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Unsteady Magnetohydrodynamics PDE of Monge–Ampère Type: Symmetries, Closed-Form Solutions, and Reductions

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Abstract: The paper studies an unsteady equation with quadratic nonlinearity in second derivatives, that occurs in electron magnetohydrodynamics. In mathematics, such PDEs are referred to as parabolic Monge–Ampère equations. An overview of the Monge–Ampère type equations is given, in which their unusual qualitative features are noted. For the first time, the Lie group analysis of the considered highly nonlinear PDE with three independent variables is carried out. An eleven-parameter transformation is found that preserves the form of the equation. Some one-dimensional reductions allowing to obtain self-similar and other invariant solutions that satisfy ordinary differential equations are described. A large number of new additive, multiplicative, generalized, and functional separable solutions are obtained. Special attention is paid to the construction of exact closed-form solutions, including solutions in elementary functions (in total, more than 30 solutions in elementary functions were obtained). Two-dimensional symmetry and non-symmetry reductions leading to simpler partial differential equations with two independent variables are considered (including stationary Monge–Ampère type equations, linear and nonlinear heat type equations, and nonlinear filtration equations). The obtained results and exact solutions can be used to evaluate the accuracy and analyze the adequacy of numerical methods for solving initial boundary value problems described by highly nonlinear partial differential equations.

Keywords: magnetohydrodynamics equations; parabolic Monge–Ampère equations; highly nonlinear PDEs; symmetries of PDEs; exact solutions; solutions in elementary functions; closed-form solutions; invariant solutions; generalized and functional separable solutions; one- and two-dimensional reductions

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1. Introduction

1°. In plasma electron magnetohydrodynamics, an approach has been developed in which the electron component of the plasma is represented as a set of random or regularly distributed point electron vortices. Local unsteady motions in the corresponding two-dimensional vortex system are described by the nonlinear equation [1–3]:

$$u_t = u_{xx}u_{yy} - u_{xy}^2, \tag{1}$$

in the right side of which an unessential constant factor is omitted.

In the special case, for $u_t = 0$, Equation (1) reduces to the stationary homogeneous Monge–Ampère equation

$$u_{xx}u_{yy} - u_{xy}^2 = 0. \tag{2}$$

First integrals, exact solutions, symmetries, and invariant transformations of this equation and its multidimensional generalization can be found, for example, in [4–10]. It is important to note that the general solution of Equation (2) can be presented in parametric form [4,10].

The quadratic combination of second derivatives

$$u_{xx}u_{yy} - u_{xy}^2 \equiv \det\begin{bmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{bmatrix},$$

which is included in Equations (1) and (2) and defined as the determinant of the matrix of second derivatives, will be called nonlinearity of the Monge–Ampère type.

For the first time, nonstationary equations with nonlinearity of the Monge–Ampère type for spatial variables appeared in [11], in which, in particular, the equation was given

$$u_t = \det[u_{x_ix_j}] - f(\mathbf{x}, t), \tag{3}$$

where $\mathbf{x} = (x_1, \dots, x_n)$ and $u_{x_ix_j} = \frac{\partial^2 u}{\partial x_i \partial x_j}$. The matrix of second derivatives $[u_{x_ix_j}]$ included in Equation (3) describes the local curvature of a function of many variables and is called the Hessian matrix.

In the case of two spatial variables $x_1 = x$ and $x_2 = y$ in Equation (3), we should put $\mathbf{x} = (x, y)$ and $\det[u_{x_ix_j}] = u_{xx}u_{yy} - u_{xy}^2$. For $n = 2$ and $f \equiv 0$, Equation (3) becomes Equation (1).

Equations (1) and (3) and more complex related nonlinear PDEs containing the first time derivative u_t and a combination of the second derivatives in spatial coordinates of the form $\det[u_{x_ix_j}]$ are called *parabolic Monge–Ampère equations*. Such PDEs were considered in many works (see, for example, [11–28]), in which, mainly, questions of the existence and uniqueness of some classes of solutions for various internal and external initial-boundary value problems were studied, and geometric applications were also discussed.

It should be noted that in [11,12,14,16,18,22,23,26] the parabolic Monge–Ampère equation

$$u_t \det[u_{x_ix_j}] = -f(\mathbf{x}, t),$$

which has a more complex nonlinearity than PDE (3), was studied.

In the case of two spatial variables, the parabolic Monge–Ampère equations have the form

$$u_t = F(x, y, t, u, u_x, u_y, u_{xx}u_{yy} - u_{xy}^2). \tag{4}$$

2°. At $u_t = F_t = 0$, the parabolic equations (4) are reduced to the stationary Monge–Ampère equations, which are usually written in the form resolved with respect to the combination of the highest derivatives

$$u_{xx}u_{yy} - u_{xy}^2 = \Phi(x, y, u, u_x, u_y). \tag{5}$$

Very complete information about stationary Monge–Ampère equations, which are often encountered in differential geometry, can be found in the book [29]. In the special case, for $\Phi = a = \text{const} < 0$, the general solution of Equation (5) can be represented in parametric form [4,10]. Various transformations, exact solutions, and linearizable equations of the form (5) are given in [7,9,10,30–34].

Equation (5) is highly nonlinear (quadratic with respect to the higher derivatives) and has properties unusual for quasi-linear equations of mathematical physics that are linear with respect to the higher derivatives. In particular, even for the simplest right-hand side $\Phi = a = \text{const}$, the qualitative features of this equation depend on the sign of the constant a, since for $a > 0$, Equation (5) is an equation of elliptic type, and for $a < 0$ it is an equation of hyperbolic type [10,29,35].

The above-mentioned qualitative features of the stationary Monge–Ampère equations significantly complicate the procedure of their numerical integration. Therefore, sometimes for the numerical solution of the stationary Monge–Ampère equation (5), various modifications of the time-relaxation method are used using the parabolic Monge–Ampère equation, which is obtained by adding the nonstationary first derivative u_t to the right side of (5). The indicated approach involving the parabolic Monge–Ampère equations (1) and (3) was used in [36–39] for the numerical integration of boundary value problems described by the corresponding stationary Monge–Ampère equations.

3°. The symmetries of the parabolic Monge–Ampère equation (1) have not been studied previously, and reductions and exact solutions of this highly nonlinear PDE have been little considered. In the works [40,41] by the classical method of separation of variables, exact solutions of Equation (1) were obtained in the form of a product and a sum of functions of different arguments, by analogy with how this is performed for linear second-order PDEs. A generalized separable solution of polynomial form in the spatial variables x and y with coefficients depending on time t is described in [42] (for more details on this solution, see Item 9° in Section 8).

Reductions of the equation under consideration are equations with fewer independent variables or equations of lower order, all solutions of which are solutions of the given equation. Reductions play a key role in constructing exact solutions to differential equations and usually lead to lower-order or lower-dimensional equations. The most important for nonlinear partial differential equations are one-dimensional reductions, using which it is possible to present their solutions in terms of solutions of much simpler ordinary differential equations.

In this paper, exact closed-form solutions of nonlinear partial differential equations are understood as solutions that are expressed in terms of:

(i) elementary functions,
(ii) elementary functions and indefinite integrals (solutions in quadrature),
(iii) solutions of ODE or ODE systems.

In addition to exact closed-form solutions, we will also consider very meaningful solutions that can be expressed in terms of solutions of linear PDEs.

It is important to note that exact solutions of nonlinear partial differential equations of mathematical physics play an important role in mathematical standards, which are widely used to evaluate accuracy, verify and develop various numerical, asymptotic, and approximate analytical methods.

Symmetries, reductions, and exact solutions of nonlinear partial differential equations of mathematical physics are most often constructed using the classical method of symmetry reductions [5,9,43–48], the direct method of symmetry reductions [10,49–55], the nonclassical symmetries methods [50,52,56–65], methods of generalized separation of variables [10,42,55,66–73], methods of functional separation of variables [10,55,60,70,74–83], the method of differential constraints [10,51,55,84–87], and some other exact analytical methods [10,55,88–95]. On methods for constructing exact solutions of nonlinear delay PDEs and functional PDEs, see, for example, [96–104].

Let us note a couple of characteristic qualitative features of Equation (1) that distinguish it from the vast majority of other equations of mathematical physics that do not explicitly depend on independent variables. This equation has neither simple solutions of the traveling wave type $u = U(ax + by + ct)$, nor more complex solutions of the form $u = V(ax + by, t)$, where a, b, c are any constants, and $U(z)$ and $V(\xi, t)$ are some functions other than constants. This circumstance complicates to some extent the search for exact solutions of the PDE under consideration.

4°. In this paper, to find exact solutions to the nonlinear magnetohydrodynamics equation (1), we used the classical method of symmetry reductions [5,9,43,44], methods of generalized and functional separation of variables [10,42,55], as well as given in [10,34], exact solutions of simpler than the original PDE, intermediate reduced equations with fewer independent variables. In some cases, exact solutions were obtained by combining the classical method and methods of generalized and/or functional separation of variables (the order in which these methods are applied may vary, and the solutions found in this way will be non-invariant). Special attention is paid to the construction of simple exact solutions that are expressed in terms of elementary functions or in closed form (in quadratures). It is most convenient to use such solutions as test problems to evaluate the accuracy and verify the adequacy of numerical methods for solving highly nonlinear partial differential equations.

