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Geophysical Monge–Ampère-Type Equation: Symmetries and Exact Solutions

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Abstract

This paper studies a mixed PDE containing the second time derivative and a quadratic nonlinearity of the Monge–Ampère type in two spatial variables, which is encountered in geophysical fluid dynamics. The Lie group symmetry analysis of this highly nonlinear PDE is performed for the first time. An invariant point transformation is found that depends on fourteen arbitrary constants and preserves the form of the equation under consideration. One-dimensional symmetry reductions leading to self-similar and some other invariant solutions that described by single ODEs are considered. Using the methods of generalized and functional separation of variables, as well as the principle of structural analogy of solutions, a large number of new non-invariant closed-form solutions are obtained. In general, the extensive list of all exact solutions found includes more than thirty solutions that are expressed in terms of elementary functions. Most of the obtained solutions contain a number of arbitrary constants, and several solutions additionally include two arbitrary functions. Two-dimensional reductions are considered that reduce the original PDE in three independent variables to a single simpler PDE in two independent variables (including linear wave equations, the Laplace equation, the Tricomi equation, and the Guderley equation) or to a system of such PDEs. A number of specific examples demonstrate that the type of the mixed, highly nonlinear PDE under consideration, depending on the choice of its specific solutions, can be either hyperbolic or elliptic. To analyze the equation and construct exact solutions and reductions, in addition to Cartesian coordinates, polar, generalized polar, and special Lorentz coordinates are also used. In conclusion, possible promising directions for further research of the highly nonlinear PDE under consideration and related PDEs are formulated. It should be noted that the described symmetries, transformations, reductions, and solutions can be utilized to determine the error and estimate the limits of applicability of numerical and approximate analytical methods for solving complex problems of mathematical physics with highly nonlinear PDEs.

Keywords: Monge–Ampère-type equations; highly nonlinear PDEs; mixed PDEs; geophysical fluid dynamics; Lie group symmetry analysis; methods of generalized and functional separation of variables; principle of structural analogy of solutions; exact solutions; solutions in elementary functions; symmetry and non-symmetry reductions

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Contents

1. Introduction

1°. Highly nonlinear partial differential equations of the Monge–Ampère type, containing a quadratic combination in the second derivatives of the form $u_{xx}u_{yy} - u_{xy}^2$, are encountered in differential geometry [1–4], gas dynamics [5–7], elasticity and plasticity theory [8–10], magnetohydrodynamics [11–13], two-phase mechanics [14], meteorology and geophysics [15], optimization problems [3], and some other applications [4,16].

The equations of gas dynamics for plane one-dimensional flows with variable entropy are reduced to a special class of Monge–Ampère equations with two independent variables [5,6]:

$$u_{xx}u_{yy} - u_{xy}^2 = f(x, y), \tag{1}$$

where $u = u(x, y)$ is the desired function, and $f(x, y)$ is the given function.

General solutions of the homogeneous Monge–Ampère Equation (1) with $f(x, y) \equiv 0$ and the nonhomogeneous Monge–Ampère Equation (1) with $f(x, y) = -A$, where $A > 0$ is a free constant, and admit parametric representations [17] (see also [16]).

Symmetries, equivalence transformations, and invariant solutions of Equation (1) were considered in [7,18,19]. In [20,21], some polynomial exact solutions of Equation (1) with quadratic and more complicated polynomial right-hand sides were obtained. In [16,22], many non-invariant solutions with generalized and functional separation of variables of Equation (1) are described (special attention was paid to PDEs of a fairly general form, depending on one or two arbitrary functions of one argument).

In [7,23], it was shown that the nonlinear Equation (1) admits an exact linearization for $f(x, y) = f_1(x)$ and $f(x, y) = x^{-4}f_2(y/x)$, where $f_1(x)$ and $f_2(z)$ are arbitrary functions. In [24], it was proved that the Monge–Ampère-type equation

$$u_{xx}u_{yy} - u_{xy}^2 = f(x, u_y)$$

can be linearized using the contact Euler transform for any function of two arguments $f(x, z)$.

Note that in the general case, the strongly nonlinear Equation (1) is a mixed-type PDE, since for $f(x,y) > 0$, it is an elliptic equation, and for $f(x,y) < 0$, it is a hyperbolic equation (see, for example, [16]).

In [16,22,25,26], exact solutions of more complicated than (1) highly nonlinear coupled PDEs with two independent variables were considered.

Some invariant and non-invariant exact solutions of multidimensional Monge–Ampère-type equations, depending on n spatial variables and containing strong nonlinearity in the second derivatives of the form

$$\det[u_{x_i x_j}], \tag{2}$$

where $u_{x_i x_j} = \frac{\partial^2 u}{\partial x_i \partial x_j}$ $(i,j = 1,\dots,n)$, were obtained in [27–29]. The matrix of second derivatives $[u_{x_i x_j}]$ included in (2) describes the local curvature of a function of many variables $u = u(x_1,\dots,x_n)$ and is called the Hessian matrix. In the two-dimensional case with $n = 2$ and $x_1 = x$, $x_2 = y$, expression (2) coincides with the left-hand side of Equation (1). Reductions and exact solutions of the three-dimensional and four-dimensional homogeneous Monge–Ampère equation $\det[u_{x_i x_j}] = 0$ for $n = 3$ and $n = 4$ were considered in [30,31] (see also [32], where a related nonhomogeneous PDE with a special right-hand side for $n = 4$ was studied).

In [33,34], reductions and exact solutions of some two-dimensional and multidimensional systems consisting of two equations with Monge–Ampère-type nonlinearity are described.

2°. In electron magnetohydrodynamics, a nonstationary Monge–Ampère-type equation with three independent variables is encountered [11–13]:

$$u_t = u_{xx}u_{yy} - u_{xy}^2. \tag{3}$$

Characteristic qualitative features, symmetries, reductions, and exact solutions of the highly nonlinear Equation (3) were considered in [35,36]. In [36], a large number of solutions of this PDE were found, which are expressed in terms of elementary functions. Some invariant and non-invariant exact solutions of more complicated related equations of Monge–Ampère type were obtained in [24,37,38].

Equation (3), as well as related highly nonlinear PDEs containing the first derivative with respect to time u_t and a quadratic combination of second derivatives with respect to spatial variables of the form $u_{xx}u_{yy} - u_{xy}^2$ or $\det[u_{x_i x_j}]$, are called the parabolic Monge–Ampère equations. Geometric applications and questions of existence and uniqueness of various classes of solutions of the corresponding initial-boundary value problems with such PDEs were considered, for example, in [39–56].

In [57], solutions with additive and multiplicative separation of variables of multidimensional parabolic equations of the Monge–Ampère type of the form $f u_t = \det[u_{x_i x_j}]$ are described, in which the functional coefficient f depends in a special way on $x_1,\dots,x_n,t,u$ (see also [58], where exact solutions of a more complex related PDE were obtained).

3°. In this paper, we will consider a more complex, than (3), highly nonlinear PDE containing the second derivative with respect to time:

$$u_{tt} = u_{xx}u_{yy} - u_{xy}^2, \tag{4}$$

which we will further call the geophysical Monge–Ampère-type equation (a clarification of this name will be given below).

In addition to Equation (4), to a lesser extent, a nonlinear equation of the form

$$f(x)u_{tt} = u_{xx}u_{yy} - u_{xy}^2, \tag{5}$$

where $f(x)$ is an arbitrary function, will also be studied.

It is important to note that the highly nonlinear PDE (4), by renaming the desired function u by $-u$, is reduced to an equation that was derived in [59] in relation to problems of geophysical fluid dynamics. In the cited article, the simple polynomial solution of Equation (4) was obtained as follows:

$$u = \tfrac{1}{2}x^2 t - \tfrac{1}{6}t^3 - \tfrac{1}{2}y^2. \tag{6}$$

Apparently, the nonlinear Equation (4) was first formally introduced in [60], where it was noted that it can have blow-up solutions.

In this paper, it will be shown, using a number of specific examples, that the type of the highly nonlinear Equation (4), depending on the choice of its specific solutions, can be either hyperbolic or elliptic. In other words, this equation is a PDE of mixed type.

Further, by exact closed-form solutions of nonlinear PDEs, as in [36,38,61], we mean solutions that are expressed in terms of (i) elementary functions, (ii) elementary functions and indefinite integrals, and (iii) solutions of ODE or ODE systems.

To analyze symmetries and find exact solutions to nonlinear PDEs, the classical method of symmetry reductions [18,62–67], the direct method of symmetry reductions [16,61,68–73], the nonclassical symmetries methods [69,71,74–83], methods of generalized separation of variables [16,60,73,84–89], methods of functional separation of variables [16,73,78,87,90–98], and the method of differential constraints [16,70,73,99–102] are most often used (see also some other exact analytical methods [16,61,103–110]). On methods for constructing exact solutions of nonlinear PDEs with constant and variable delay as well as some other nonlinear functional PDEs, see, for example, [110–118].

4°. In this paper, to find exact solutions to the nonlinear PDE (4) encountered in geophysical fluid dynamics, we mainly used the classical classical method of symmetry reductions [18,62–64] and methods of generalized or functional separation of variables [16,60,73]. In a number of cases, exact solutions were obtained by applying various combinations of the above methods.

Remark 1. *To construct exact solutions of mixed nonlinear PDE* (4)*, we will also partially use the principle of structural analogy of solutions (see, for example, [61,107,108]). Namely, the structure of exact solutions of Equation* (4) *in some cases was determined by the structure of exact solutions of the related simpler Equation* (3)*, which are found, for example, in [36,38].*

2. Symmetries of the Monge–Ampère Mixed-Type PDE. Reproduction Formula

Applying the technique of Lie group analysis [62–64], we look for the symmetry operators of the nonlinear PDE (4) in the form

$$X = \zeta^1(x,y,t,u)\frac{\partial}{\partial x} + \zeta^2(x,y,t,u)\frac{\partial}{\partial y} + \zeta^3(x,y,t,u)\frac{\partial}{\partial t} + \zeta^4(x,y,t,u)\frac{\partial}{\partial u}\,.$$

Using the invariance criterion [62], for the four desired functions ζ^1, ζ^2, ζ^3, and ζ^4, one can derive the following overdetermined linear homogeneous system consisting of nineteen defining PDEs:

$$\zeta^1_t = 0\,, \qquad \zeta^1_u = 0\,, \qquad \zeta^2_t = 0\,, \qquad \zeta^2_u = 0\,,$$

$$
\begin{aligned}
&\zeta^3_x = 0\,, \qquad \zeta^3_y = 0\,, \qquad \zeta^3_u = 0\,, \\
&\zeta^1_{xx} = 0\,, \qquad \zeta^1_{xy} = 0\,, \qquad \zeta^1_{yy} = 0\,, \\
&\zeta^2_{xx} = 0\,, \qquad \zeta^2_{xy} = 0\,, \qquad \zeta^2_{yy} = 0\,, \\
&\zeta^3_{tt} = 0\,, \qquad \zeta^4_u - 2(\zeta^1_x + \zeta^2_y - \zeta^3_t) = 0\,, \\
&\zeta^4_{xx} = 0\,, \qquad \zeta^4_{xy} = 0\,, \qquad \zeta^4_{yy} = 0\,, \qquad \zeta^4_{tt} = 0\,.
\end{aligned}
\tag{7}
$$

It is not difficult to prove that the general solution of the overdetermined PDE system (7) is given by the formulas

$$
\begin{aligned}
\zeta^1 &= c_1 x + c_2 y + c_3\,, \\
\zeta^2 &= c_4 x + c_5 y + c_6\,, \\
\zeta^3 &= c_7 t + c_8\,, \\
\zeta^4 &= (c_9 x + c_{11} y + c_{12})t + 2(c_1 + c_5 - c_7)u + c_{10} x + c_{13} y + c_{14},
\end{aligned}
$$

where c_j $(j = 1, \ldots, 14)$ are arbitrary constants. This leads to two propositions, formulated below.

