Hall/CRC Press, Boca Raton, 2018.

21. O.O. Vaneeva, A.G. Johnpillai, R.O. Popovych, C. Sophocleous, Extended group analysis of variable coefficient reaction-diffusion equations with power nonlinearities, J. Math. Anal. Appl. 330(2) (2007) 1363–1386.

22. O.O. Vaneeva, R.O. Popovych, C. Sophocleous, Enhanced group analysis and exact solutions of variable coefficient semilinear diffusion equations with a power source, Acta Appl. Math. 106(1) (2009) 1–46.

23. O.O. Vaneeva, R.O. Popovych, C. Sophocleous, Extended group analysis of variable coefficient reaction-diffusion equations with exponential nonlinearities, J. Math. Anal. Appl. 396 (2012) 225–242.

24. O. Vaneeva, A. Zhalij, Group classification of variable coefficient quasilinear reaction-diffusion equations, Publ. L'Institute Mathématique (Nouvelle série) 94(108) (2013) 81–90.

25. A.D. Polyanin, Functional separable solutions of nonlinear reaction-diffusion equations with variable coefficients, Appl. Math. Comput. 347 (2019) 282–292.

26. B.H. Bradshaw-Hajek, R.J. Moitsheki, Symmetry solutions for reaction-diffusion equations with spatially dependent diffusivity, Appl. Math. Comput. 254 (2015) 30–38.

27. M.L. Gandarias, J.L. Romero, J.M. Díaz, Nonclassical symmetry reductions of a porous medium equation with convection, J. Phys. A: Math. Gen. 32 (1999) 1461–1473.

28. R.O. Popovych, N.M. Ivanova, New results on group classification of nonlinear diffusion-convection equations, J. Physics A: Math. Gen. 37(30) (2004) 7547–7565.

29. N.M. Ivanova, C. Sophocleous, On the group classification of variable-coefficient nonlinear diffusion-convection equations, J. Comput. Appl. Math. 197(2) (2006) 322–344.

30. N.M. Ivanova, Exact solutions of diffusion-convection equations, Dynamics of PDE 5(2) (2008) 139–171.

31. O.O. Vaneeva, R.O. Popovych, C. Sophocleous, Group analysis of variable coefficient diffusion-convection equations, I. Enhanced group classification, Lobachevskii J. Math. 31(2) (2010) 100–122.

32. P. Basarab-Horwath, V. Lahno, R. Zhdanov, The structure of Lie algebras and the classification problem for partial differential equations, Acta Appl. Math. 69 (2001) 43–94.

33. V.I. Lagno, S.V. Spichak, V.I. Stognii, Symmetry analysis of evolution type equations, Institute of Computer Sciences, Moscow–Izhevsk, 2004 (in Russian).

34. H. Jia, W.X.X. Zhao, Z. Li, Separation of variables and exact solutions to nonlinear diffusion equations with x-dependent convection and absorption, J. Math. Anal. Appl. 339 (2008) 982–995.

35. E. Pereira, E. Suazo, J. Trespalacios, Riccati–Ermakov systems and explicit solutions for variable coefficient reaction-diffusion equations, Appl. Math. Comput. 329 (2018) 278–296.

36. N.A. Kudryashov, R.B. Rybka, A. G. Sboev, Analytical properties of the perturbed FitzHugh-Nagumo model, Appl. Math. Lett. 76 (2018) 142–147.

37. R. Cherniha, V. Davydovych, Nonlinear Reaction-Diffusion Systems: Conditional Symmetry, Exact Solutions and Their Applications in Biology, Springer, 2017.

38. S.V. Meleshko, S. Moyo, On the complete group classification of the reaction-diffusion equation with a delay, J. Math. Anal. Appl. 338 (2008) 448–466.

39. A.D. Polyanin, A.I. Zhurov, Exact solutions of linear and non-linear differential-difference heat and diffusion equations with finite relaxation time, Int. J. Non-Linear Mech. 54 (2013) 115–126.

40. A.D. Polyanin, A.I. Zhurov, Exact separable solutions of delay reaction-diffusion equations and other nonlinear partial functional-differential equations, Commun. Nonlinear Sci. Numer. Simul. 19(3) (2014) 409–416.

41. A.D. Polyanin, A.I. Zhurov, Functional constraints method for constructing exact solutions to delay reaction-diffusion equations and more complex nonlinear equations, Commun. Nonlinear Sci. Numer. Simul., 19(3) (2014) 417–430.

42. A.D. Polyanin, A.I. Zhurov, New generalized and functional separable solutions to non-linear delay reaction-diffusion equations, Int. J. Non-Linear Mech. 59 (2014) 16–22.

43. A.D. Polyanin, A.I. Zhurov, Nonlinear delay reaction-diffusion equations with varying transfer coefficients: Exact methods and new solutions, Appl. Math. Letters 37 (2014) 43–48.

44. A.D. Polyanin, A.I. Zhurov, The functional constraints method: Application to non-linear delay reaction-diffusion equations with varying transfer coefficients, Int. J. Non-Linear Mech. 67 (2014) 267–277.

45. A.D. Polyanin, A.I. Zhurov, Non-linear instability and exact solutions to some delay reaction-diffusion systems, Int. J. Non-Linear Mech. 62 (2014) 33–40.

46. A.D. Polyanin, A.G. Sorokin, Nonlinear delay reaction-diffusion equations: Traveling-wave solutions in elementary functions, Appl. Math. Letters 46 (2015) 38–43.