2. Symmetries of the Magnetohydrodynamics Equation, Admissible Operators, and Invariant Transformations

We are looking for the symmetry operators of Equation (1) in the form

$$X = \xi^1(x, y, t, u)\frac{\partial}{\partial x} + \xi^2(x, y, t, u)\frac{\partial}{\partial y} + \xi^3(x, y, t, u)\frac{\partial}{\partial t} + \eta(x, y, t, u)\frac{\partial}{\partial u}\,.$$

Applying the invariance criterion [5], we obtain the following overdetermined linear homogeneous system of determining equations

$$\begin{aligned}
&\xi^1_t = 0\,, \quad \xi^1_u = 0\,, \quad \xi^2_t = 0\,, \quad \xi^2_u = 0\,,\\
&\xi^3_x = 0\,, \quad \xi^3_y = 0\,, \quad \xi^3_u = 0\,,\\
&\eta_t = 0\,, \quad \eta_u - 2\xi^1_x - 2\xi^2_y + \xi^3_t = 0\,,\\
&\xi^1_{xx} = 0\,, \quad \xi^1_{xy} = 0\,, \quad \xi^1_{yy} = 0\,,\\
&\xi^2_{xx} = 0\,, \quad \xi^2_{xy} = 0\,, \quad \xi^2_{yy} = 0\,,\\
&\xi^3_{tt} = 0\,, \quad \eta_{xx} = 0\,, \quad \eta_{xy} = 0\,, \quad \eta_{yy} = 0\,.
\end{aligned} \tag{6}$$

It is not difficult to show that the joint solution of the overdetermined system of Equation (6) is given by the formulas

$$\begin{aligned}
\xi^1 &= c_1 x + c_2 y + c_3\,,\\
\xi^2 &= c_4 x + c_5 y + c_6\,,\\
\xi^3 &= c_7 t + c_8\,,
\end{aligned}$$

$$\eta = (2c_1 + 2c_5 - c_7)u + c_9 x + c_{10} y + c_{11},$$

where c_j $(j = 1, \dots, 11)$ are arbitrary constants. From which two propositions follow.

Proposition 1. *The basis of the Lie algebra of symmetry operators of Equation* (1) *has the form*

$$\begin{gathered}
X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = \frac{\partial}{\partial t}, \qquad X_4 = \frac{\partial}{\partial u}, \\
X_5 = y\frac{\partial}{\partial x}, \qquad X_6 = x\frac{\partial}{\partial y}, \qquad X_7 = x\frac{\partial}{\partial u}, \qquad X_8 = y\frac{\partial}{\partial u}, \\
X_9 = x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}, \qquad X_{10} = y\frac{\partial}{\partial y} + 2u\frac{\partial}{\partial u}, \qquad X_{11} = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}.
\end{gathered}$$

Proposition 2. *The transformation*

$$\begin{gathered}
\bar{x} = a_1 x + b_1 y + c_1, \qquad \bar{y} = a_2 x + b_2 y + c_2, \qquad \bar{t} = d_1 t + d_2, \\
\bar{u} = \frac{(a_1 b_2 - a_2 b_1)^2}{d_1} u + a_3 x + b_3 y + c_3 \qquad (a_1 b_2 - a_2 b_1 \neq 0, \;\; d_1 \neq 0)
\end{gathered} \tag{7}$$

transforms any solution $u = \varphi(x, y, t)$ of Equation (1) *into an eleven-parameter family of solutions*

$$u = \frac{d_1}{(a_1 b_2 - a_2 b_1)^2} \Big[\varphi(a_1 x + b_1 y + c_1, a_2 x + b_2 y + c_2, d_1 t + d_2) - a_3 x - b_3 y - c_3 \Big], \tag{8}$$

where a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2 are arbitrary constants.

It is important to note that in the Formula (8) the free parameters can be complex numbers if the solution obtained in this way is valid (for details, see [94]). Section 4 provides a specific example of using this approach.

Remark 1. *The transformation* (7) *leaves the form of Equation* (1) *invariant. The Formula* (8) *makes it possible to obtain multi-parameter families of solutions from simpler solutions.*

3. Two-Dimensional Similarity Reductions

The regular procedure for constructing two-dimensional similarity reductions of partial differential equations is described in [5,44]. In our work, we will confine ourselves to the most informative examples of the construction of two-dimensional reductions of the parabolic Monge–Ampère equation (1) based on the symmetries described above.

1°. Passing in (1) to variables of the traveling wave type

$$u = U(\xi, \eta), \quad \xi = x - a_1 t, \quad \eta = y - a_2 t, \tag{9}$$

where a_1 and a_2 are arbitrary constants, we arrive at a two-dimensional Monge–Ampère type equation with constant coefficients

$$U_{\xi\xi} U_{\eta\eta} - U_{\xi\eta}^2 + a_1 U_\xi + a_2 U_\eta = 0. \tag{10}$$

The solution of the form (9) is invariant with respect to the one-parameter group of transformations given by the symmetry operator

$$Y = a_1 X_1 + a_2 X_2 + X_3 = a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + \frac{\partial}{\partial t}.$$

Equation (10) admits exact solutions, quadratic in any independent variable, of the form

$$U_1 = f_1(\xi)\eta^2 + g_1(\xi)\eta + h_1(\xi),\\ U_2 = f_2(\eta)\xi^2 + g_2(\eta)\xi + h_2(\eta), \tag{11}$$

where the functions f_i, g_i, h_i ($i = 1, 2$) are described by the corresponding one-dimensional systems of three ODEs, which are omitted here.

Remark 2. *The consistent use of Formulas (9) and (11) leads to exact non-invariant solutions obtained by combining the classical method and the method of generalized separation of variables.*

Remark 3. *Generalized separable solutions that are quadratic in one or more variables, similar to (11), are often used to construct exact solutions to reaction-diffusion equations and some other nonlinear PDEs (see, for example, [10,42,55,71]).*

2°. Passing in (1) to variables of self-similar type

$$u = t^{-2\alpha-2\beta-1}U(\xi,\eta), \quad \xi = xt^\alpha, \quad \eta = yt^\beta, \tag{12}$$

where α and β are arbitrary constants, we obtain a two-dimensional equation of the Monge–Ampère type with variable coefficients for the lower derivatives

$$U_{\xi\xi}U_{\eta\eta} - U_{\xi\eta}^2 - \alpha\xi U_\xi - \beta\eta U_\eta + (2\alpha + 2\beta + 1)U = 0\,. \tag{13}$$

The solution of the form (12) is invariant with respect to the one-parameter group of transformations given by the symmetry operator

$$Y = \alpha X_9 + \beta X_{10} - X_{11} = \alpha x\frac{\partial}{\partial x} + \beta y\frac{\partial}{\partial y} - t\frac{\partial}{\partial t} + (2\alpha + 2\beta + 1)u\frac{\partial}{\partial u}\,.$$

Equation (13) admits exact non-invariant solutions, quadratic with respect to any independent variable, of the form (11). The values of $\alpha = \beta = 0$ in (12) correspond to a multiplicative separable solution.

3°. Passing in (1) to variables of limit self-similar type

$$u = e^{-2(\alpha+\beta)t}U(\xi,\eta), \quad \xi = xe^{\alpha t}, \quad \eta = ye^{\beta t}, \tag{14}$$

where α and β are arbitrary constants, we obtain another two-dimensional equation of the Monge–Ampère type with variable coefficients for the lower derivatives

$$U_{\xi\xi}U_{\eta\eta} - U_{\xi\eta}^2 - \alpha\xi U_\xi - \beta\eta U_\eta + 2(\alpha + \beta)U = 0\,. \tag{15}$$

The solution of the form (14) is invariant with respect to the one-parameter group of transformations given by the symmetry operator

$$Y = \alpha X_9 + \beta X_{10} - X_3 = \alpha x\frac{\partial}{\partial x} + \beta y\frac{\partial}{\partial y} - \frac{\partial}{\partial t} + 2(\alpha + \beta)u\frac{\partial}{\partial u}\,.$$

Equation (15) admits exact non-invariant solutions, quadratic with respect to any independent variable, of the form (11).

4°. Passing in (1) to invariant variables

$$u = \frac{1}{t}U(\xi,\eta), \quad \xi = x + \lambda_1 \ln t, \quad \eta = y + \lambda_2 \ln t, \tag{16}$$

where λ_1 and λ_2 are arbitrary constants, we obtain a two-dimensional equation of the Monge–Ampère type with constant coefficients for the lower derivatives

$$U_{\xi\xi}U_{\eta\eta} - U_{\xi\eta}^2 - \lambda_1 U_\xi - \lambda_2 U_\eta + U = 0\,.$$

The solution of the form (16) is invariant with respect to the one-parameter group of transformations given by the symmetry operator

$$Y = \lambda_1 X_1 + \lambda_2 X_2 - X_{11} = \lambda_1 \frac{\partial}{\partial x} + \lambda_2 \frac{\partial}{\partial y} - t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}.$$

Equation (16) admits traveling wave solutions, as well as non-invariant solutions quadratic in any independent variable of the form (11).

The values of $\lambda_1 = \lambda_2 = 0$ in (16) correspond to a multiplicative separable solution.

5°. Passing in (1) to other invariant variables

$$u = x^2 U(\xi, \eta), \quad \xi = t + \alpha \ln x, \quad \eta = y + \beta \ln x, \tag{17}$$

where λ_1 and λ_2 are arbitrary constants, we obtain a two-dimensional equation of the Monge–Ampère type, which is omitted here. The values of $\alpha = \beta = 0$ in (17) correspond to a multiplicative separable solution.