Proposition 1. *The basis of the Lie algebra of symmetry operators for the Monge–Ampère-type PDE* (4) *can be written in the form*

$$
\begin{aligned}
&X_1 = \frac{\partial}{\partial x}\,, \qquad X_2 = \frac{\partial}{\partial y}\,, \qquad X_3 = \frac{\partial}{\partial t}\,, \qquad X_4 = \frac{\partial}{\partial u}\,, \\
&X_5 = y\frac{\partial}{\partial x}\,, \qquad X_6 = x\frac{\partial}{\partial y}\,, \qquad X_7 = x\frac{\partial}{\partial u}\,, \qquad X_8 = y\frac{\partial}{\partial u}\,, \\
&X_9 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}\,, \qquad X_{10} = y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t}\,, \qquad X_{11} = t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}\,, \\
&X_{12} = t\frac{\partial}{\partial u}\,, \qquad X_{13} = xt\frac{\partial}{\partial u}\,, \qquad X_{14} = yt\frac{\partial}{\partial u}\,.
\end{aligned}
$$

Proposition 2. *The transformation*

$$
\begin{aligned}
&\bar{x} = A_1 x + B_1 y + C_1\,, \qquad \bar{y} = A_2 x + B_2 y + C_2\,, \qquad \bar{t} = D_1 t + D_2\,, \\
&\bar{u} = \frac{(A_1 B_2 - A_2 B_1)^2}{D_1^2}\, u + t(A_3 x + B_3 y + C_3) + A_4 x + B_4 y + C_4,
\end{aligned}
\tag{8}
$$

where A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, C_1, C_2, C_3, C_4, D_1, and D_2 are free parameters satisfying two conditions, $A_1 B_2 - A_2 B_1 \neq 0$ and $D_1 \neq 0$, leaves the form of Equation (4) *invariant.*

Below are simple consequences for two one-parameter transformations that follow from Proposition 2.

Corollary 1. *The rotation transformation of spatial variables:*

$$
\bar{x} = x\cos\beta - y\sin\beta, \qquad \bar{y} = y\cos\beta + x\sin\beta, \qquad \bar{u} = u,
\tag{9}
$$

where β is a free parameter, leaves the highly nonlinear PDE (4) *invariant.*

Note that the rotation transformation (9) also leaves invariant the simpler Laplace equation $u_{xx} + u_{yy} = 0$, which is a linear elliptic PDE.

Corollary 2. *The Lorentz transformation of spatial variables:*

$$\bar{x} = x\cosh\beta - y\sinh\beta, \quad \bar{y} = y\cosh\beta - x\sinh\beta, \quad \bar{u} = u, \tag{10}$$

where β is a free parameter, leaves the highly nonlinear PDE (4) invariant.

Note that the Lorentz transformation (10) also leaves invariant the simpler wave equation $u_{xx} - u_{yy} = 0$, which is a linear hyperbolic PDE.

Proposition 3. *Transformation (8) transforms an arbitrary solution $u = \Phi(x, y, t)$ of the nonlinear PDE (4) into a fourteen-parameter family of solutions*

$$\begin{aligned} u = \frac{D_1^2}{(A_1B_2 - A_2B_1)^2}\big[&\Phi(A_1x + B_1y + C_1, A_2x + B_2y + C_2, D_1t + D_2) \\ &- t(A_3x + B_3y + C_3) - A_4x - B_4y - C_4\big]. \end{aligned} \tag{11}$$

The reproduction formula (11) makes it possible to obtain complex multiparameter solutions using the more simple solutions. Note that in Formula (11), the free parameters can take complex values, provided that the solutions obtained are real (see details [108]). Section 4 provides examples of using this approach.

3. Two-Dimensional Similarity Reductions

The classical procedure for finding symmetry reductions in PDEs is presented in [62,64]. In this section, we will limit ourselves to a brief description of the most important cases of constructing two-dimensional reductions for the Monge–Ampère-type PDE with three independent variables (4) using the symmetries found in Section 2.

1°. Equation (4) admits a symmetry solution of the form

$$u = U(\varrho, \vartheta), \quad \varrho = x - \alpha t, \quad \vartheta = y - \beta t, \tag{12}$$

where α and β are free parameters, ϱ and ϑ are traveling wave-type variables, and the new desired function $U = U(\varrho, \vartheta)$ satisfies the two-dimensional Monge–Ampère PDE

$$U_{\varrho\varrho}U_{\vartheta\vartheta} - U_{\varrho\vartheta}^2 - \alpha^2 U_{\varrho\varrho} - 2\alpha\beta U_{\varrho\vartheta} - \beta^2 U_{\vartheta\vartheta} = 0. \tag{13}$$

The symmetry solution (12) is invariant under the transformation group, which is specified by the operator

$$Y = \alpha X_1 + \beta X_2 + X_3 = \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y} + \frac{\partial}{\partial t}.$$

The PDE (13) admits exact solutions, quadratic in one independent variable, of the form

$$\begin{aligned} U_1 &= f_1(\varrho)\vartheta^2 + g_1(\varrho)\vartheta + h_1(\varrho), \\ U_2 &= f_2(\vartheta)\varrho^2 + g_2(\vartheta)\varrho + h_2(\vartheta), \end{aligned} \tag{14}$$

where the functions f_i, g_i, h_i ($i = 1, 2$) are described by ODE systems that are not presented here.

Remark 2. *The successive use of Formulas (12) and (14) leads to non-invariant solutions of Equation (4), obtained by combining the classical method of group analysis and the method of generalized separation of variables.*

Remark 3. *Generalized separable solutions that are quadratic in one or more independent variables, like solutions (14), are often used to construct exact solutions of reaction–diffusion equations, wave-type equations, and some other nonlinear partial differential equations (see, e.g., [16,60,61,73]).*

2°. Equation (4) admits a symmetry solution of the form

$$u = t^{-2(\alpha+\beta+1)}U(\varrho, \vartheta), \quad \varrho = xt^{\alpha}, \quad \vartheta = yt^{\beta}, \tag{15}$$

where α and β are free parameters, ϱ and ϑ are self-similar variables, and the new desired function $U = U(\varrho, \vartheta)$ satisfies the two-dimensional Monge–Ampère PDE

$$\begin{aligned} U_{\varrho\vartheta}^2 - U_{\varrho\varrho}U_{\vartheta\vartheta} + \alpha^2\varrho^2 U_{\varrho\varrho} + \beta^2\vartheta^2 U_{\vartheta\vartheta} + 2\alpha\beta\varrho\vartheta U_{\varrho\vartheta} - \alpha\varrho(3\alpha + 4\beta + 5)U_{\varrho} \\ - \beta\vartheta(4\alpha + 3\beta + 5)U_{\vartheta} + 2(2\alpha + 2\beta + 3)(\alpha + \beta + 1)U = 0\,. \end{aligned} \tag{16}$$

The values of $\alpha = \beta = 0$ in (15) correspond to a multiplicative separable solution.

The symmetry solution (15) is invariant under the transformation group, which is specified by the operator

$$Y = \alpha X_9 + \beta X_{10} - (\alpha + \beta + 1)X_{11} = \alpha x\frac{\partial}{\partial x} + \beta y\frac{\partial}{\partial y} - t\frac{\partial}{\partial t} + 2(\alpha + \beta + 1)u\frac{\partial}{\partial u}\,.$$

The PDE (16) admits non-invariant solutions of the form (14), quadratic in one independent variable.

3°. Equation (4) admits a symmetry solution of the form

$$u = e^{-2(\alpha+\beta)t}U(\varrho, \vartheta), \quad \varrho = xe^{\alpha t}, \quad \vartheta = ye^{\beta t}, \tag{17}$$

where α and β are free parameters, ϱ and ϑ are limit self-similar variables, and the new desired function $U = U(\varrho, \vartheta)$ satisfies the two-dimensional Monge–Ampère PDE

$$\begin{aligned} U_{\varrho\vartheta}^2 - U_{\varrho\varrho}U_{\vartheta\vartheta} + \alpha^2\varrho^2 U_{\varrho\varrho} + \beta^2\vartheta^2 U_{\vartheta\vartheta} + 2\alpha\beta\varrho\vartheta U_{\varrho\vartheta} \\ - \alpha\varrho(3\alpha + 4\beta)U_{\varrho} - \beta\vartheta(4\alpha + 3\beta)U_{\vartheta} + 4(\alpha + \beta)^2 U = 0\,. \end{aligned} \tag{18}$$

The symmetry solution (17) is invariant under the transformation group, which is specified by the operator

$$Y = -X_3 + \alpha X_9 + \beta X_{10} = \alpha x\frac{\partial}{\partial x} + \beta y\frac{\partial}{\partial y} - \frac{\partial}{\partial t} + 2(\alpha + \beta)u\frac{\partial}{\partial u}\,.$$

The PDE (18) admits non-invariant solutions of the form (14), quadratic in one independent variable.

4°. Equation (4) admits a symmetry solution of the form

$$u = t^{-2}U(\varrho, \vartheta), \quad \varrho = x + \alpha\ln t, \quad \vartheta = y + \beta\ln t, \tag{19}$$

where α and β are free parameters, and the new desired function $U = U(\varrho, \vartheta)$ satisfies the two-dimensional Monge–Ampère PDE

$$U_{\varrho\varrho}U_{\vartheta\vartheta} - U_{\varrho\vartheta}^2 - \alpha^2 U_{\varrho\varrho} - \beta^2 U_{\vartheta\vartheta} - 2\alpha\beta U_{\varrho\vartheta} + 5\alpha U_{\varrho} + 5\beta U_{\vartheta} - 6U = 0. \tag{20}$$

The values of $\alpha = \beta = 0$ in (19) correspond to a multiplicative separable solution.

The symmetry solution (19) is invariant under the transformation group, which is specified by the operator

$$Y = \alpha X_1 + \beta X_2 - X_{11} = \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} - t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u}.$$

The PDE (20) has a traveling wave solution, as well as non-invariant solutions of the form (14), quadratic in one independent variable.