47. A.D. Polyanin, A.I. Zhurov, The generating equations method: Constructing exact solutions to delay reaction-diffusion systems and other non-linear coupled delay PDEs, Int. J. Non-Linear Mech. 71 (2015) 104–115.

48. A.D. Polyanin, Generalized traveling-wave solutions of nonlinear reaction-diffusion equations with delay and variable coefficients, Appl. Math. Letters 90 (2019) 49–53.

49. G. Birkhoff, Hydrodynamics, Princeton University Press, Princeton, 1960.

50. L.V. Ovsiannikov, Group Analysis of Differential Equations, Academic Press, Boston, 1982.

51. A.D. Polyanin, V.F. Zaitsev, Handbook of Ordinary Differential Equations: Exact Solutions, Methods, and Problems, CRC Press, Boca Raton, 2018.

52. N.N. Yanenko, The compatibility theory and methods of integration of systems of nonlinear partial differential equations, In: Proc. All-Union Math. Congress, Nauka, Leningrad, 2 (1964) 613–621.

53. G.W. Bluman, J.D. Cole, The general similarity solution of the heat equation, J. Math. Mech. 18 (1969) 1025–1042.

54. P.J. Olver, Direct reduction and differential constraints, Proc. Roy. Soc. London, Ser. A 444 (1994) 509–523.

55. O.V. Kaptsov, Determining equations and differential constraints, Nonlinear Math. Phys. 2(3-4) (1995) 283–291.

56. O.V. Kaptsov, Linear determining equations for differential constraints, Sbornik: Mathematics 189(12) (1998) 1839–1854.

57. S.V. Meleshko, Differential constraints and one-parameter Lie–Bäcklund groups, Sov. Math. Dokl., 28 (1983) 37–41.

58. A.F. Sidorov, V.P. Shapeev, N.N. Yanenko, Method of Differential Constraints and its Applications in Gas Dynamics, Nauka, Novosibirsk, 1984 (in Russian).

59. V.K. Andreev, O.V. Kaptsov, V.V. Pukhnachov, A.A. Rodionov, Applications of Group-Theoretical Methods in Hydrodynamics, Kluwer, Dordrecht, 1998.

60. M. Kuranishi, Lectures on Involutive Systems on Partial Differential Equations, Publ. Soc. Math., Sao Paulo, 1967.

61. J.F. Pommaret, Systems in Partial Differential Equations and Lie Pseudogroups, Gordon & Breach Sci. Publ., New York, 1978.

62. D. Levi, P. Winternitz, Nonclassical symmetry reduction: Example of the Boussinesq equation, J. Phys. A 22 (1989) 2915–2924.

63. M.C. Nucci, P.A. Clarkson, The nonclassical method is more general than the direct method for symmetry reductions. An example of the Fitzhugh–Nagumo equation, Phys. Lett. A, 164 (1992) 49–56.

64. E. Pucci, Similarity reductions of partial differential equations. J. Phys. A: Math. General, 25 (1992) 2631–2640.

65. P.A. Clarkson, Nonclassical symmetry reductions for the Boussinesq equation, Chaos, Solitons & Fractals 5 (1995) 2261–2301.

66. P.J. Olver, E.M. Vorob'ev, Nonclassical and conditional symmetries, In: CRC Handbook of Lie Group Analysis of Differential Equations, Vol. 3 (ed. N. H. Ibragimov), CRC Press, Boca Raton, 1996, pp. 291–328.

67. P.A. Clarkson, D.K. Ludlow, T.J. Priestley, The classical, direct and nonclassical methods for symmetry reductions of nonlinear partial differential equations, Methods Appl.Anal. 4(2) (1997) 173–195.

68. G. Saccomandi, A personal overview on the reduction methods for partial differential equations, Note di Matematica 23(2) (2004/2005) 217–248.

69. E. Pucci, G. Saccomandi, On the weak symmetry groups of partial differential equations, J. Math. Anal. Appl. 163 (1992) 588–598.

70. G.I. Burde, The construction of special explicit solutions of the boundary-layer equations. Steady flows, Quart. J. Mech. Appl. Math., 47(2) (1994) 247–260.

71. A.D. Polyanin, A.I. Zhurov, Unsteady axisymmetric boundary-layer equations: Transformations, properties, exact solutions, order reduction and solution method, Int. J. Non-Linear Mech. 74 (2015) 40–50.

72. A.D. Polyanin, A.I. Zhurov, Direct functional separation of variables and new exact solutions to axisymmetric unsteady boundary-layer equations, Commun. Nonlinear Sci. Numer. Simulat., 31 (2016) 11–20.

73. A.D. Polyanin, A.I. Zhurov, Functional and generalized separable solutions to unsteady Navier–Stokes equations, Int. J. Non-Linear Mech. 79 (2016) 88–98.

74. P.A. Clarkson, M.D. Kruskal, New similarity reductions of the Boussinesq equation, J. Math. Phys. 30 (1989) 2201–2213.

75. A.V. Aksenov, A.A. Kozyrev, Reductions of the stationary boundary layer equation with a pressure gradient, Doklady Mathematics, 87(2) (2013) 236–239.

76. D.K. Ludlow, P.A. Clarkson, A.P. Bassom, Similarity reductions and exact solutions for the two-dimensional incompressible Navier–Stokes equations, Studies Appl. Math. 103 (1999) 183–240.

77. D.K. Ludlow, P.A. Clarkson, A.P. Bassom, New similarity solutions of the unsteady incompressible boundary-layer equations, Quart. J. Mech. and Appl. Math. 53 (2000) 175–206.