The solution of the form (17) is invariant with respect to the one-parameter group of transformations given by the symmetry operator

$$Y = -\beta X_2 - \alpha X_3 + X_9 = x\frac{\partial}{\partial x} - \beta\frac{\partial}{\partial y} - \alpha\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u}.$$

6°. Equation (1) using invariant variables

$$u = e^{(\alpha - 2\beta)x} U(\xi, \eta), \quad \xi = te^{\alpha x}, \quad \eta = ye^{\beta x}, \tag{18}$$

is reduced to a two-dimensional Monge–Ampère type equation, which is omitted here.

The solution of the form (18) is invariant with respect to the one-parameter group of transformations given by the symmetry operator

$$Y = X_1 - \beta X_{10} - \alpha X_{11} = \frac{\partial}{\partial x} - \beta y\frac{\partial}{\partial y} - \alpha t\frac{\partial}{\partial t} + (\alpha - 2\beta)u\frac{\partial}{\partial u}.$$

7°. Equation (1) using invariant variables

$$u = e^{-2\alpha x} U(\xi, \eta), \quad \xi = x + \beta t, \quad \eta = ye^{\alpha x}, \tag{19}$$

is reduced to a two-dimensional Monge–Ampère type equation, which is omitted here.

The solution of the form (19) is invariant with respect to the one-parameter group of transformations given by the symmetry operator

$$Y = \beta X_1 - X_3 - \alpha\beta X_{10} + X_9 = \beta\frac{\partial}{\partial x} - \alpha\beta y\frac{\partial}{\partial y} - \frac{\partial}{\partial t} - 2\alpha\beta u\frac{\partial}{\partial u}.$$

Remark 4. *More complex two-dimensional reductions of Equation* (1) *can be obtained by applying to the solutions* (9), (12), (14), (16), (17)–(19) *the reproduction Formula* (8).

4. One-Dimensional Similarity Reductions and Exact Solutions

The standard procedure for obtaining one-dimensional similarity reductions of PDEs is outlined in [5]. In our work, we will give only a few illustrative examples of constructing invariant solutions (in terms of ODEs) of the parabolic Monge–Ampère equation (1) based on the symmetries described above. We will also present several exact solutions of this PDE in elementary functions.

1°. The simplest invariant solution of Equation (1), which allows a scaling transformation, is a solution in the form of a product of the corresponding degrees of independent variables

$$u = \frac{x^2 y^2}{12t}. \tag{20}$$

Below, we consider several more complex invariant solutions that can be obtained from the solution (20) using simple methods described in [93,94].

The solution (20) is a special case of a wider family of invariant solutions of the form

$$u = \frac{x^2}{t} f(z), \quad z = y + \beta \ln t, \tag{21}$$

where β is an arbitrary constant, and the function $f = f(z)$ is described by the ODE

$$2ff''_{zz} - 4(f'_z)^2 - \beta f'_z + f = 0.$$

The solution (20) is a special case of another broader family of invariant solutions of the form

$$u = \frac{x^2}{t} g(\xi), \quad \xi = y + \lambda \ln x, \tag{22}$$

where λ is an arbitrary constant, and the function $f = f(z)$ satisfies the ODE

$$\lambda g'_\xi g''_{\xi\xi} - 2gg''_{\xi\xi} + 4(g'_\xi)^2 = g.$$

The solution (20) is also a special case of another broader family of invariant solutions of the form

$$u = x^2 y^2 \varphi(\eta), \quad \eta = t + \gamma \ln y, \tag{23}$$

where γ is an arbitrary constant, and the function $\varphi = \varphi(\eta)$ is described by the ODE

$$2\gamma^2 \varphi\varphi''_{\eta\eta} - 4\gamma^2(\varphi'_\eta)^2 - 10\gamma\varphi\varphi'_\eta - \varphi'_\eta - 12\varphi^2 = 0.$$

In solutions (21)–(23), the spatial variables x and y can be swapped or use the Formula (8).

Let us apply the Formula (8) with $a_1 = a_2 = b_2 = d_1 = 1$, $b_1 = -1$, $c_1 = c_2 = d_2 = a_3 = b_3 = c_3 = 0$ to solution (20). As a result, we obtain a solution of a more complex form,

$$u = \frac{(x^2 - y^2)^2}{48t}. \tag{24}$$

Following [93,94], we show how another exact solution can be obtained from solution (24) by using a complex parameter. Equation (1) does not change if we make the transformation $\bar{u} = -u$, $\bar{x} = ix$, where $i^2 = -1$ (this is equivalent to choosing the complex parameter $a_1 = i$ in Formula (8)). Having made the same transformation in (24), we arrive at the following simple solution with axial symmetry:

$$u = -\frac{(x^2 + y^2)^2}{48t}. \tag{25}$$

2°. Equation (1) using invariant variables

$$u = x^{2-2\beta} t^{-2\alpha-1} V(\zeta), \quad \zeta = x^\beta t^\alpha y \tag{26}$$

is reduced to a second-order ODE

$$\big[\beta(\beta+1)\zeta V'_\zeta - 2(\beta-1)(2\beta-1)V\big] V''_{\zeta\zeta} + (\beta-2)^2(V'_\zeta)^2 + \alpha\zeta V'_\zeta - (2\alpha+1)V = 0\,.$$

The solution of the form (26) is an invariant solution for the two-parameter transformation group given by symmetry operators

$$Y_1 = \alpha X_9 - \beta X_{11} = \alpha x \frac{\partial}{\partial x} - \beta t \frac{\partial}{\partial t} + (2\alpha + \beta) u \frac{\partial}{\partial u}\,,$$
$$Y_2 = \alpha X_{10} - X_{11} = \alpha y \frac{\partial}{\partial y} - t \frac{\partial}{\partial t} + (2\alpha + 1) u \frac{\partial}{\partial u}\,.$$

3°. Equation (1) using invariant variables

$$u = e^{-2\alpha t} x^{2-2\beta} V(\zeta), \quad \zeta = e^{\alpha t} x^{\beta} y \tag{27}$$

is reduced to a second-order ODE

$$\big[\beta(\beta+1)\zeta V'_\zeta - 2(\beta-1)(2\beta-1)V\big] V''_{\zeta\zeta} + (\beta-2)^2 (V'_\zeta)^2 + \alpha\zeta V'_\zeta - 2\alpha V = 0\,.$$

The solution of the form (27) is an invariant solution for the two-parameter transformation group given by symmetry operators

$$Y_1 = -X_3 + \alpha X_{10} = \alpha y \frac{\partial}{\partial y} - \frac{\partial}{\partial t} + 2\alpha u \frac{\partial}{\partial u}\,,$$
$$Y_2 = X_9 - \beta X_{10} = x \frac{\partial}{\partial x} - \beta y \frac{\partial}{\partial y} - 2(\beta - 1) u \frac{\partial}{\partial u}\,.$$

4°. Equation (1) using invariant variables

$$u = t^{-2\alpha-1} e^{-2\beta x} V(\zeta), \quad \zeta = t^{\alpha} e^{\beta x} y \tag{28}$$

is reduced to a second-order ODE

$$\beta^2 (\zeta V'_\zeta - 4V) V''_{\zeta\zeta} + \beta^2 (V'_\zeta)^2 + \alpha\zeta V'_\zeta - (2\alpha+1)V = 0\,.$$

The solution of the form (28) is an invariant solution for the two-parameter transformation group given by symmetry operators

$$Y_1 = X_1 - \beta X_{10} = \frac{\partial}{\partial x} - \beta y \frac{\partial}{\partial y} - 2\beta u \frac{\partial}{\partial u}\,,$$
$$Y_2 = \alpha X_{10} - X_{11} = \alpha y \frac{\partial}{\partial y} - t \frac{\partial}{\partial t} + (2\alpha + 1) u \frac{\partial}{\partial u}\,.$$

5°. Equation (1) using invariant variables

$$u = e^{-2\alpha t - 2\beta x} V(\zeta), \quad \zeta = e^{\alpha t + \beta x} y \tag{29}$$

is reduced to a second-order ODE

$$\beta^2 (\zeta V'_\zeta - 4V) V''_{\zeta\zeta} + \beta^2 (V'_\zeta)^2 + \alpha\zeta V'_\zeta - 2\alpha V = 0\,.$$

The solution of the form (29) is an invariant solution for the two-parameter transformation group given by symmetry operators

$$Y_1 = X_1 - \beta X_{10} = \frac{\partial}{\partial x} - \beta y \frac{\partial}{\partial y} - 2\beta u \frac{\partial}{\partial u}\,,$$
$$Y_2 = X_3 - \alpha X_{10} = \frac{\partial}{\partial t} - \alpha y \frac{\partial}{\partial y} - 2\alpha u \frac{\partial}{\partial u}\,.$$

Remark 5. *More complex two-dimensional reductions of Equation* (1) *can be obtained by applying to the solutions* (22), (23), (26)–(29) *the reproduction Formula* (8)

5. Exact Solutions with Multiplicative Separation of Variables

1°. Equation (1) has the multiplicative separable solution [40],

$$u = t^{-1}U(x, y),$$

which is a special case of solution (12) for $\alpha = \beta = 0$. Here, the function $U = U(x, y)$ is described by the stationary Monge–Ampère equation

$$U_{xx}U_{yy} - U_{xy}^2 + U = 0. \tag{30}$$

If $U = F(x, y)$ is a solution of Equation (30), then the function

$$U = \frac{1}{(a_1b_2 - a_2b_1)^2}F(a_1x + b_1y + c_1, a_2x + b_2y + c_2),$$

where a_j, b_j, c_j ($j = 1$, 2) are arbitrary constants, is also a solution of this equation.

Invariant solutions of the reduced Equation (30) are described below in Items 2° and 3°.