5°. Equation (4) admits a symmetry solution of the form

$$u = x^2 U(\varrho, \vartheta), \quad \varrho = t + \alpha \ln x, \quad \vartheta = y + \beta \ln x, \tag{21}$$

where α and β are free parameters, and the new desired function $U = U(\varrho, \vartheta)$ satisfies the two-dimensional Monge–Ampère PDE

$$\alpha^2(U_{\varrho\varrho}U_{\vartheta\vartheta} - U_{\varrho\vartheta}^2) - U_{\varrho\varrho} + (3\alpha U_\varrho - \beta U_\vartheta + 2U)U_{\vartheta\vartheta} - 4\alpha U_\vartheta U_{\varrho\vartheta} - 4U_\vartheta^2 = 0. \tag{22}$$

The values of $\alpha = \beta = 0$ in (21) correspond to a multiplicative separable solution.

The symmetry solution (21) is invariant under the transformation group, which is specified by the operator

$$Y = -\beta X_2 - \alpha X_3 + X_9 - X_{11} = x\frac{\partial}{\partial x} - \beta\frac{\partial}{\partial y} - \alpha\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u}.$$

The PDE (22) has a traveling wave solution.

6°. Equation (4) admits a symmetry solution of the form

$$u = e^{2(\alpha-\beta)x} U(\varrho, \vartheta), \quad \varrho = te^{\alpha x}, \quad \vartheta = ye^{\beta x}, \tag{23}$$

where α and β are free parameters, and the new desired function $U = U(\varrho, \vartheta)$ satisfies the two-dimensional Monge–Ampère PDE

$$\begin{aligned}\alpha^2\varrho^2(U_{\varrho\varrho}U_{\vartheta\vartheta} - U_{\varrho\vartheta}^2) - U_{\varrho\varrho} + \left[\alpha(5\alpha - 4\beta)\varrho U_\varrho - \beta^2\vartheta U_\vartheta + 4(\alpha-\beta)^2 U\right]U_{\vartheta\vartheta} \\ - 2\alpha(2\alpha-\beta)\varrho U_\vartheta U_{\varrho\vartheta} - (2\alpha-\beta)^2 U_\vartheta^2 = 0.\end{aligned} \tag{24}$$

The symmetry solution (23) is invariant under the transformation group, which is specified by the operator

$$Y = X_1 - \beta X_{10} - (\alpha - \beta)X_{11} = \frac{\partial}{\partial x} - \beta y\frac{\partial}{\partial y} - \alpha t\frac{\partial}{\partial t} + 2(\alpha-\beta)u\frac{\partial}{\partial u}.$$

7°. Equation (4) admits a symmetry solution of the form

$$u = e^{-2\alpha x} U(\varrho, \vartheta), \quad \varrho = x + \beta t, \quad \vartheta = ye^{\alpha x}, \tag{25}$$

where α and β are free parameters, and the new desired function $U = U(\varrho, \vartheta)$ satisfies the two-dimensional Monge–Ampère PDE

$$U_{\varrho\varrho}U_{\vartheta\vartheta} - U_{\varrho\vartheta}^2 - \beta^2 U_{\varrho\varrho} - \alpha(4U_\varrho + \alpha\vartheta U_\vartheta - 4\alpha U)U_{\vartheta\vartheta} + 2\alpha U_\vartheta U_{\varrho\vartheta} - \alpha^2 U_\vartheta^2 = 0. \tag{26}$$

The symmetry solution (25) is invariant under the transformation group, which is specified by the operator

$$Y = \beta X_1 - X_3 - \alpha\beta X_{10} + \alpha\beta X_{11} = \beta\frac{\partial}{\partial x} - \alpha\beta y\frac{\partial}{\partial y} - \frac{\partial}{\partial t} - 2\alpha\beta u\frac{\partial}{\partial u}.$$

8°. Equation (4) admits a symmetry solution of the form

$$u = U(z,t), \quad z = y + ax^2, \tag{27}$$

where a is a free parameter, and the new desired function $U = U(z,t)$ satisfies the two-dimensional Monge–Ampère PDE

$$U_{tt} - 2aU_z U_{zz} = 0.$$

Exact solutions of this equation are discussed further in Section 10.

The symmetry solution (27) is invariant under the transformation group, which is specified by the operator

$$Y = X_1 + X_3 - 2aX_6 = \frac{\partial}{\partial t} + \frac{\partial}{\partial x} - 2ax\frac{\partial}{\partial y}.$$

9°. Equation (4) admits a symmetry solution of the form

$$u = U(r,t), \quad r = \sqrt{x^2 + y^2}, \tag{28}$$

where r is the polar radius, and the new desired function $U = U(r,t)$ satisfies the two-dimensional Monge–Ampère PDE

$$U_{tt} - r^{-1}U_r U_{rr} = 0.$$

Exact solutions of this equation are discussed further in Items 3°–7° of Section 11.

The physical meaning of solution (28) is that it is invariant with respect to the rotation transformation of spatial variables (9) (i.e., this solution is spatially isotropic). Solutions with the same property are typical for many elliptic PDEs, in particular for the Laplace equation (see also Corollary 1).

The symmetry solution (28) is invariant under the transformation group, which is specified by the operator

$$Y = X_3 + X_5 - X_6 = \frac{\partial}{\partial t} + y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

10°. Equation (4) admits a symmetry solution of the form

$$u = U(\zeta,t), \quad \zeta = \sqrt{x^2 - y^2}, \tag{29}$$

where the new desired function $U = U(\zeta,t)$ satisfies the two-dimensional Monge–Ampère PDE

$$U_{tt} + \zeta^{-1}U_\zeta U_{\zeta\zeta} = 0.$$

Renaming, here, U to $-U$ and ζ to r, we obtain the equation from Item 8°.

The physical meaning of solution (29) is that it is invariant with respect to the Lorentz transformation of spatial variables (10). Solutions with the same wave property are typical for many hyperbolic PDEs (see also Corollary 2).

The symmetry solution (29) is invariant under the transformation group, which is specified by the operator

$$Y = X_3 + X_5 + X_6 = \frac{\partial}{\partial t} + y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

11°. Equation (4) admits a symmetry solution of the form

$$u = U(\eta, t), \quad \eta = xy, \tag{30}$$

where the new desired function $U = U(\eta, t)$ satisfies the two-dimensional Monge–Ampère PDE

$$U_{tt} + 2\eta U_\eta U_{\eta\eta} + U_\eta^2 = 0.$$

Solution (30) is self-similar and symmetric with respect to the spatial variables x and y. It also is invariant under the transformation group, which is specified by the operator

$$Y = X_3 + X_9 - X_{10} = \frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

Remark 4. *More complicated solutions of the highly nonlinear PDE* (4) *can be obtained by applying the reproduction formula* (11) *to the solutions* (12), (15), (17), (19), (21), (23), (25), (27), (28), (29), *and* (30).

Remark 5. *From the solution forms described above in Items 9° and 10°, it follows that that the considered highly nonlinear PDE* (4) *simultaneously has properties characteristic of both elliptic and hyperbolic equations. This is a very extraordinary property, not shared by second-order linear and quasilinear PDEs (see, for example, handbooks [16,18,22,119]). Another unusual property of Equation* (4)*, which is invariant under arbitrary constant shifts with respect to all independent variables, is that it has no non-degenerate traveling wave solutions of the form $u = F(k_1x + k_2y - \lambda t)$, where k_1, k_2, and λ are free parameters.*

4. One-Dimensional Similarity Reductions and Invariant Solutions

The regular technique for obtaining one-dimensional similarity reductions in PDEs is described in [62–64]. In this section, we restrict ourselves to several illustrative examples of constructing invariant solutions of the Monge–Ampère-type PDE (4) using the symmetries described above. We will also give a few simple solutions of this PDE in terms of elementary functions.

1°. The simplest invariant solution of the nonlinear PDE (4), which admits a scaling transformation, is a multiplicative separable solution of the form

$$u = -\frac{x^2y^2}{2t^2}. \tag{31}$$

Below, we consider some other invariant solutions that can be obtained from solution (31), applying simple methods outlined in [61,107,108].

Solution (31) is a special case of a broader family of invariant solutions of the following form:

$$u = \frac{x^2}{t^2} f(z), \quad z = y + \beta \ln t, \tag{32}$$

where β is a free parameter, and the function $f = f(z)$ satisfies the second-order ODE

$$(2f - \beta^2) f''_{zz} - 4(f'_z)^2 + 5\beta f'_z - 6f = 0.$$

Solution (32) is invariant under the two-dimensional Lie algebra of symmetry operators

$$Y_1 = \beta X_2 - X_{11} = \beta\frac{\partial}{\partial y} - t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u}, \qquad Y_2 = X_9 - X_{11} = x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}.$$

Solution (31) is a special case of another broader family of invariant solutions of the form

$$u = \frac{x^2}{t^2} g(\xi), \quad \xi = y + \lambda \ln x, \tag{33}$$

where λ is a free parameter, and the function $g = g(\xi)$ satisfies the second-order ODE

$$(\lambda g'_\xi - 2g) g''_{\xi\xi} + 4(g'_\xi)^2 + 6g = 0.$$

Solution (33) is invariant under the two-dimensional Lie algebra of symmetry operators

$$Y_1 = \lambda X_2 - X_9 = \lambda \frac{\partial}{\partial y} - x \frac{\partial}{\partial x} - t \frac{\partial}{\partial t}, \qquad Y_2 = X_{11} = t \frac{\partial}{\partial t} - 2u \frac{\partial}{\partial u}.$$

Solution (31) is also a special case of another broader family of invariant solutions of the form

$$u = x^2 y^2 h(\eta), \quad \eta = t + \gamma \ln y, \tag{34}$$

where γ is a free parameter, and the function $h = h(\eta)$ satisfies the second-order ODE

$$(2\gamma^2 h - 1) h''_{\eta\eta} - 4\gamma^2 (h'_\eta)^2 - 10\gamma h h'_\eta - 12h^2 = 0.$$

Solution (34) is invariant under the two-dimensional Lie algebra of symmetry operators

$$Y_1 = \gamma X_3 - X_{10} = \gamma \frac{\partial}{\partial t} - y \frac{\partial}{\partial y} + 2u \frac{\partial}{\partial u}, \qquad Y_2 = X_9 - X_{11} = x \frac{\partial}{\partial x} + 2u \frac{\partial}{\partial u}.$$

Applying Formula (11) with $A_1 = A_2 = B_2 = D_1 = 1$, $B_1 = -1$, $A_3 = A_4 = B_3 = B_4 = C_1 = C_2 = D_2 = C_3 = C_4 = 0$ to solution (31), we obtain a solution of the more complex form

$$u = -\frac{(x^2 - y^2)^2}{8t^2}. \tag{35}$$

Following [107,108], we construct, using a complex parameter, another solution based on solution (35). The PDE (4) is invariant under the transformation

$$\bar{x} = ix, \qquad \bar{y} = y, \qquad \bar{t} = t, \qquad \bar{u} = -u,$$

where $i^2 = -1$ (this corresponds to the use of a purely imaginary parameter $A_1 = i$ in the reproduction Formula (11)). Using this complex transformation, we obtain from solution (35) another solution to the PDE (4):

$$u = \frac{(x^2 + y^2)^2}{8t^2}. \tag{36}$$

2°. Using the invariant variables

$$u = x^{2-2\beta} t^{-2\alpha-2} V(\zeta), \quad \zeta = x^\beta t^\alpha y, \tag{37}$$

where α and β are free parameters, we obtain from the PDE (4) the second-order nonlinear ODE

$$\begin{aligned}&\left[\beta(\beta+1)\zeta V'_\zeta - 2(\beta-1)(2\beta-1)V + \alpha^2\zeta^2\right] V''_{\zeta\zeta} \\ &\qquad + (\beta-2)^2 (V'_\zeta)^2 - \alpha(3\alpha+5)\zeta V'_\zeta + 2(\alpha+1)(2\alpha+3)V = 0.\end{aligned}$$

Solution (37) is invariant under the two-dimensional Lie algebra of symmetry operators

$$Y_1 = \alpha X_9 - (\alpha+\beta)X_{11} = \alpha x\frac{\partial}{\partial x} - \beta t\frac{\partial}{\partial t} + 2(\alpha+\beta)u\frac{\partial}{\partial u},$$
$$Y_2 = \alpha X_{10} - (\alpha+1)X_{11} = \alpha y\frac{\partial}{\partial y} - t\frac{\partial}{\partial t} + 2(\alpha+1)u\frac{\partial}{\partial u}.$$

3°. Using the invariant variables

$$u = e^{-2\alpha t}x^{2-2\beta}V(\zeta), \quad \zeta = e^{\alpha t}x^{\beta}y, \tag{38}$$

where α and β are free parameters, and we obtain from the PDE (38) the second-order nonlinear ODE

$$\left[\beta(\beta+1)\zeta V'_\zeta - 2(\beta-1)(2\beta-1)V + \alpha^2\zeta^2\right]V''_{\zeta\zeta} + (\beta-2)^2(V'_\zeta)^2 - 3\alpha^2\zeta V'_\zeta + 4\alpha^2 V = 0.$$

Solution (38) is invariant under the two-dimensional Lie algebra of symmetry operators

$$Y_1 = X_9 - \beta X_{10} + (\beta-1)X_{11} = x\frac{\partial}{\partial x} - \beta y\frac{\partial}{\partial y} - 2(\beta-1)u\frac{\partial}{\partial u},$$
$$Y_2 = -X_3 + \alpha X_{10} - \alpha X_{11} = \alpha y\frac{\partial}{\partial y} - \frac{\partial}{\partial t} + 2\alpha u\frac{\partial}{\partial u}.$$

4°. Using the invariant variables

$$u = t^{-2\alpha-2}e^{-2\beta x}V(\zeta), \quad \zeta = t^{\alpha}e^{\beta x}y, \tag{39}$$

where α and β are free parameters, we obtain from the PDE (39) the second-order nonlinear ODE

$$(\beta^2\zeta V'_\zeta - 4\beta^2 V + \alpha^2\zeta^2)V''_{\zeta\zeta} + \beta^2(V'_\zeta)^2 - \alpha(3\alpha+5)\zeta V'_\zeta + 2(\alpha+1)(2\alpha+3)V = 0.$$

Solution (39) is invariant under the two-dimensional Lie algebra of symmetry operators

$$Y_1 = X_1 - \beta X_{10} + \beta X_{11} = \frac{\partial}{\partial x} - \beta y\frac{\partial}{\partial y} - 2\beta u\frac{\partial}{\partial u},$$
$$Y_2 = \alpha X_{10} - (\alpha+1)X_{11} = \alpha y\frac{\partial}{\partial y} - t\frac{\partial}{\partial t} + 2(\alpha+1)u\frac{\partial}{\partial u}.$$

5°. Using the invariant variables

$$u = e^{-2\alpha t-2\beta x}V(\zeta), \quad \zeta = e^{\alpha t+\beta x}y, \tag{40}$$

where α and β are free parameters, we obtain from the PDE (40) the second-order nonlinear ODE

$$(\beta^2\zeta V'_\zeta - 4\beta^2 V + \alpha^2\zeta^2)V''_{\zeta\zeta} + \beta^2(V'_\zeta)^2 - 3\alpha^2\zeta V'_\zeta + 4\alpha^2 V = 0.$$

Solution (39) is invariant under the two-dimensional Lie algebra of symmetry operators

$$Y_1 = X_1 - \beta X_{10} + \beta X_{11} = \frac{\partial}{\partial x} - \beta y\frac{\partial}{\partial y} - 2\beta u\frac{\partial}{\partial u},$$
$$Y_2 = X_3 - \alpha X_{10} + \alpha X_{11} = \frac{\partial}{\partial t} - \alpha y\frac{\partial}{\partial y} - 2\alpha u\frac{\partial}{\partial u}.$$

Remark 6. *More complicated solutions of the highly nonlinear PDE* (4) *can be obtained by applying the reproduction Formula* (11) *to the solutions* (33), (34), (37), (38), (39), *and* (40).

5. Multiplicative Separable Solutions

1°. The Monge–Ampère-type PDE (4) has the multiplicative separable solution

$$u = t^{-2} U(x, y),$$

which is a particular case of solution (15) for $\alpha = \beta = 0$. Here, the new desired function $U = U(x, y)$ satisfies the stationary Monge–Ampère equation

$$U_{xx} U_{yy} - U_{xy}^2 - 6U = 0. \tag{41}$$

Proposition 4. *Let $U = F(x, y)$ be a solution of Equation* (41). *Then, the function*

$$U = \frac{1}{(a_1 b_2 - a_2 b_1)^2} F(a_1 x + b_1 y + c_1, a_2 x + b_2 y + c_2),$$

where a_j, b_j, c_j ($j = 1, 2$) are arbitrary constants, is also a solution of this PDE.

2°. The PDE (41), in turn, admits the multiplicative separable solution

$$U = x^2 \varphi(y), \tag{42}$$

where the function $\varphi = \varphi(y)$ satisfies the autonomous second-order ODE

$$\varphi \varphi''_{yy} - 2(\varphi'_y)^2 - 3\varphi = 0. \tag{43}$$

Remark 7. *Solution* (42) *is invariant under a one-parameter group of transformations which is specified by the operator*

$$Y = x \frac{\partial}{\partial x} + 2U \frac{\partial}{\partial U},$$

admitted by Equation (41).

Substituting $Z(\varphi) = (\varphi'_y)^2$ reduces Equation (43) to the first-order linear ODE

$$\varphi Z'_\varphi - 4Z - 6\varphi = 0,$$

the general solution of which is

$$Z = C_1 \varphi^4 - 2\varphi, \tag{44}$$

where C_1 is an arbitrary constant. Replacing Z in (44) by $(\varphi'_y)^2$, we obtain a first-order separable ODE whose general solution can be written in implicit form.

Remark 8. *The ODE* (43) *has a one-parameter particular solution $\varphi = -\frac{1}{2}(y + C_2)^2$, which corresponds to the case $C_1 = 0$ in* (44).

3°. The PDE (41) has the solution

$$U = x^2 \varphi(\xi), \qquad \xi = y + \gamma \ln x, \tag{45}$$

where γ is a free parameter, and the function $\varphi = \varphi(\xi)$ satisfies the autonomous second-order ODE

$$(\gamma \varphi'_\xi - 2\varphi)\varphi''_{\xi\xi} + 4(\varphi'_\xi)^2 + 6\varphi = 0.$$

Remark 9. *Solution (45) is invariant under a one-parameter group of transformations which is specified by the operator*

$$Y = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + 2U\frac{\partial}{\partial U},$$

admitted by Equation (41).

4°. Equation (41) has a generalized separable solution that is quadratic with respect to any independent variable, e.g.,

$$U = x^2\varphi(y) + x\psi(y) + \chi(y), \tag{46}$$

where the functions $\varphi = \varphi(y)$, $\psi = \psi(y)$, and $\chi = \chi(y)$ are described by the following system of ODEs:

$$\begin{aligned} &\varphi\varphi''_{yy} - 2(\varphi'_y)^2 - 3\varphi = 0, \\ &\varphi\psi''_{yy} - 2\varphi'_y\psi'_y - 3\psi = 0, \\ &2\varphi\chi''_{yy} - (\psi'_y)^2 - 6\chi = 0. \end{aligned} \tag{47}$$

The non-invariant solution (46) is a generalization of solution (42). It can be seen that the first equation of system (47) coincides with the nonlinear ODE (43), and the second and third equations are linear with respect to the sought functions.

It is not difficult to prove the following proposition.

Proposition 5. *Let $\varphi = \varphi(y)$ be a solution of the first Equation (47). Then, the corresponding general solution of the second Equation (47) is given by the formula*

$$\psi = C_1\varphi + C_2 y\varphi, \tag{48}$$

where C_1 and C_2 are arbitrary constants.

Note that system (47) admits a particular solution:

$$\begin{aligned} &\varphi = -\tfrac{1}{2}y^2, \quad \psi = 0, \\ &\chi = \sqrt{y}\big[A_1\cos\big(\tfrac{\sqrt{23}}{2}\ln y\big) + A_2\sin\big(\tfrac{\sqrt{23}}{2}\ln y\big)\big], \end{aligned} \tag{49}$$

where A_1 and A_2 are arbitrary constants.

Remark 10. *Using Formula (48), it is possible to obtain a more complex solution of system (46) than (49) with the same function $\varphi = -\frac{1}{2}y^2$, namely the following:*

$$\begin{aligned} &\varphi = -\tfrac{1}{2}y^2, \\ &\psi = C_1 y^2 + C_3 y^3, \\ &\chi = \sqrt{y}\big[C_3\cos\big(\tfrac{\sqrt{23}}{2}\ln y\big) + C_4\sin\big(\tfrac{\sqrt{23}}{2}\ln y\big)\big] - \tfrac{1}{2}y^2(C_2 y + C_1), \end{aligned}$$

where $C_1, \ldots, C_4$ are arbitrary constants.

5°. Equation (4) admits the degenerate multiplicative separable solution

$$u = (At + B)W(x, y),$$

where A and B are arbitrary constants, and the function $W = W(x, y)$ is any solution of the homogeneous Monge–Ampère Equation (1) with $f(x, y) \equiv 0$.

6. Reductions with Additive and Generalized Separation of Variables Leading to Two-Dimensional Monge–Ampère Equations. Exact Solutions

1°. The Monge–Ampère-type PDE (4) has additive separable solutions

$$u = -\tfrac{1}{2}At^2 + Bt + w(x, y), \tag{50}$$

where A and B are arbitrary constants, and the function w satisfies the stationary nonhomogeneous Monge–Ampère equation with a constant right-hand side:

$$w_{xx}w_{yy} - w_{xy}^2 = -A. \tag{51}$$

The qualitative features of the PDE (51) depend on the sign of the constant A, since for $A > 0$, this PDE is a hyperbolic equation, and for $A < 0$, it is an elliptic equation [1,2,16].