2°. Equation (30), in turn, admits the multiplicative separable solution

$$U = x^2\varphi(y), \tag{31}$$

where the function $\varphi = \varphi(y)$ satisfies the autonomous ODE

$$2\varphi\varphi''_{yy} - 4(\varphi'_y)^2 = -\varphi. \tag{32}$$

The substitution $Z(\varphi) = (\varphi'_y)^2$ reduces this equation to the first-order linear ODE

$$\varphi Z'_\varphi - 4Z + \varphi = 0,$$

the general solution of which has the form

$$Z = C_1\varphi^4 + \tfrac{1}{3}\varphi, \tag{33}$$

where C_1 is an arbitrary constant. Replacing Z in (33) with $(\varphi'_y)^2$, we obtain an ODE with separable variables, the general solution of which can be represented in implicit form.

The ODE (32) admits a particular solution $\varphi = \frac{1}{12}(y + C_2)^2$, which corresponds to the value $C_1 = 0$ in (33).

3°. Equation (30) has an invariant solution of the form

$$U = x^2\varphi(\xi), \quad \xi = y + \gamma \ln x,$$

where γ is an arbitrary constant, and the function $\varphi = \varphi(\xi)$ satisfies the autonomous ODE

$$\gamma\varphi'_\xi\varphi''_{\xi\xi} - 2\varphi\varphi''_{\xi\xi} + 4(\varphi'_\xi)^2 - \varphi = 0.$$

4°. Equation (30) has a generalized separable solution that is quadratic with respect to any independent variable, e.g.,

$$U = x^2\varphi(y) + x\psi(y) + \chi(y), \tag{34}$$

where the functions $\varphi = \varphi(y)$, $\psi = \psi(y)$, and $\chi = \chi(y)$ are described by the ODE system

$$\begin{aligned} &2\varphi\varphi''_{yy} - 4(\varphi'_y)^2 = -\varphi, \\ &2\varphi\psi''_{yy} - 4\varphi'_y\psi'_y = -\psi, \\ &2\varphi\chi''_{yy} - (\psi'_y)^2 = -\chi. \end{aligned} \tag{35}$$

The non-invariant solution (34) is a generalization of solution (31). It can be seen that the first equation of system (35) coincides with the nonlinear ODE (32), and the second and third equations are linear with respect to the sought functions.

The following statement is true.

Let $\varphi = \varphi(y)$ be a solution to the first equation (35). Then, the corresponding general solution to the second equation is given by the formula

$$\psi = C_1\varphi + C_2 y\varphi, \tag{36}$$

where C_1 and C_2 are arbitrary constants.

The proof of this statement is carried out by directly substituting expression (36) into the second equation of system (35) taking into account the fact that the function φ satisfies the first equation (35).

Note that the system (35) admits a particular solution,

$$\begin{aligned} &\varphi = \tfrac{1}{12}(y + A_1)^2, \quad \psi = 0,\\ &\chi = \sqrt{|y|}\big[A_2 \cos\big(\tfrac{\sqrt{23}}{2}\ln|y + A_1|\big) + A_3 \sin\big(\tfrac{\sqrt{23}}{2}\ln|y + A_1|\big)\big], \end{aligned} \tag{37}$$

where A_1, A_2, and A_2 are arbitrary constants.

Remark 6. *Using the Formula (36), it is possible to obtain a more complex solution of the system (34) than (37) with the same function $\varphi = \frac{1}{12}(y + A_1)^2$. In this case, the linear inhomogeneous third ODE of the system (34) is the Euler equation, and its particular solution is sought by the method of indefinite coefficients in the form of a sum of three terms of the power form $B_j y^{k_j}$, where B_j and k_j are the sought ones coefficients.*

5°. More complex solutions to Equation (30) are discussed further in Sections 11 and 12.

6. Reductions with Additive and Generalized Separation of Variables Leading to Stationary Monge–Ampère Equations, Exact Solutions

1°. Equation (1) has additive separable solutions of the form [32]:

$$u = -At + w(x, y), \tag{38}$$

where A is an arbitrary constant, and the function w is described by the stationary nonhomogeneous Monge–Ampère equation with a constant right-hand side,

$$w_{xx}w_{yy} - w_{xy}^2 = -A. \tag{39}$$

Remark 7. *Solution (38) can be obtained using a linear combination of the Lie symmetries X_3 and X_4 (see Section 2).*

2°. It is easy to verify that Equation (1) admits an additive separable solution of the form (38), which is expressed in elementary functions

$$u = C_1x^2 + C_2xy + C_3y^2 + C_4x + C_5y + (4C_1C_3 - C_2^2)t + C_6,$$

where $C_1, \ldots, C_6$ are arbitrary constants.

3°. Using the results of [10] one can obtain, for example, the following exact non-invariant solutions of the form (38) of Equation (1):

$$u = -At \pm \frac{\sqrt{A}}{C_2}x(C_1x + C_2y) + \varphi(C_1x + C_2y) + C_3x + C_4y + C_5,$$

$$u = -At + \frac{1}{x + C_1}\left(C_2y^2 + C_3y + \frac{C_3^2}{4C_2}\right) - \frac{A}{12C_2}(x^3 + 3C_1x^2) + C_4y + C_5x + C_6,$$

$$u = -At \pm \frac{2\sqrt{A}}{3C_1C_2}(C_1x - C_2^2y^2 + C_3)^{3/2} + C_4x + C_5y + C_6,$$

where $C_1, \ldots, C_6$ are arbitrary constants, and $\varphi = \varphi(z)$ is an arbitrary function.

Remark 8. *For $A > 0$, the general solution of the nonhomogeneous Monge–Ampère equation (39) can be represented in parametric form [4,10].*

4°. Equation (1) admits more complex solutions, than (38), with a generalized separation of variables of the form

$$u = -(ax + by + c)t + w(x, y),$$

where a, b, and c are arbitrary constants, and the function w is described by the stationary nonhomogeneous Monge–Ampère equation with a variable right-hand side,

$$w_{xx}w_{yy} - w_{xy}^2 = -ax - by - c. \tag{40}$$

For $b = c = 0$, Equation (40) has, for example, the following generalized separable solutions:

$$\begin{aligned}
w &= \pm\frac{2}{3a}y(ax)^{3/2} + C_1y + \varphi(x),\\
w &= C_1y^2 + C_2xy + C_3y - \frac{a}{12C_1}x^3 + \frac{C_2^2}{4C_1}x^2 + C_4x + C_5,\\
w &= C_1\frac{y^2}{x} + C_2y - \frac{a}{24C_1}x^4 + C_3x + C_4,
\end{aligned}$$

where $\varphi(x)$ is an arbitrary function, and $C_1, \ldots, C_4$ are arbitrary constants.

7. Reduction with Generalized Separation of Variables Leading to the Linear Heat Equation

1°. Equation (1) admits generalized separable solutions of the form

$$u = \tfrac{1}{2}ay^2 + bxy + \tfrac{1}{2}cx^2 + dy + (ac - b^2)t + U(x, t), \tag{41}$$

where a, b, c, and d are arbitrary constants, and the function $U = U(x, t)$ is described by the linear heat equation

$$U_t = aU_{xx}. \tag{42}$$

2°. For solutions to Equation (42) under arbitrary initial and boundary conditions, see, for example, in [105,106]. In [106], there is a large list of exact solutions to the linear heat equation, which are expressed in elementary functions. To illustrate what has been said, we present several simple solutions to Equation (42), which contain exponential and trigonometric functions:

$$\begin{aligned}
U &= C_1 \exp(a\mu^2 t \pm \mu x) + C_2,\\
U &= C_1 \exp(-a\mu^2 t)\cos(\mu x) + C_2,\\
U &= C_1 \exp(-a\mu^2 t)\sin(\mu x) + C_2,\\
U &= C_1 \exp(-\mu x)\cos(\mu x - 2a\mu^2 t) + C_2,\\
U &= C_1 \exp(-\mu x)\sin(\mu x - 2a\mu^2 t) + C_2,
\end{aligned} \tag{43}$$

where C_1, C_2, and μ are arbitrary constants.

Formula (41) for $b = c = 0$ together with the second and third functions (43) gives two solutions periodic in the spatial variable x, and together with the fourth and fifth functions (43) describes two periodic solutions in time t.

8. Generalized Separable Solutions in the Form of Polynomials in One Spatial Variable

1°. Equation (1) has generalized separable solutions that are quadratic with respect to any spatial variable, e.g.,

$$u = y^2 f(x,t) + y g(x,t) + h(x,t), \tag{44}$$

where the functions $f = f(x,t)$, $g = g(x,t)$, and $h = h(x,t)$ are described by the following system of PDEs with two independent variables:

$$\begin{aligned} f_t &= 2f f_{xx} - 4f_x^2, \\ g_t &= 2f g_{xx} - 4f_x g_x, \\ h_t &= 2f h_{xx} - g_x^2. \end{aligned} \tag{45}$$

Here the first equation for f is nonlinear and isolated (i.e., does not depend on other equations), and the other two equations are linear with respect to the desired functions g and h (moreover, the equation for g is homogeneous, and the equation for h is nonhomogeneous).

Solution (41), leading to the linear heat equation (42), is a special case of the generalized separable solution (44) with

$$f(x,t) = \tfrac{1}{2}a, \quad g(x,t) = bx + d, \quad h(x,t) = \tfrac{1}{2}cx^2 + (ac - b^2)t + U(x,t).$$

2°. The last two equations of the system (45) have the obvious solution $g = C_2$, $h = C_3$, where C_2 and C_3 are arbitrary constants. A more general statement is also true.