2°. It is easy to verify that Equation (4) admits an additive separable solution of the form (50), which is expressed in elementary functions

$$u = C_1x^2 + C_2xy + C_3y^2 + C_4x + C_5y + \tfrac{1}{2}(4C_1C_3 - C_2^2)t^2 + C_6t + C_7,$$

where $C_1, \ldots, C_7$ are arbitrary constants.

3°. Using the results of [16], one can obtain, for example, the following exact non-invariant solutions of the form (50) of Equation (4):

$$\begin{aligned}
u &= -\frac{1}{2}A^2t^2 + Bt \pm \frac{A}{C_2}x(C_1x + C_2y) + \Phi(C_1x + C_2y + C_3) + C_4x + C_5y + C_6,\\
u &= -\frac{1}{2}At^2 + Bt + \frac{1}{x + C_1}\left(C_2y^2 + C_3y + \frac{C_3^2}{4C_2}\right) - \frac{A}{12C_2}(x^3 + 3C_1x^2) + C_4x + C_5y + C_6,\\
u &= -\frac{1}{2}A^2t^2 + Bt \pm \frac{2A}{3C_1C_2}(C_1x - C_2^2y^2 + C_3)^{3/2} + C_4x + C_5y + C_6,
\end{aligned}$$

where A, B, $C_1, \ldots, C_6$ are arbitrary constants, and $\Phi = \Phi(\zeta)$ is an arbitrary function.

Remark 11. *For $A > 0$, the general solution of the nonhomogeneous Monge–Ampère PDE (51) can be represented in parametric form [16,17]:*

$$x = \frac{\alpha - \beta}{2\sqrt{A}}, \quad y = \frac{\Psi'(\beta) - \Phi'(\alpha)}{2\sqrt{A}},$$
$$w = \frac{(\alpha + \beta)[\Psi'(\beta) - \Phi'(\alpha)] + 2\Phi(\alpha) - 2\Psi(\beta)}{4\sqrt{A}},$$

where α and β are arbitrary constants, and $\Phi = \Phi(\alpha)$ and $\Psi = \Psi(\beta)$ are arbitrary functions.

For $A < 0$, the nonlinear PDE (51) is reduced by the Euler contact transformation to a second-order linear PDE with constant coefficients [24], which is reduced to the Laplace equation by scaling new independent variables.

4°. Equation (4) admits a more complex, than (50), generalized separable solution:

$$u = -\tfrac{1}{2}(a_1x + b_1y + c_1)t^2 + (a_2x + b_2y + c_2)t + w(x, y),$$

where a_1, a_2, b_1, b_2, c_1, c_2 are arbitrary constants, and the function w satisfies the stationary Monge–Ampère PDE with a nonhomogeneous right-hand side,

$$w_{xx}w_{yy} - w_{xy}^2 = -a_1x - b_1y - c_1. \tag{52}$$

For $b_1 = c_1 = 0$, the PDE (52) has generalized separable solutions:

$$w = \pm\frac{2}{3a_1}y(a_1x)^{3/2} + C_1y + \Phi(x),$$
$$w = C_1y^2 + C_2xy + C_3y - \frac{a_1}{12C_1}x^3 + \frac{C_2^2}{4C_1}x^2 + C_4x + C_5,$$
$$w = C_1\frac{y^2}{x} + C_2y - \frac{a_1}{24C_1}x^4 + C_3x + C_4,$$

where $\Phi(x)$ is an arbitrary function, and $C_1, \dots, C_5$ are arbitrary constants.

7. Reductions with Generalized Separation of Variables Leading to Linear PDEs

1°. The Monge–Ampère-type PDE (4) admits the generalized separable solution

$$u = \tfrac{1}{2}ay^2 + bxy + \tfrac{1}{2}cx^2 + dy + \tfrac{1}{2}(ac - b^2)t^2 + kyt + U(x,t), \tag{53}$$

where a, b, c, d, and k are arbitrary constants, and the new desired function $U = U(x,t)$ satisfies a constant-coefficient linear PDE:

$$U_{tt} = aU_{xx}. \tag{54}$$

For $a > 0$, the PDE (54) is the classical linear wave equation, the general solution of which has the form

$$U = \Phi\big(x - \sqrt{a}\,t\big) + \Psi\big(x + \sqrt{a}\,t\big), \tag{55}$$

where $\Phi(z_1)$ and $\Psi(z_2)$ are arbitrary functions.

For $a < 0$, the PDE (54) is an elliptic equation that reduces to the Laplace equation by substituting $z = \sqrt{-a}\,t$ (for solutions to this linear PDE, see, for example, [119,120]).

Below are several simple solutions to Equation (54) for $a = -1$, which are expressed in elementary functions:

$$U = A(x^3 - 3xt^2) + B(3x^2t - t^3),$$
$$U = \frac{Ax + Bt}{x^2 + t^2} + C,$$
$$U = (Ae^{\mu x} + Be^{-\mu x})(C\cos\mu t + D\sin\mu t),$$
$$U = (A\cos\mu x + B\sin\mu x)(Ce^{\mu t} + De^{-\mu t}),$$
$$U = A\ln\big[(x - x_0)^2 + (t - t_0)^2\big] + B,$$

where A, B, C, D, x_0, y_0, and μ are arbitrary constants.

Remark 12. *Setting $A = 0$, $B = 1/6$ in the first solution for U, we obtain solution (6).*

2°. Equation (4) also admits another generalized separable solution

$$u = \tfrac{1}{2}(at + b)y^2 + (ct + d)y + U(x,t), \tag{56}$$

where a, b, c, and d are arbitrary constants, and the function $U = U(x,t)$ satisfies a linear PDE with a variable coefficient:

$$U_{tt} = (at + b)U_{xx}. \tag{57}$$

The PDE (57) is an equation of mixed type, which is hyperbolic for $at + b > 0$, and elliptic for $at + b < 0$ (i.e., when passing t through the point $t_0 = -b/a$, the type of the reduced PDE changes).

Below are some exact solutions of the Tricomi equation [119]:

$$U_{tt} + tU_{xx} = 0,$$

which describes near-sonic flows of gas and is a special case of the PDE (57) for $a = -1$ and $b = 0$.

1. Simple exact solutions:

$$U = A(3x^2 - t^3),$$
$$U = B(x^3 - xt^3),$$
$$U = C(6tx^2 - t^4),$$

where A, B, and C are arbitrary constants.

2. Generalized separable solutions with even powers of x [119]:

$$U = \sum_{k=0}^{n} f_k(t)x^{2k},$$

where the functions $f_k = f_k(t)$ are defined by the recurrence relations

$$f_n(t) = A_n t + B_n, \quad f_{k-1}(t) = A_k t + B_k - 2k(2k-1)\int_0^t (t-s)sf_k(s)\,ds,$$

where A_k, B_k are arbitrary constants ($k = n, \dots, 1$).

3. Generalized separable solutions with odd powers of x [119]:

$$U = \sum_{k=0}^{n} g_k(t)x^{2k+1},$$

where the functions $g_k = g_k(t)$ are defined by the recurrence relations

$$g_n(t) = A_n t + B_n, \quad g_{k-1}(t) = A_k t + B_k - 2k(2k+1)\int_0^t (t-s)sg_k(s)\,ds,$$

where A_k, B_k are arbitrary constants ($k = n, \dots, 1$).

4. Multiplicative separable solutions:

$$U = \sqrt{t}\,\big[C_1 J_{1/3}(2\lambda t^{3/2}) + C_2 Y_{1/3}(2\lambda t^{3/2})\big]\big[C_3\sinh(3\lambda x) + C_4\cosh(3\lambda x)\big],$$
$$U = \sqrt{t}\,\big[C_1 I_{1/3}(2\lambda t^{3/2}) + C_2 K_{1/3}(2\lambda t^{3/2})\big]\big[C_3\sin(3\lambda x) + C_4\cos(3\lambda x)\big],$$

where C_1, C_2, C_3, C_4, and λ are arbitrary constants, $J_{1/3}(z)$ and $Y_{1/3}(z)$ are Bessel functions, and $I_{1/3}(z)$ and $K_{1/3}(z)$ are modified Bessel functions.

8. Polynomial Solutions in One Spatial Variable

1°. The Monge–Ampère-type PDE (4) has generalized separable solutions that are quadratic in any spatial variable (x and y can be interchanged):

$$u = y^2 f(x,t) + yg(x,t) + h(x,t), \tag{58}$$

where the functions $f = f(x,t)$, $g = g(x,t)$, and $h = h(x,t)$ are described by the following PDE system:

$$\begin{aligned} &f_{tt} - 2ff_{xx} + 4f_x^2 = 0, \\ &g_{tt} - 2fg_{xx} + 4f_x g_x = 0, \\ &h_{tt} - 2fh_{xx} + g_x^2 = 0. \end{aligned} \tag{59}$$

Remark 13. *Solution* (53)*, leading to the linear PDE* (54)*, is a special case of solution* (58) *with*

$$f(x,t) = \tfrac{1}{2}a, \quad g(x,t) = bx + d + kt, \quad h(x,t) = \tfrac{1}{2}cx^2 + \tfrac{1}{2}(ac - b^2)t^2 + U(x,t).$$

2°. The last two PDEs of the system (59) have a simple solution $g = C_2 t + C_3$, $h = C_4 t + C_5$, where C_2, C_3, C_4, and C_5 are arbitrary constants. It is not difficult to prove a more general proposition.

Proposition 6. *Let $f = f(x,t)$ be any solution to the first PDE* (59)*. Then, the last two PDEs of system* (59) *admit a particular solution:*

$$g = C_1 f + C_2 t + C_3, \quad h = \tfrac{1}{4}C_1^2 f + C_4 t + C_5, \tag{60}$$

where $C_1, \ldots, C_5$ are arbitrary constants.

3°. The simplest solution to the first PDE of system (59) is

$$f = \tfrac{1}{2}a, \tag{61}$$

where a is an arbitrary constant. In this case, for $a > 0$, the last two PDEs of system (59) are linear wave equations

$$\begin{aligned} &g_{tt} - a g_{xx} = 0, \\ &h_{tt} - a h_{xx} + g_x^2 = 0. \end{aligned} \tag{62}$$

The first is homogeneous, and the second is nonhomogeneous. To obtain a solution to system (62), we move on to new characteristic variables

$$\xi = x - \sqrt{a}\,t, \quad \eta = x + \sqrt{a}\,t.$$

As a result, we receive

$$\begin{aligned} &g_{\xi\eta} = 0, \\ &4ah_{\xi\eta} - (g_\xi + g_\eta)^2 = 0. \end{aligned}$$

Sequentially integrating these PDEs, we find the general solution of system (62):

$$\begin{aligned} &g = \varphi_1(\xi) + \psi_1(\eta), \quad \xi = x - \sqrt{a}\,t, \quad \eta = x + \sqrt{a}\,t, \\ &h = \varphi_2(\xi) + \psi_2(\eta) + \frac{1}{2a}\varphi_1(\xi)\psi_1(\eta) + \frac{1}{4a}\eta \int \left(\frac{d\varphi_1}{d\xi}\right)^2 d\xi + \frac{1}{4a}\xi \int \left(\frac{d\psi_1}{d\eta}\right)^2 d\eta, \end{aligned} \tag{63}$$

where $\varphi_1 = \varphi_1(\xi)$, $\psi_1 = \psi_1(\eta)$, $\varphi_2 = \varphi_2(\xi)$, $\psi_2 = \psi_2(\eta)$ are arbitrary functions.

4°. The first PDE (59) has a simple stationary solution:

$$f = \frac{1}{C_1 x + C_2},$$

as well as a more complex nonstationary four-parameter solution with generalized separation of variables

$$f = -\frac{(x+C_1)^2}{2(t+C_2)^2} + (t+C_2)^{1/2}\big[C_3\cos\big(\tfrac{\sqrt{7}}{2}\ln(t+C_2)\big) + C_4\sin\big(\tfrac{\sqrt{7}}{2}\ln(t+C_2)\big)\big],$$

where C_1, C_2, C_3, C_4 are arbitrary constants.

5°. Using the substitution $f = 1/\theta$, we transform the first equation of system (59) to the form

$$\theta\theta_{tt} - 2\theta_{xx} - 2\theta_t^2 = 0.$$

This nonlinear PDE admits a generalized separable solution, linear in t, of the form $\theta = \varphi(x)t + \psi(x)$. Therefore, the first PDE (59) has a solution

$$f = \frac{1}{\varphi(x)t + \psi(x)}, \tag{64}$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(x)$ are described by a simple ODE system,

$$\varphi''_{xx} = 0, \quad \psi''_{xx} = -\varphi^2. \tag{65}$$

The general solution of this system is expressed in elementary functions:

$$\varphi = C_1x + C_2, \quad \psi = -\frac{1}{12C_1^2}(C_1x+C_2)^4 + C_3x + C_4,$$

where C_1, C_2, C_3, C_4 are arbitrary constants.

7°. The PDE system (59) admits a traveling wave solution of the form

$$f = f(\xi), \quad g = g(\xi), \quad h = h(\xi), \quad \xi = \alpha x - \beta t,$$

where α and β are arbitrary constants.

8°. The PDE (4) has generalized separable solutions which are fourth-degree polynomials in any spatial variable (x and y can be interchanged):

$$u = y^4F(x,t) + y^2G(x,t) + H(x,t), \tag{66}$$

where the functions $F = F(x,t)$, $G = G(x,t)$, and $H = H(x,t)$ satisfy the following overdetermined system of PDEs:

$$\begin{aligned}
&3FF_{xx} - 4F_x^2 = 0,\\
&F_{tt} - 2GF_{xx} - 12FG_{xx} + 16F_xG_x = 0,\\
&G_{tt} - 2GG_{xx} - 12FH_{xx} + 4G_x^2 = 0,\\
&H_{tt} - 2GH_{xx} = 0.
\end{aligned} \tag{67}$$

Remark 14. *The previously obtained simple solutions* (31), (35), *and* (36) *are special cases of a solution of the form* (66).

The first PDE of system (67) can be satisfied if we take $F = f_0(t)$. Then, from the other three PDEs of this system, we sequentially obtain

$$\begin{aligned}
G &= g_2x^2 + g_1x + g_0,\\
H &= h_4x^4 + h_3x^3 + h_2x^2 + h_1x + h_0,
\end{aligned} \tag{68}$$

where the functions $f_0 = f_0(t)$, $g_0 = g_0(t)$, $g_1 = g_1(t)$, and $h_j = h_j(t)$ are described by the ODE system

$$\begin{aligned}
&24f_0g_2 - f_0'' = 0,\\
&g_2'' + 12g_2^2 - 144f_0h_4 = 0,\\
&g_1'' + 12g_2g_1 - 72f_0h_3 = 0,\\
&g_2'' - 4g_2^2 + 4g_1^2 - 24f_0h_2 = 0,\\
&h_4'' - 24g_2h_4 = 0,\\
&h_3'' - 12g_2h_3 - 24g_1h_4 = 0,\\
&h_2'' - 4g_2h_2 - 12g_1h_3 - 24g_2h_4 = 0,\\
&h_1'' - 4g_1h_2 - 12g_2h_3 = 0,\\
&h_0'' - 4g_2h_2 = 0.
\end{aligned}$$

9°. If in Formulas (66) and (68), we remove the odd components in the variable x, setting $g_1 = h_3 = h_1 = 0$, then, we arrive at the solution of the form

$$u = f_0y^4 + h_4x^4 + g_2x^2y^2 + g_0y^2 + h_2x^2 + h_0, \tag{69}$$

whose time-dependent functional coefficients satisfy the second-order ODE system

$$\begin{aligned}
&f_0'' - 24f_0g_2 = 0,\\
&h_4'' - 24g_2h_4 = 0,\\
&g_2'' + 12g_2^2 - 144f_0h_4 = 0,\\
&g_0'' - 4g_0g_2 - 24f_0h_2 = 0,\\
&h_2'' - 4g_2h_2 - 24g_0h_4 = 0,\\
&h_0'' - 4g_0h_2 = 0.
\end{aligned} \tag{70}$$

The first three ODEs of system (70) form an independent subsystem. The last three ODEs of this system have the trivial particular solution $g_0 = h_2 = h_0 = 0$.

The ODE system (70) admits the exact solution

$$\begin{aligned}
&f_0(t) = C_1t + C_2, \quad h_4 = g_2 = 0, \quad h_2(t) = C_3t + C_4,\\
&g_0(t) = 2C_1C_3t^4 + 4(C_1C_4 + C_2C_3)t^3 + 12C_2C_4t^2 + C_5t + C_6,\\
&h_0(t) = \tfrac{4}{21}C_1C_3^2t^7 + \tfrac{4}{5}C_3(C_1C_4 + \tfrac{2}{3}C_2C_3)t^6 + \tfrac{4}{5}C_4(C_1C_4 + 4C_2C_3)t^5\\
&\qquad + (4C_2C_4^2 + \tfrac{1}{3}C_3C_5)t^4 + \tfrac{2}{3}(C_3C_6 + C_4C_5)t^3 + 2C_4C_6t^2 + C_7t + C_8,
\end{aligned}$$

where $C_1, \ldots, C_8$ are arbitrary constants.

It can be shown that the ODE system (70) also has solutions proportional to $(t + C)^{-2}$,

$$\begin{aligned}
&f_0(t) = \frac{A_1}{(t+C)^2}, \qquad g_0(t) = \frac{A_2}{(t+C)^2}, \qquad g_2(t) = \frac{A_3}{(t+C)^2},\\
&h_0(t) = \frac{A_4}{(t+C)^2}, \qquad h_2(t) = \frac{A_5}{(t+C)^2}, \qquad h_4(t) = \frac{A_6}{(t+C)^2},
\end{aligned}$$

where C is an arbitrary constant, and the factors A_j $(j = 1, \ldots, 6)$ satisfy the system of algebraic equations

$$\begin{aligned}
&A_1(4A_3 - 1) = 0,\\
&A_6(4A_3 - 1) = 0,\\
&24A_1A_6 - A_3(2A_3 + 1) = 0,\\
&A_2(2A_3 - 3) + 12A_1A_5 = 0,\\
&A_5(2A_3 - 3) + 12A_2A_6 = 0,\\
&2A_2A_5 - 3A_4 = 0.
\end{aligned}$$

In particular, system (70) admits the following solution:

$$\begin{aligned}
&f_0 = \frac{A}{8(t+C)^2}, \quad h_4 = \frac{1}{8A(t+C)^2}, \quad g_2 = \frac{1}{4(t+C)^2},\\
&g_0 = h_2 = h_0 = 0,
\end{aligned}$$

where A and C are arbitrary constants $(A \neq 0)$.

10°. One can also look for more complex polynomial solutions in a spatial variable of the form

$$u = \sum_{n=0}^{4} F_n(x,t)y^n, \tag{71}$$

which generalize (66). In particular, Equation (4) has exact solutions of the form (71) with

$$F_n(x,t) = \sum_{k=0}^{n} A_{nk} f_{nk}(t) x^k,$$

where the functions $f_{nk}(t)$ are described by a corresponding autonomous system of second-order ODEs.

9. Reductions in Traveling Wave Variables to Two-Dimensional Mixed-Type Equations. Linearizable PDEs

1°. The Monge–Ampère-type PDE (4) admits complex generalized separable solutions of the combined type

$$\begin{aligned}
&u = C_1x^2 + C_2xy + C_3y^2 + C_4t^2 + C_5xt + C_6yt + C_7x + C_8y + C_9t + U(\xi,\eta),\\
&\xi = a_1x + b_1y + \lambda_1 t, \quad \eta = a_2x + b_2y + \lambda_2 t,
\end{aligned} \tag{72}$$

where C_i, a_j, b_j, λ_j $(i = 1, \ldots, 9;\ j = 1, 2)$ are arbitrary constants, and ξ and η are new traveling wave-type variables, and the new desired function $U = U(\xi,\eta)$ satisfies a two-dimensional Monge–Ampère-type equation:

$$\begin{aligned}
&(a_1b_2 - a_2b_1)^2(U_{\xi\xi}U_{\eta\eta} - U_{\xi\eta}^2) + [2(a_1^2C_3 + b_1^2C_1 - a_1b_1C_2) - \lambda_1^2]U_{\xi\xi}\\
&\quad + 2[2a_1a_2C_3 + 2b_1b_2C_1 - (a_1b_2 + a_2b_1)C_2 - \lambda_1\lambda_2]U_{\xi\eta} +\\
&\quad + [2(a_2^2C_3 + b_2^2C_1 - a_2b_2C_2) - \lambda_2^2]U_{\eta\eta} + 4C_1C_3 - C_2^2 - 2C_4 = 0.
\end{aligned} \tag{73}$$

2°. The PDE (73) has exact solutions that are quadratic in any new independent variable of the form

$$\begin{aligned}
&U_1 = f_1(\xi)\eta^2 + g_1(\xi)\eta + h_1(\xi),\\
&U_2 = f_2(\eta)\xi^2 + g_2(\eta)\xi + h_2(\eta),
\end{aligned}$$

where the functions f_i, g_i, h_i $(i = 1, 2)$ are described by the corresponding ODE systems, which are omitted here.

3°. Equation (73) becomes a second-order linear PDE with constant coefficients if the condition connecting the free parameters is satisfied as follows: $a_1b_2 - b_1a_2 = 0$.