Statement. Let $f = f(x,t)$ be any solution to the first equation (45). Then, the last two equations of the system (45) admit partial solutions

$$g = C_1 f + C_2, \quad h = \tfrac{1}{4}C_1^2 f + C_3, \tag{46}$$

where C_1, C_2, and C_3 are arbitrary constants.

This statement is proven by substituting functions (46) into the last two equations of system (45) and comparing the resulting expressions with the first equation (45).

3°. The simplest solution to the first equation (45) is the constant

$$f = \tfrac{1}{2}a, \tag{47}$$

where a is an arbitrary constant. In this case, the last two PDEs (45) are linear heat equations

$$\begin{aligned} g_t &= a g_{xx}, \\ h_t &= a h_{xx} - g_x^2. \end{aligned} \tag{48}$$

the first of which is homogeneous, and the second is nonhomogeneous. Solutions to these equations can be obtained for arbitrary initial and boundary conditions (see, for example, [105,106]).

Note that the first equation (48) has a degenerate stationary solution $g = bx$, where b is an arbitrary constant. Some other simple solutions to this equation, which are expressed in elementary functions, are given by the right-hand sides of the Formula (43).

Let us present simple solutions of the traveling wave type to Equation (48) containing exponential functions:

$$
\begin{aligned}
g &= C_1 \exp(kx + ak^2 t), \\
h &= C_2 \exp(kx + ak^2 t) + \frac{C_1^2}{2a} \exp[2(kx + ak^2 t)],
\end{aligned} \tag{49}
$$

where C_1, C_2, and k are arbitrary constants. Substituting (47) and (49) into Formula (44), we obtain an exact solution to the original equation (1).

4°. The first equation (45) has the stationary solution

$$f = \frac{1}{C_1 x + C_2},$$

as well as a more complex non-stationary three-parameter exact solution with generalized separation of variables

$$f = \frac{(x + C_1)^2}{12(t + C_2)} + C_3 (t + C_2)^{1/3},$$

where C_1, C_2, and C_3 are arbitrary constants.

5°. The first equation of system (45) is simplified using the substitution $f = 1/\theta$, which leads to the following PDE

$$\theta\theta_t = 2\theta_{xx},$$

that admits a fairly obvious generalized separable solution, linear in t, of the form $\theta = \varphi(x)t + \psi(x)$. It follows that the first equation (45) has a solution

$$f = \frac{1}{\varphi(x)t + \psi(x)}, \tag{50}$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(x)$ are described by the ODE system

$$2\varphi''_{xx} = \varphi^2, \quad 2\psi''_{xx} = \varphi\psi. \tag{51}$$

System (51) has the following exact solution, which is expressed in elementary functions:

$$\varphi = \frac{12}{(x + C_1)^2}, \quad \psi = C_2 (x + C_1)^3 + \frac{C_3}{(x + C_1)^2},$$

where C_1, C_2, and C_3 are arbitrary constants.

Let $\varphi = \varphi(x)$ be a solution to the first ODE of the system (51). Then, the second ODE of this system has a particular solution $\psi = \varphi(x)$. Since the second ODE is linear with respect to ψ, its general solution is given by the formula [107]:

$$\psi = C_3 \varphi + C_3 \varphi \int \frac{dx}{\varphi^2},$$

where C_3 and C_4 are arbitrary constants. The general solution of the first (autonomous) ODE of system (51) is easy to represent in implicit or equivalent parametric form [107]. Taking into account the above and using Formula (51), we obtain the general solution of system (51) in parametric form

$$
\begin{aligned}
x &= \pm \int_\xi^\infty \frac{d\zeta}{\left(\frac{1}{3}\zeta^3 + C_1\right)^{1/2}} + C_2, \\
\varphi &= \xi, \\
\psi &= C_3 \xi + C_4 \xi \int_\xi^\infty \frac{d\zeta}{\left(\frac{1}{3}\zeta^3 + C_1\right)^{1/2}}.
\end{aligned}
$$

where $0 \le \xi < \infty$ and $C_1 \ge 0$.

6°. By replacing $f = \xi^{-1/2}$, the first equation (45) is reduced to a nonlinear heat equation of the special form

$$\xi_t = 2(\xi^{-1/2}\xi_x)_x, \tag{52}$$

which admits a number of invariant and more complex solutions (a review of these solutions is given, for example, in [10,108]).

Remark 9. *In [109,110], a solution to Equation (52) of the form $\xi = [\varphi(x)t + \psi(x)]^2$ was obtained, which leads to solution (50).*

7°. The system of PDEs (45) admits an exact solution of the traveling wave type

$$f = f(z), \quad g = g(z), \quad h = h(z), \quad z = kx - \lambda t,$$

where k and λ are arbitrary constants. The solution of the corresponding autonomous ODE for the function f (which reduces to a linear ODE of the first order) can be expressed in elementary functions.

8°. PDE (1) has generalized separable solutions in the form of a fourth-degree polynomial with respect to the spatial variable,

$$u = y^4 F(x,t) + y^2 G(x,t) + H(x,t), \tag{53}$$

where the functions $F = F(x,t)$, $G = G(x,t)$, and $H = H(x,t)$ are described by the following overdetermined PDE system with two independent variables:

$$\begin{aligned} &3FF_{xx} - 4F_x^2 = 0, \\ &F_t = 12FG_{xx} + 2GF_{xx} - 16F_xG_x, \\ &G_t = 12FH_{xx} + 2GG_{xx} - 4G_x^2, \\ &H_t = 2GH_{xx}. \end{aligned} \tag{54}$$

Here, the first equation for F is nonlinear and isolated (i.e., does not depend on other equations); it can be considered as an ODE that implicitly depends on t.

Note that the solutions in elementary functions (20), (24), and (25) are particular solutions of the form (53).

The first equation (54) can be satisfied, for example, by setting $F = f_0(t)$. Then, from the second and other PDEs of this system, we will consistently obtain

$$\begin{aligned} &G = g_2x^2 + g_1x + g_0, \\ &H = h_4x^4 + h_3x^3 + h_2x^2 + h_1x + h_0, \end{aligned} \tag{55}$$

where the functions $f_0 = f_0(t)$, $g_i = g_i(t)$, and $h_j = h_j(t)$ are described by a corresponding ODE system.

9°. If in Formulas (53) and (55), we remove the odd components in the variable x, setting $g_1 = h_3 = h_1 = 0$, then, we arrive at the solution discussed in [42]:

$$u = f_0y^4 + h_4x^4 + g_2x^2y^2 + g_0y^2 + h_2x^2 + h_0, \tag{56}$$

whose time-dependent functional coefficients are described by the ODE system

$$
\begin{aligned}
f_0' &= 24f_0g_2,\\
h_4' &= 24g_2h_4,\\
g_2' &= 144f_0h_4 - 12g_2^2,\\
g_0' &= 4g_0g_2 + 24f_0h_2,\\
h_2' &= 4g_2h_2 + 24g_0h_4,\\
h_0' &= 4g_0h_2.
\end{aligned}
\tag{57}
$$

The first three ODEs of the system (57) form an independent subsystem, the general solution of which can be represented in closed form (expressed in quadratures). The last three ODEs of this system admit a trivial particular solution $g_0 = h_2 = h_0 = 0$.

System (57) has a simple solution of the form

$$
f_0 = a, \quad h_4 = g_2 = 0, \quad g_0 = 24abt + c, \quad h_2 = b, \quad h_0 = 48ab^2t^2 + 4bct + d,
$$

where a, b, c, and d are arbitrary constants, which determines the following exact solution of the original equation (1):

$$
u = ay^4 + (24abt + c)y^2 + bx^2 + 48ab^2t^2 + 4bct + d.
$$

Note that this solution was not considered in [42].

It can be shown that the ODE system (57) also has solutions when all the desired functional coefficients in the right part of (56) are proportional to $(t + C)^{-1}$, where C is an arbitrary constant. In particular, system (57) admits the following solution [42]:

$$
\begin{aligned}
&f_0 = \frac{A}{48(t+C)}, \quad h_4 = \frac{1}{48A(t+C)}, \quad g_2 = -\frac{1}{24(t+C)},\\
&g_0 = h_2 = h_0 = 0,
\end{aligned}
$$

where A and C are arbitrary constants $(A \neq 0)$.

Remark 10. *One can also look for more complex polynomial solutions of the form* $u = \sum_{k=0}^{4} F_k(x,t)y^k$ *which generalize* (53).

9. Reductions to a Monge–Ampère Type Equation in Traveling Wave Variables, Linearizable PDEs, and Two-Phase Solutions

1°. Equation (1) admits solutions with generalized separation of variables of the combined type

$$
\begin{aligned}
&u = C_1x^2 + C_2xy + C_3y^2 + C_4x + C_5y + C_6t + U(\xi,\eta),\\
&\xi = a_1x + b_1y + \lambda_1t, \qquad \eta = a_2x + b_2y + \lambda_2t,
\end{aligned}
\tag{58}
$$

where C_i, a_j, b_j, λ_j $(i = 1, \dots, 6;\ j = 1,\ 2)$ are arbitrary constants, and ξ and η are new traveling wave type variables, and the function $U = U(\xi,\eta)$ is described by a two-dimensional Monge-Ampère type equation

$$
\begin{aligned}
C_6 + \lambda_1U_\xi + \lambda_2U_\eta = {} & 4C_1C_3 - C_2^2 + (a_1b_2 - b_1a_2)^2(U_{\xi\xi}U_{\eta\eta} - U_{\xi\eta}^2)\\
&+ 2(a_1^2C_3 + b_1^2C_1 - a_1b_1C_2)U_{\xi\xi} + 2(a_2^2C_3 + b_2^2C_1 - a_2b_2C_2)U_{\eta\eta}\\
&+ 2[(2a_1a_2C_3 + 2b_1b_2C_1 - (a_1b_2 + b_1a_2)C_2]U_{\xi\eta}.
\end{aligned}
\tag{59}
$$

2°. Equation (59) admits exact solutions, quadratic with respect to any independent variable of the traveling wave type, of the form

$$
\begin{aligned}
U_1 &= f_1(\xi)\eta^2 + g_1(\xi)\eta + h_1(\xi),\\
U_2 &= f_2(\eta)\xi^2 + g_2(\eta)\xi + h_2(\eta),
\end{aligned}
$$

where the functions f_i, g_i, h_i ($i = 1, 2$) are described by the corresponding ODE systems, which are omitted here.