Let us consider, in more detail, the special case (72) and (73), putting

$$a_1 = a, \quad b_1 = b, \quad \lambda_1 = \lambda, \quad a_2 = b_2 = 0, \quad \lambda_2 = 1, \quad \eta = t,$$

which leads to a solution of the form

$$\begin{aligned} &u = C_1x^2 + C_2xy + C_3y^2 + C_4t^2 + C_5xt + C_6yt + C_7x + C_8y + C_9t + U(\xi, t), \\ &\xi = ax + by + \lambda t, \end{aligned} \tag{74}$$

where C_i $(i = 1, \dots, 6)$, a, b, and λ are arbitrary constants. In this case, the function $U = U(\xi, t)$ satisfies the following second-order linear PDE:

$$U_{tt} + 2\lambda U_{\xi t} + [\lambda^2 - 2(a^2C_3 + b^2C_1 - abC_2)]U_{\xi\xi} = 4C_1C_3 - C_2^2 - 2C_4. \tag{75}$$

The type of this equation may vary depending on the values of the free parameters. Namely, the PDE (75) is (see, for example, [119,120]) as follows:

$$\begin{array}{lll} \textit{parabolic} & \text{if} & \Delta = a^2C_3 + b^2C_1 - abC_2 = 0, \\ \textit{hyperbolic} & \text{if} & \Delta = a^2C_3 + b^2C_1 - abC_2 > 0, \\ \textit{elliptic} & \text{if} & \Delta = a^2C_3 + b^2C_1 - abC_2 < 0. \end{array}$$

Remark 15. *The general solution of the hyperbolic Equation* (75) *(for $\Delta > 0$) is given by the formula*

$$\begin{aligned} U &= F_1(\xi - \sigma_1 t) + F_2(\xi - \sigma_2 t) + \frac{4C_1C_3 - C_2^2 - 2C_4}{2\lambda}\xi t, \\ \sigma_{1,2} &= \lambda \pm \sqrt{2(a^2C_3 + b^2C_1 - abC_2)}, \end{aligned}$$

where $F_1(z_1)$ and $F_2(z_2)$ are arbitrary functions.

Remark 16. *The highly nonlinear PDE* (4) *under consideration belongs to the class of partially linearizable PDEs [36] (or, according to [121], to the class of conditionally integrable PDEs), since it admits solutions that are described by linear PDEs* (54) *and* (75)*.*

4°. Let us equate to zero the constant factors at the second derivatives $U_{\xi\xi}$, $U_{\xi\eta}$, and $U_{\eta\eta}$, which enter linearly into the PDE (73):

$$\begin{aligned} 2(a_1^2C_3 + b_1^2C_1 - a_1b_1C_2) - \lambda_1^2 = 0, \\ 2(a_2^2C_3 + b_2^2C_1 - a_2b_2C_2) - \lambda_2^2 = 0, \\ (2a_1a_2C_3 + 2b_1b_2C_1 - (a_1b_2 + b_1a_2)C_2 - \lambda_1\lambda_2 = 0, \end{aligned} \tag{76}$$

and then divide the remaining terms of the equation by $(a_1b_2 - b_1a_2)^2$. As a result, we obtain an nonhomogeneous stationary Monge–Ampère equation of the form (51), which, as was indicated earlier in Remark 11, can be linearized.

Relations (76) can be considered as a linear nonhomogeneous system of three algebraic equations with respect to the coefficients C_1, C_2, and C_3.

10. Reductions and Exact Solutions Based on a New Variable, Parabolic in Spatial Coordinates

1°. In variables, one of which is time, and the other is specified by a parabolic function in spatial variables, we can observe the following:

$$u = U(z,t), \quad z = y + ax^2, \tag{77}$$

where $a \neq 0$ is a free parameter, and the Monge–Ampère-type PDE (4) is reduced to the Guderley type equation

$$U_{tt} = 2aU_z U_{zz}, \tag{78}$$

which is used to describe transonic gas flows [16].

Some exact solutions of the reduced nonlinear PDE (78) are described below.

2°. The two-dimensional PDE (78) has additive separable solutions

$$U = \pm C_1(z + C_2)^{3/2} + \tfrac{9}{8}aC_1^2t^2 + C_3t + C_4,$$

where C_1, C_2, C_3, and C_4 are arbitrary constants. In this case, we receive

$$u = \pm C_1(y + ax^2 + C_2)^{3/2} + \tfrac{9}{8}aC_1^2t^2 + C_3t + C_4.$$

3°. The PDE (78) admits the simple multiplicative separable solution

$$U = \frac{(z + C_2)^3}{6a(t + C_1)^2},$$

where C_1 and C_2 are arbitrary constants. In this case, we have

$$u = \frac{(y + ax^2 + C_2)^3}{6a(t + C_1)^2}.$$

4°. The PDE (78) has an exact solution of the form

$$U = C_1t^2 + C_2t + W(\xi), \quad \xi = z + \lambda t \equiv y + ax^2 + \lambda t,$$

where C_1 ($C_1 \neq 0$)), C_2, and λ are arbitrary constants, and the function $W = W(\xi)$ satisfies the integrable ODE

$$(2aW'_\xi - \lambda^2)W''_{\xi\xi} - 2C_1 = 0,$$

the general solution of which can be written as follows:

$$W = \pm\frac{[8aC_1(\xi + C_3) + \lambda^4]^{3/2}}{24a^2C_1} + \frac{\lambda^2\xi}{2a} + C_4,$$

where C_3, C_4 are arbitrary constants.

5°. The PDE (78) has the self-similar solution

$$U = t^{-3\beta-2}V(\zeta), \quad \zeta = zt^\beta, \tag{79}$$

where β is an arbitrary constant, and the function $V = V(\xi)$ satisfies the ODE

$$(2aV'_\zeta - \beta^2\zeta^2)V''_{\zeta\zeta} + 5\beta(\beta+1)\zeta V'_\zeta - 3(\beta+1)(3\beta+2)V = 0. \tag{80}$$

Solution (79) is invariant under a one-parameter group of transformations which is specified by the operator

$$Y = \beta z \frac{\partial}{\partial z} - t \frac{\partial}{\partial t} + (3\beta + 2) U \frac{\partial}{\partial U},$$

admitted by the PDE (78).

For $\beta = -1$, ODE (80) has a simple one-parameter particular solution

$$V = \frac{1}{6a} \zeta^3 + C.$$

6°. The PDE (78) admits the invariant solution

$$U = t^{-2} f(\eta), \quad \eta = z + \lambda \ln t, \tag{81}$$

where λ is an arbitrary constant, and the function $f = f(\eta)$ satisfies the ODE

$$(2a f'_\eta - \lambda^2) f''_{\eta\eta} + 5\lambda f'_\eta - 6f = 0.$$

Solution (81) is invariant under a one-parameter group of transformations which is specified by the operator

$$Y = \lambda \frac{\partial}{\partial z} - t \frac{\partial}{\partial t} + 2U \frac{\partial}{\partial U},$$

admitted by the PDE (78).

7°. The PDE (78) also admits another invariant solution

$$U = e^{-3\beta t} g(\tau), \quad \tau = e^{\beta t} z, \tag{82}$$

where β is an arbitrary constant, and the function $g = g(\tau)$ satisfies the ODE

$$(2a g'_\tau - \beta^2 \tau^2) g''_{\tau\tau} + 5\beta^2 \tau g'_\tau - 9\beta^2 g = 0.$$

Solution (82) is invariant under a one-parameter group of transformations which is specified by the operator

$$Y = \beta z \frac{\partial}{\partial z} - \frac{\partial}{\partial t} + 3\beta U \frac{\partial}{\partial U},$$

admitted by the PDE (78).

8°. The PDE (78) has a generalized separable solution in the form of a cubic polynomial in z:

$$U = \psi_1(t) + \psi_2(t) z + \psi_3(t) z^2 + \psi_4(t) z^3, \tag{83}$$

where the functions $\psi_n(t)$ ($n = 1, \ldots, 4$) are described by the second-order nonlinear ODE system

$$\begin{aligned} &\psi_1'' - 4a\psi_2\psi_3 = 0, \\ &\psi_2'' - 4a(3\psi_2\psi_4 + 2\psi_3^2) = 0, \\ &\psi_3'' - 36a\psi_3\psi_4 = 0, \\ &\psi_4'' - 36a\psi_4^2 = 0. \end{aligned} \tag{84}$$

This system is integrated backwards, starting with the last equation, the general solution of which can be expressed in an implicit form.

Using a simple solution of the last ODE (84):

$$\psi_4 = \frac{1}{6a}(t + C_1)^{-2},$$

we obtain the following seven-parameter exact solution of the resulting system (84):

$$\begin{aligned}
\psi_1 &= \tfrac{4}{3}a^2C_2^3(t+C_1)^{-2} + 2aC_2C_4(t+C_1)^{-1} + \tfrac{4}{1053}a^2C_3^3(t+C_1)^{13} + \\
&\quad + \tfrac{8}{27}a^2C_2C_3^2(t+C_1)^8 + \tfrac{2}{21}aC_3C_5(t+C_1)^7 + \tfrac{1}{3}aC_3C_4(t+C_1)^4 + \\
&\quad + 4a^2C_2^2C_3(t+C_1)^3 + 2aC_2C_5(t+C_1)^2 + C_6(t+C_1) + C_7, \\
\psi_2 &= 2aC_2^2(t+C_1)^{-2} + C_4(t+C_1)^{-1} + \tfrac{4}{27}aC_3^2(t+C_1)^8 + 4aC_2C_3(t+C_1)^3 + C_5(t+C_1)^2, \\
\psi_3 &= C_2(t+C_1)^{-2} + C_3(t+C_1)^3, \\
\psi_4 &= \frac{1}{6a}(t+C_1)^{-2},
\end{aligned}$$

where $C_1, \ldots, C_7$ are arbitrary constants.

9°. The PDE (78) has a generalized separable solution of a more exotic form

$$U = \vartheta_1(t) + \vartheta_2(t)z^{3/2} + \vartheta_3(t)z^3, \tag{85}$$

where the functions $\vartheta_n(t)$ $(n = 1, 2, 3)$ are described by the second-order nonlinear ODE system

$$\begin{aligned}
&\vartheta_1'' - \tfrac{9}{4}a\vartheta_2^2 = 0, \\
&\vartheta_2'' - \tfrac{45}{2}a\vartheta_2\vartheta_3 = 0, \\
&\vartheta_3'' - 36a\vartheta_3^2 = 0.
\end{aligned} \tag{86}$$

This system is integrated backwards, starting with the last equation, which coincides with the last ODE of system (84).

Using a simple solution of the last ODE of system (86),

$$\vartheta_3 = \frac{1}{6a}(t+C_1)^{-2},$$

we obtain the following five-parameter exact solution of this system:

$$\begin{aligned}
\vartheta_1 &= \tfrac{9}{8}aC_3^2(t+C_1)^{-1} + \tfrac{3}{56}aC_3^2(t+C_1)^7 + \tfrac{3}{4}C_2C_3(t+C_1)^3 + C_4t + C_5, \\
\vartheta_2 &= C_2(t+C_1)^{-3/2} + C_3(t+C_1)^{5/2}, \\
\vartheta_3 &= \frac{1}{6a}(t+C_1)^{-2},
\end{aligned}$$

where $C_1, \ldots, C_5$ are arbitrary constants.