3°. Let us consider the special case (58) and (59), putting

$$a_1 = a, \quad b_1 = b, \quad \lambda_1 = \lambda, \quad a_2 = b_2 = 0, \quad \lambda_2 = 1, \quad \eta = t,$$

which corresponds to a solution of the form

$$u = C_1x^2 + C_2xy + C_3y^2 + C_4x + C_5y + C_6t + U(\xi, t), \quad \xi = ax + by + \lambda t, \tag{60}$$

where C_i ($i = 1, \ldots, 6$), a, b, and λ are arbitrary constants. In this case, the function $U = U(\xi, t)$ is described by the linear convective heat equation with constant source

$$U_t = 2(a^2C_3 + b^2C_1 - abC_2)U_{\xi\xi} - \lambda U_\xi + 4C_1C_3 - C_2^2 - C_6. \tag{61}$$

Remark 11. *The considered Equation (1), according to the terminology introduced in [111], refers to conditionally integrable PDEs, since it admits solutions that are described by linear PDEs (42) and (61). Perhaps in such cases, it is more accurate to use the term partially linearizable equations.*

4°. In particular, taking in (60) and (61), the function U with one argument ξ, and setting $C_6 = 4C_1C_3 - C_2^2$, we arrive at a linear ODE with constant coefficients,

$$2(a^2C_3 + b^2C_1 - abC_2)U''_{\xi\xi} - \lambda U'_\xi = 0,$$

the general solution, which is expressed through the exponent

$$U(\xi) = A_1 \exp\left[\frac{\lambda}{2(a^2C_3 + b^2C_1 - abC_2)}\xi\right] + A_2,$$

where A_1 and A_2 are arbitrary constants.

5°. Let us consider a two-phase solution (58) with additive separation of variables of the form

$$U(\xi, \eta) = V(\xi) + W(\eta), \quad a_1 = a, \quad a_2 = ka, \quad b_1 = b, \quad b_2 = kb, \tag{62}$$

where k is an arbitrary constant. As a result, we arrive at a functional differential equation with separable variables,

$$\begin{aligned} &C_6 - 4C_1C_3 + C_2^2 + \lambda_1 V'_\xi + \lambda_2 W'_\eta \\ &\qquad = 2(a^2C_3 + b^2C_1 - abC_2)V''_{\xi\xi} + 2k^2(a^2C_3 + b^2C_1 - abC_2)W''_{\eta\eta}, \end{aligned}$$

which reduces to two simple independent linear ODEs for determining the functions $V = V(\xi)$ and $W = W(\eta)$:

$$\begin{aligned} &2(a^2C_3 + b^2C_1 - abC_2)V''_{\xi\xi} - \lambda_1 V'_\xi = B, \\ &2k^2(a^2C_3 + b^2C_1 - abC_2)W''_{\eta\eta} - \lambda_2 W'_\eta = -B + C_6 - 4C_1C_3 + C_2^2, \end{aligned} \tag{63}$$

where B is an arbitrary constant.

In particular, setting $B = 0$, $C_6 = 4C_1C_3 - C_2^2$ in (63), we first find $V = V(\xi)$ and $W = W(\eta)$, and then using Formula (62), we determine the required function $U(\xi, \eta)$ included in the solution (58):

$$\begin{gathered} U(\xi, \eta) = A_1 \exp\left(\frac{\lambda_1\xi}{2s}\right) + A_2 \exp\left(\frac{\lambda_2\eta}{2k^2s}\right) + A_3, \quad s = a^2C_3 + b^2C_1 - abC_2, \\ \xi = ax + by + \lambda_1 t, \quad \eta = k(ax + by) + \lambda_2 t, \end{gathered}$$

where A_1, A_2, and A_3 are arbitrary constants.

10. Reduction Using a New Variable, Parabolic in Spatial Coordinates, Exact Solutions

1°. In variables, one of which is time, and the other is specified by a parabolic function in spatial variables,

$$u = U(z,t), \quad z = y + ax^2, \tag{64}$$

where a is an arbitrary constant, Equation (1) is reduced to the two-dimensional PDE

$$U_t = 2aU_z U_{zz}, \tag{65}$$

which is found in the theory of nonlinear filtration of fluids in porous media.

Note that solutions of the form (64) refer to functional separable solutions (see, for example, [10,55]).

Some exact solutions of the reduced Equation (65) are described below.

2°. The reduced Equation (65) admits the additive separable solution

$$\begin{aligned} U &= \pm C_1(z+C_2)^{3/2} + \tfrac{9}{4}aC_1^2 t + C_3 \\ &= \pm C_1(y+ax^2+C_2)^{3/2} + \tfrac{9}{4}aC_1^2 t + C_3, \end{aligned}$$

where C_1, C_2, and C_3 are arbitrary constants.

3°. Equation (65) has a simple solution in the form of a product of power functions of different arguments

$$U = -\frac{(z+C_2)^3}{36a(t+C_1)} = -\frac{(y+ax^2+C_2)^3}{36a(t+C_1)},$$

where C_1 and C_2 are arbitrary constants.

4°. Equation (65) has the traveling wave solution

$$U = \frac{\lambda}{4a}\xi^2 + C_1\xi + C_2, \quad \xi = z + \lambda t \equiv y + ax^2 + \lambda t, \tag{66}$$

where C_1, C_2, and λ are arbitrary constants.

Remark 12. *More general than* (66)*, the solution of Equation* (65) *can be obtained if we look for a solution in the form*

$$U = Ct + W(\xi), \quad \xi = z + \lambda t \equiv y + ax^2 + \lambda t.$$

5°. Equation (65) admits the self-similar solution

$$U = t^{-3\beta-1}V(\zeta), \quad \zeta = zt^\beta,$$

where β is an arbitrary constant, and the function $V = V(\zeta)$ is described by the non-autonomous ODE

$$2aV'_\zeta V''_{\zeta\zeta} = -(3\beta+1)V + \beta\zeta V'_\zeta.$$

For $\beta = -1/3$, the general solution of the last equation has the form

$$V = -\frac{1}{36a}\zeta^3 + C_1\zeta + C_2,$$

where C_1 and C_2 are arbitrary constants.

6°. Equation (65) has an invariant solution of the form

$$U = t^{-1}f(\eta), \quad \eta = z + \lambda \ln t,$$

where λ is an arbitrary constant, and the function $f = f(\eta)$ is described by the autonomous ODE

$$2af'_\eta f''_{\eta\eta} = -f + \lambda f'_\eta.$$

7°. Equation (65) also admits another invariant solution

$$U = e^{-3\beta t} g(\tau), \quad \tau = e^{\beta t} z,$$

where β is an arbitrary constant, and the function $g = g(\tau)$ is described by the non-autonomous ODE

$$2ag'_\tau g''_{\tau\tau} = -3\beta g + \beta \tau g'_\tau.$$

8°. The reduced equation (65) admits an exact solution in the form of a cubic polynomial in z:

$$U = \psi_1(t) + \psi_2(t)z + \psi_3(t)z^2 + \psi_4(t)z^3, \tag{67}$$

where the functions $\psi_n(t)$ $(n = 1, \dots, 4)$ are described by the ODE system

$$\begin{aligned}
\psi'_1 &= 4a\psi_2\psi_3, \\
\psi'_2 &= 4a(3\psi_2\psi_4 + 2\psi_3^2), \\
\psi'_3 &= 36a\psi_3\psi_4, \\
\psi'_4 &= 36a\psi_4^2.
\end{aligned}$$

This system is integrated in reverse order, starting with the last equation. Its general solution is described by the formulas

$$\begin{aligned}
\psi_1 &= -12aC_2C_3(t + C_1)^{-1/3} + 48a^2C_2^3(t + C_1)^{-1} + C_4, \\
\psi_2 &= C_3(t + C_1)^{-1/3} - 12aC_2^2(t + C_1)^{-1}, \\
\psi_3 &= C_2(t + C_1)^{-1}, \quad \psi_4 = -\frac{1}{36a}(t + C_1)^{-1},
\end{aligned}$$

where C_1, C_2, C_3, and C_4 are arbitrary constants.

9°. The reduced equation (65) also has an exact solution with a generalized separation of variables of a more exotic form

$$U = \vartheta_1(t) + \vartheta_2(t)z^{3/2} + \vartheta_3(t)z^3, \tag{68}$$

where the functions $\vartheta_n(t)$ $(n = 1,\ 2,\ 3)$ satisfy the ODE system

$$\begin{aligned}
\vartheta'_1 &= \tfrac{9}{4}a\vartheta_2^2, \\
\vartheta'_2 &= \tfrac{45}{2}a\vartheta_2\vartheta_3, \\
\vartheta'_3 &= 36a\vartheta_3^2.
\end{aligned}$$

This system is integrated in reverse order, starting with the last equation. Its general solution has the form

$$\begin{aligned}
\theta_1 &= -9aC_2^2(t + C_1)^{-1/4} + C_3, \\
\theta_2 &= C_2(t + C_1)^{-5/8}, \\
\theta_3 &= -\frac{1}{36a}(t + C_1)^{-1},
\end{aligned}$$

where C_1, C_2, and C_3 are arbitrary constants.

Remark 13. *To construct generalized separable solutions* (67) *and* (68)*, we used invariant subspaces of the nonlinear differential operator* $F(U) = U_z U_{zz}$*, which is included in the right side of Equation* (65) *(for details see [42,55]).*

11. Reduction Using a New Variable, Quadratic in Two Spatial Coordinates, Exact Solutions

1°. In variables, one of which is time and the other is quadratic with respect to both spatial variables,

$$u = U(z,t), \quad z = ax^2 + bxy + cy^2 + kx + sy, \tag{69}$$

where a, b, c, k, and s are arbitrary constants, Equation (1) is reduced to a two-dimensional nonstationary equation

$$\begin{gathered} U_t = 2(Az+B)U_z U_{zz} + AU_z^2; \\ A = 4ac - b^2, \quad B = as^2 + ck^2 - bks. \end{gathered} \tag{70}$$

Solutions of the form (69) refer to functional separable solutions (see, for example, [10,55]).

Note that depending on the coefficients a, b, c of the quadratic form in (69), the curve $z = \text{const}$ can be an ellipse (for $A = 4ac - b^2 > 0$), a hyperbola (for $A < 0$), or a parabola (for $A = 0$).

The degenerate case $A = 0$ leads to an equation of the form (65), which was discussed in detail in the previous section. Next, we will consider the non-degenerate case when $A = 4ac - b^2 \neq 0$.

2°. The transformaation

$$t = t, \quad z = \frac{\sqrt{|A|}}{2}\rho^2 - \frac{B}{A}, \quad U = \operatorname{sign}(A) W(\rho,t),$$

leads to Equation (70) to the canonical form

$$W_t = \rho^{-1} W_\rho W_{\rho\rho}. \tag{71}$$

Let us consider some classes of exact solutions that Equation (71) admits.

3°. Equation (71) has an exact solution with additive separation of variables

$$W = C_1 C_2^2 t + C_2 \int \sqrt{C_1\rho^2 + C_3}\, d\rho + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants. The integral on the right side is expressed in terms of different elementary functions

$$\int \sqrt{C_1\rho^2 + C_3}\, d\rho = \\ = \begin{cases} \dfrac{1}{2}\rho\sqrt{C_1\rho^2 + C_3} + \dfrac{C_3}{2\sqrt{C_1}} \ln\left(\sqrt{C_1}\rho + \sqrt{C_1\rho^2 + C_3}\right), & \text{if } C_1 > 0; \\ \dfrac{1}{2}\rho\sqrt{C_1\rho^2 + C_3} + \dfrac{C_3}{2\sqrt{-C_1}} \arctan\dfrac{\sqrt{-C_1}\rho}{\sqrt{C_1\rho^2 + C_3}}, & \text{if } C_1 < 0; \\ \sqrt{C_3}\rho, & \text{if } C_1 = 0,\ C_3 > 0. \end{cases}$$

4°. Equation (71) admits solutions in the form of a product of functions of different arguments

$$W = t^{-1} f(\rho),$$

where the function $f = f(\rho)$ is described by the non-autonomous ODE

$$\rho^{-1} f'_\rho f''_\rho = -f,$$

which has the simple particular solution

$$f = -\tfrac{1}{48}\rho^4.$$

Returning now to the original variables, we obtain a five-parameter solution in elementary functions of Equation (1):

$$u = -\frac{1}{12(4ac - b^2)t}\left[ax^2 + bxy + cy^2 + kx + sy + \frac{as^2 + ck^2 - bks}{4ac - b^2}\right]^2.$$

5°. Equation (71) admits the self-similar solution

$$W = t^{-4\gamma-1}F(z), \quad z = t^{\gamma}\rho,$$

where γ is an arbitrary constant, and the function $F = F(z)$ is described by the ODE

$$z^{-1}F'_z F''_{zz} = -(4\gamma + 1)F + \gamma z F'_z.$$

For $\gamma = -1/1$, the general solution of the last equation has the form

$$F = -\tfrac{1}{48}z^4 + C_1 z + C_2,$$

where C_1 and C_2 are arbitrary constants.

6°. Equation (71) has the invariant solution

$$W = e^{-4\lambda t}\Phi(\zeta), \quad \zeta = \rho e^{\lambda t},$$

where λ is an arbitrary constant, and the function $\Phi = \Phi(\zeta)$ satisfies the generalized homogeneous ODE

$$\zeta^{-1}\Phi'_\zeta \Phi''_{\zeta\zeta} = -4\lambda\Phi + \lambda\zeta\Phi'_\zeta,$$

whose order can be lowered by one [107].

7°. Equation (71) also has the generalized separable solution of polynomial form

$$W = \theta_1(t) + \theta_2(t)\rho^2 + \theta_3(t)\rho^4,$$

where the functions $\theta_n(t)$ ($n = 1, 2, 3$) satisfy the ODE system

$$\begin{aligned} \theta'_1 &= 4\theta_2^2, \\ \theta'_2 &= 32\theta_2\theta_3, \\ \theta'_3 &= 48\theta_3^2. \end{aligned}$$

This system is integrated in reverse order, starting with the last equation. Its general solution is described by the formulas

$$\begin{aligned} \theta_1 &= -12C_2^2(t + C_1)^{-1/3} + C_3, \\ \theta_2 &= C_2(t + C_1)^{-2/3}, \\ \theta_3 &= -\tfrac{1}{48}(t + C_1)^{-1}, \end{aligned}$$

where C_1, C_2, and C_2 are arbitrary constants.

12. Reductions and Exact Solutions in Polar Coordinates

1°. At the point (x_0, y_0), where x_0 and y_0 are arbitrary constants, we introduce polar coordinates r, φ using the formulas

$$x = x_0 + r\cos\varphi, \quad y = y_0 + r\sin\varphi.$$

As a result, the original equation (1) is transformed to the form

$$u_t = r^{-2} u_{rr}(u_{\varphi\varphi} + r u_r) - [(r^{-1} u_\varphi)_r]^2. \tag{72}$$

This equation will be used in the future to construct reductions and exact solutions of the equation under consideration.

2°. From the results of group analysis of the transformed magnetohydrodynamics equation (72), it follows that the transformation

$$\begin{aligned} &\bar{r} = ar, \quad \bar{\varphi} = \varphi + b, \quad \bar{t} = pt + q, \\ &\bar{u} = \frac{a^4}{p} u + c_1 r \cos\varphi + c_2 r \sin\varphi + c_3, \end{aligned} \tag{73}$$

where a, b, c_1, c_2, c_3, p, and q are arbitrary constants, leads Equation (72) to an equation of exactly the same form.

The seven-parameter invariant transformation (73) allows using simpler solutions of Equation (72) to construct its more complex exact solutions. Namely, if $u = F(r, \varphi, t)$ is a solution to Equation (72), then the function

$$u = \frac{p}{a^4} F(ar, \varphi + b, pt + q) + s_1 r \cos\varphi + s_2 r \sin\varphi + s_3,$$

where $s_1 = -c_1 p/a^4$, $s_2 = -c_2 p/a^4$, and $s_3 = -c_3 p/a^4$ are arbitrary constants, is also a solution of this equation.

3°. Equation (72) admits radially symmetric solutions independent of the angular variable, which are described by the two-dimensional equation

$$u_t = r^{-1} u_r u_{rr},$$

which, up to obvious renotations, coincides with Equation (71). Therefore, it allows five exact solutions, described earlier in Items 3°–7° from Section 11.

4°. Passing in (72) to new variables of mixed type

$$u = t^{-4\alpha-1} U(\xi, \eta), \quad \xi = r t^\alpha, \quad \eta = \varphi + \beta \ln t, \tag{74}$$

where α and β are arbitrary constants, we arrive at the two-dimensional equation

$$\xi^{-2} U_{\xi\xi}(U_{\eta\eta} + \xi U_\xi) - [(\xi^{-1} U_\eta)_\xi]^2 + (4\alpha + 1)U - \alpha\xi U_\xi - \beta U_\eta = 0.$$

The values of $\alpha = \beta = 0$ in (74) correspond to a solution with multiplicative separation of variables of the form $u = t^{-1} U(r, \varphi)$.

The value $\beta = 0$ in (74) corresponds to a self-similar solution in the variables r and t.

5°. Passing in (72) to other variables of mixed type

$$u = e^{-4\gamma} U(\xi, \eta), \quad \xi = e^{\gamma t} r, \quad \eta = \varphi - \lambda t, \tag{75}$$

where γ and λ are arbitrary constants, we arrive at the two-dimensional equation

$$\xi^{-2} U_{\xi\xi}(U_{\eta\eta} + \xi U_\xi) - [(\xi^{-1} U_\eta)_\xi]^2 + 4\gamma U - \gamma\xi U_\xi + \lambda U_\eta = 0.$$

The value $\gamma = 0$ in (75) corresponds to a solution of the traveling wave type in the variables φ and t.

6°. Equation (72) also has exact solutions with separation of variables of the form

$$u = r^4 U(\varphi, t),$$

where the function $U = U(\varphi, t)$ is described by the two-dimensional equation

$$U_t = 12UU_{\varphi\varphi} - 9U_\varphi^2 + 48U^2. \tag{76}$$

7°. Since Equation (76) does not depend explicitly on independent variables, it has a solution of the traveling wave type

$$U = U(\eta), \quad \eta = \varphi - \lambda t,$$

where λ is an arbitrary constant, and the function $U = U(\eta)$ is described by the autonomous ODE

$$12UU''_{\eta\eta} - 9(U'_\eta)^2 - 48U^2 + \lambda U'_\eta = 0.$$

8°. Equation (76) also admits a multiplicative separable solution of the form

$$U = (t + C)^{-1} V(\varphi),$$

where C is an arbitrary constant, and the function $V = V(\varphi)$ is described by the autonomous ODE

$$12VV''_{\varphi\varphi} - 9(V'_\varphi)^2 + 48V^2 + V = 0. \tag{77}$$

The substitution $Z(V) = (V'_\varphi)^2$ leads (77) to a first-order linear ODE $6VZ'_V - 9Z + 48V^2 + V = 0$, the general solution of which is written as follows:

$$Z = C_1 V^{3/2} - 16V^2 + \tfrac{1}{3}V,$$

where C_2 is an arbitrary constant. Integrating further, we find the general solution to Equation (77) in implicit form

$$\int \frac{dV}{\left(C_1 V^{3/2} - 16V^2 + \frac{1}{3}V\right)^{1/2}} = \pm\varphi + C_2.$$

9°. Equation (76) has generalized separable solutions of the form

$$v = f(t) + g(t)[A\cos(4\varphi) + B\sin(4\varphi)],$$

where A and B are arbitrary constants, and the functions $f(t)$ and $g(t)$ are described by the ODE system

$$\begin{aligned} f'_t &= 48f^2 - 144(A^2 + B^2)g^2, \\ g'_t &= -96fg. \end{aligned} \tag{78}$$

Eliminating the variable t from (78), we arrive at the homogeneous first-order ODE

$$f'_g = -\tfrac{1}{2}(f/g) + \tfrac{3}{2}(A^2 + B^2)(g/f), \tag{79}$$

which, using the substitution $w = f/g$, is reduced to an ODE with separable variables. Integrating this equation, we obtain the general solution to ODE (79) in the form

$$f^2 = (A^2 + B^2)\left(g^2 + \frac{C_1}{g}\right), \tag{80}$$

where C_1 is an arbitrary constant. Eliminating f from the second ODE (78) using (80), we arrive at an ODE with separable variables, the general solution of which can be represented in implicit form using elementary functions

$$z + \sqrt{z^2 + C_1} = C_2 \exp\left[\pm 144\sqrt{(A^2 + B^2)}\, t\right], \quad z = g^{3/2}.$$

Using this relation, it is possible to express the function $g = g(t)$ explicitly.

Remark 14. *To construct generalized separable solutions of the form* (77)*, we used invariant subspaces of the nonlinear differential operator* $F[v] = 12vv_{\varphi\varphi} - 9v_{\varphi}^2 + 48v^2$*, included in the right side of Equation* (76) *(for details see [42,55]).*

13. Reductions and Exact Solutions in Elliptic and Hyperbolic Coordinates

1°. At the point (x_0, y_0), we introduce elliptic coordinates r, φ according to the formulas

$$x = x_0 + ar\cos\varphi, \quad y = y_0 + br\sin\varphi,$$

where x_0 and y_0 are arbitrary constants, a and b are any positive constants. As a result, the original equation (1) takes the form

$$u_t = (ab)^{-2}\{r^{-2}u_{rr}(u_{\varphi\varphi} + ru_r) - [(r^{-1}u_{\varphi})_r]^2\}.$$

Using a simple substitution $u = (ab)^2\bar{u}$ (or $t = (ab)^2\bar{t}$), this equation is reduced to Equation (72), the exact solutions of which are described in Section 12.

2°. In hyperbolic coordinates ζ, ψ, which are introduced by the formulas

$$x = x_0 + a\zeta\cosh\psi, \quad y = y_0 + b\zeta\sinh\psi,$$

where x_0 and y_0 are arbitrary constants, a and b are non-zero constants, Equation (1) takes the form

$$u_t = -(ab)^{-2}\{\zeta^{-2}u_{\zeta\zeta}(u_{\psi\psi} + \zeta u_{\zeta}) - [(\zeta^{-1}u_{\psi})_{\zeta}]^2\}.$$

Using a simple substitution $u = -(ab)^2\bar{u}$, this equation is reduced to Equation (72), the exact solutions of which are described in Section 12.

14. Using a Special Point Transformation to Construct Reductions and Exact Solutions

The special point transformation

$$x = \frac{\xi}{1 + \alpha\xi + \beta\eta}, \quad y = \frac{\eta}{1 + \alpha\xi + \beta\eta}, \quad u = \frac{w}{1 + \alpha\xi + \beta\eta}, \tag{81}$$

where α and β are arbitrary constants, brings Equation (1) to the form

$$w_{\xi\xi}w_{\eta\eta} - w_{\xi\eta}^2 = (1 + \alpha\xi + \beta\eta)^{-5}w_t. \tag{82}$$

Note that transformation (81) brings the stationary Monge–Ampère equation $u_{xx}u_{yy} - u_{xy}^2 = f(x, y)$ to a similar form with a different right-hand side [7,9].

Setting $\beta = 0$ in (81) and (82), we arrive at the equation

$$w_{\xi\xi}w_{\eta\eta} - w_{\xi\eta}^2 = f(\xi)w_t, \tag{83}$$

where $f(\xi) = (1 + \alpha\xi)^{-5}$.

Let us now describe some exact solutions of Equation (83) for a more general case, considering the function $f(\xi)$ arbitrary.

1°. Equation (83) admits generalized separable solutions of the form

$$w = (a\xi + b)t + Z(\xi, \eta),$$

where a and b are arbitrary constants, and the function $Z = Z(\xi, \eta)$ is described by the stationary Monge–Ampère equation

$$Z_{\xi\xi}Z_{\eta\eta} - Z_{\xi\eta}^2 = f(\xi)(a\xi + b). \tag{84}$$

PDEs of this type were considered in [10]. Equation (84) has the following generalized separable solutions in closed form:

$$Z = \pm\eta \int \sqrt{-f(\xi)(a\xi + b)}\, d\xi + \varphi(\xi),$$

$$Z = C_1\eta^2 + C_2\xi\eta + \frac{C_2^2}{4C_1}\xi^2 + \frac{1}{2C_1}\int_0^{\xi}(\xi - \zeta)f(\zeta)(a\zeta + b)\, d\zeta + C_3\xi + C_4\eta,$$

$$Z = \frac{1}{\xi + C_1}\left(C_2\eta^2 + C_3\eta + \frac{C_3^2}{4C_2}\right) + \frac{1}{2C_2}\int_0^{\xi}(\xi - \zeta)(\zeta + C_1)f(\zeta)(a\zeta + b)\, d\zeta,$$

where $\varphi(\xi)$ is an arbitrary function, $C_1, \ldots, C_4$ are arbitrary constants.

2°. Passing in (83) to variables of self-similar type

$$w = t^{-2k-1}W(\xi, \theta), \quad \theta = \eta t^k,$$

where k is a free parameter, we obtain a two-dimensional equation of the Monge–Ampère type with variable coefficients for the lower derivatives

$$W_{\xi\xi}W_{\theta\theta} - W_{\xi\theta}^2 = f(\xi)[-(2k+1)W + k\theta W_\theta].$$

3°. Passing in (83) to variables of limit self-similar type

$$w = \exp(-2\gamma t)W(\xi, \theta), \quad \theta = \exp(\gamma t)\eta,$$

where γ is a free parameter, we obtain a two-dimensional stationary equation of the Monge–Ampère type

$$W_{\xi\xi}W_{\theta\theta} - W_{\xi\theta}^2 = \gamma f(\xi)(-2W + \theta W_\theta).$$

4°. Equation (83) admits a one-dimensional invariant solution

$$w = t^{-1}\eta^2\varphi(\xi),$$

where the function $\varphi = \varphi(\xi)$ is described by the ODE

$$2\varphi\varphi''_{\xi\xi} - 4(\varphi'_\xi)^2 = -f(\xi)\varphi.$$

15. Brief Conclusions

We study a highly nonlinear partial differential equation of the form

$$u_t = u_{xx}u_{yy} - u_{xy}^2,$$

which describes local unsteady plasma motions in electron magnetohydrodynamics. Invariant multiparameter transformations that preserve the form of this equation are considered. Some one-dimensional symmetry reductions leading to ordinary differential equations are described and corresponding self-similar and other invariant solutions were obtained. Many new exact solutions with additive, multiplicative, generalized, and functional separation of variables have been found. Special attention is paid to the construction of exact closed-form solutions. More than 30 solutions in elementary functions were obtained (it is such exact solutions that are most useful for testing numerical methods). Two-dimensional symmetry and non-symmetry reductions that reduce the original equation with three independent variables to simpler single PDE or PDE system with two independent variables are described. In particular, a class of solutions is described, which are expressed in terms of solutions to a linear diffusion equation. Note that in some cases, exact solutions were obtained by combining the classical method of symmetry reductions and methods of generalized and/or functional separation of variables (the order in which these methods are applied may vary).

Note that this paper did not use the nonclassical symmetries method and the method of differential constraints. These methods are based on the analysis of the compatibility of overdetermined nonlinear PDE systems and can also be used to construct exact solutions to the highly nonlinear PDE (1) (usually these methods are more labor-intensive and lead to a larger volume of calculations than the methods used in our work).

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