10°. The nonlinear PDE (78) can be linearized using the Legendre transformation, which is defined by the formulas:

$$\begin{aligned}
&t = W_T, \quad z = W_Z, \quad U = TW_T + ZW_Z - W \quad &&\text{(direct transformation)}; \\
&T = U_t, \quad Z = U_z, \quad W = tU_t + zU_z - U \quad &&\text{(inverse transformation)},
\end{aligned} \tag{87}$$

where $U = U(t,z)$ and $W = W(T,Z)$, and the second derivatives are calculated using the formulas

$$\begin{gathered}
U_{tt} = JW_{ZZ}, \quad U_{tz} = U_{zt} = -JW_{TZ}, \quad U_{zz} = JW_{TT}, \\
J = U_{tt}U_{zz} - U_{tz}^2, \quad 1/J = W_{TT}W_{ZZ} - W_{TZ}^2.
\end{gathered}$$

As a result of the Legendre transformation, the nonlinear PDE (78) is reduced to the second-order linear PDE:

$$W_{ZZ} = 2aZW_{TT}. \tag{88}$$

If $W = W(T, Z)$ is a solution of Equation (88), then Formula (87) determines the corresponding solution in parametric form of Equation (78).

Note that when using the Legendre transform, some solutions may be lost if $J \equiv 0$.

11. Reductions and Exact Solutions Based on a New Variable, Quadratic in Two Spatial Coordinates

1°. In variables, one of which is time and the other is quadratic with respect to both spatial variables, we can observe the following:

$$u = U(z, t), \quad z = ax^2 + bxy + cy^2 + kx + sy, \tag{89}$$

where a, b, c, k, and s are free parameters, and the Monge–Ampère-type PDE (4) is reduced to the two-dimensional PDE

$$U_{tt} - 2(Az + B)U_z U_{zz} - AU_z^2 = 0, \tag{90}$$

where $A = 4ac - b^2$, $B = as^2 + ck^2 - bks$.

For $A = 0$ (this corresponds to the degenerate case), we obtain Equation (78), which is discussed in detail in Section 10. Further, we will assume that $A = 4ac - b^2 \neq 0$.

2°. The transformaation

$$t = t, \quad z = \frac{\sqrt{|A|}}{2}\rho^2 - \frac{B}{A}, \quad U = \operatorname{sign}(A) W(\rho, t),$$

leads to the PDE (90) to the simpler equation

$$W_{tt} - \rho^{-1} W_\rho W_{\rho\rho} = 0. \tag{91}$$

Some exact solutions of the nonlinear PDE (91) are described below.

3°. The PDE (91) admits an additive separable solution of the form

$$W = \tfrac{1}{2}C_1 t^2 + C_2 t \pm \int \sqrt{C_1\rho^2 + C_3}\, d\rho + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, and

$$\int \sqrt{C_1\rho^2 + C_3}\, d\rho =$$
$$= \begin{cases} \dfrac{1}{2}\rho\sqrt{C_1\rho^2 + C_3} + \dfrac{C_3}{2\sqrt{C_1}} \ln\left(\sqrt{C_1}\,\rho + \sqrt{C_1\rho^2 + C_3}\right), & \text{if } C_1 > 0; \\ \dfrac{1}{2}\rho\sqrt{C_1\rho^2 + C_3} + \dfrac{C_3}{2\sqrt{-C_1}} \arctan \dfrac{\sqrt{-C_1}\,\rho}{\sqrt{C_1\rho^2 + C_3}}, & \text{if } C_1 < 0; \\ \sqrt{C_3}\rho, & \text{if } C_1 = 0,\ C_3 \geqslant 0. \end{cases}$$

4°. The PDE (91) admits solutions in the form of a product of functions of different arguments

$$W = t^{-2} f(\rho), \tag{92}$$

where the function $f = f(\rho)$ satisfies the ODE

$$f'_\rho f''_{\rho\rho} - 6\rho f = 0,$$

which admits the particular solution

$$f = \tfrac{1}{8}\rho^4.$$

Returning to the original variables, we arrive at a multiparameter solution in elementary functions of the PDE (4):

$$u = \frac{1}{2(4ac - b^2)t^2}\left(ax^2 + bxy + cy^2 + kx + sy + \frac{as^2 + ck^2 - bks}{4ac - b^2}\right)^2.$$

Solution (92) is invariant under a one-parameter group of transformations which is specified by the operator

$$Y = t\frac{\partial}{\partial t} - 2W\frac{\partial}{\partial W},$$

admitted by the PDE (91).

5°. The PDE (91) admits the self-similar solution

$$W = t^{-4\beta-2}F(z), \quad z = t^{\beta}\rho, \tag{93}$$

where β is an arbitrary constant, and the function $F = F(z)$ satisfies the generalized homogeneous ODE

$$(F'_z - \beta^2 z^3)F''_{zz} + \beta(7\beta+5)z^2F'_z - 2(2\beta+1)(4\beta+3)zF = 0,$$

whose order can be lowered by one [122].

Solution (93) is invariant under a one-parameter group of transformations which is specified by the operator

$$Y = \beta\frac{\partial}{\partial \rho} - t\frac{\partial}{\partial t} + 2(2\beta+1)W\frac{\partial}{\partial W},$$

admitted by the PDE (91).

6°. The PDE (91) admits the invariant solution

$$W = e^{-4\lambda t}\Phi(\zeta), \quad \zeta = \rho e^{\lambda t}, \tag{94}$$

where λ is an arbitrary constant, and the function $\Phi = \Phi(\zeta)$ satisfies the ODE

$$\zeta^{-1}\Phi'_\zeta\Phi''_{\zeta\zeta} = \lambda^2(16\Phi - 7\zeta\Phi'_\zeta + \zeta^2\Phi''_{\zeta\zeta}),$$

whose order can be lowered by one [122].

Solution (94) is invariant under a one-parameter group of transformations which is specified by the operator

$$Y = \lambda\rho\frac{\partial}{\partial \rho} - \frac{\partial}{\partial t} + 4\lambda W\frac{\partial}{\partial W},$$

admitted by the PDE (91).

7°. The PDE (91) also admits an exact polynomial solution of the form

$$W = \theta_1(t) + \theta_2(t)\rho^2 + \theta_3(t)\rho^4,$$

where the functions $\theta_n(t)$ ($n = 1, 2, 3$) satisfy the second-order nonlinear system of PDEs

$$\begin{aligned} &\theta_1'' - 4\theta_2^2 = 0,\\ &\theta_2'' - 32\theta_2\theta_3 = 0,\\ &\theta_3'' - 48\theta_3^2 = 0. \end{aligned} \tag{95}$$

The ODE system (95) is integrated in reverse order, starting with the last equation, which coincides with the last equations of systems (84) and (86) for $a = 4/3$.

Using a simple solution of the last ODE of system (95),

$$\theta_3 = \tfrac{1}{8}(t + C_1)^{-2},$$

we obtain the following five-parameter exact solution of this system:

$$\begin{aligned}
\theta_1 &= \tfrac{1}{26}(t+C_1)^3\big[-(5\sqrt{17}-23)C_2^2(t+C_1)^{\sqrt{17}} + (5\sqrt{17}+23)C_3^2(t+C_1)^{-\sqrt{17}}\big] + \\
&\quad + \tfrac{4}{3}C_2C_3(t+C_1)^3 + C_4 t + C_5, \\
\theta_2 &= \sqrt{t+C_1}\,\Big[C_2(t+C_1)^{\sqrt{17}/2} + C_3(t+C_1)^{-\sqrt{17}/2}\Big], \\
\theta_3 &= \tfrac{1}{8}(t+C_1)^{-2},
\end{aligned}$$

where $C_1, \ldots, C_5$ are arbitrary constants.

12. Reductions and Exact Solutions in Polar and Generalized Polar Coordinates

1°. At the point (x_0, y_0), where x_0 and y_0 are arbitrary constants, we introduce polar coordinates r, φ using the formulas

$$x = x_0 + r\cos\varphi, \qquad y = y_0 + r\sin\varphi.$$

As a result, the original PDE (4) is transformed to the form

$$u_{tt} = r^{-2}u_{rr}(u_{\varphi\varphi} + r u_r) + [(r^{-1}u_\varphi)_r]^2. \tag{96}$$

2°. The Lie group symmetry analysis of the resulting PDE (96) (see also Section 2) shows that the transformation

$$\begin{aligned}
&\bar r = ar, \quad \bar\varphi = \varphi + b, \quad \bar t = pt + q, \\
&\bar u = \frac{a^4}{p^2}u + t(c_1 r\cos\varphi + c_2 r\sin\varphi + c_3) + c_4 r\cos\varphi + c_5 r\sin\varphi + c_6,
\end{aligned} \tag{97}$$

where a, b, c_1, c_2, c_3, c_4, c_5, c_6, p, and q are arbitrary constants, leads the PDE (96) to an equation of exactly the same form.

The invariant ten-parameter transformation (97) allows, starting from simpler solutions of the PDE (96), us to obtain its more complex exact solutions. Namely, if $u = F(r, \varphi, t)$ is a solution to the PDE (96), then the function

$$u = \frac{p^2}{a^4}\big[F(ar, \varphi + b, pt + q) - t(c_1 r\cos\varphi + c_2 r\sin\varphi + c_3) - c_4 r\cos\varphi - c_5 r\sin\varphi - c_6\big]$$

is also a solution of this PDE.

3°. The PDE (96) has two-dimensional radially symmetric solutions, which are described by the PDE

$$u_{tt} - r^{-1}u_r u_{rr} = 0,$$

which, up to obvious renotations, coincides with Equation (91). Therefore, it allows five exact solutions, described earlier in Items 3°–7° from Section 11.

4°. The PDE (96), using new mixed-type variables

$$u = t^{-4\alpha-2}U(\xi, \eta), \qquad \xi = rt^\alpha, \qquad \eta = \varphi + \beta\ln t, \tag{98}$$

where α and β are arbitrary constants, is reduced to a two-dimensional Monge–Ampère-type PDE, which is not presented here.

Solution (98) is invariant under a one-parameter group of transformations which is specified by the operator

$$Y = \alpha r\frac{\partial}{\partial r} - t\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial \varphi} + (4\alpha + 2)u\frac{\partial}{\partial u},$$

admitted by the PDE (96).

The values of $\alpha = \beta = 0$ in (98) correspond to a multiplicative separable solution of the form $u = t^{-2}U(r, \varphi)$.

For $U = U(\xi)$ in (98), we have a self-similar solution.

5°. The PDE (96), using other mixed-type variables

$$u = e^{-4\gamma t}U(\xi, \eta), \quad \xi = e^{\gamma t}r, \quad \eta = \varphi - \lambda t, \tag{99}$$

where γ and λ are arbitrary constants, also reduces to a two-dimensional Monge–Ampère-type PDE, which is not written here.

Solution (99) is invariant under a one-parameter group of transformations which is specified by the operator

$$Y = \gamma r\frac{\partial}{\partial r} - \frac{\partial}{\partial t} + \lambda\frac{\partial}{\partial \varphi} + 4\gamma u\frac{\partial}{\partial u},$$

admitted by the PDE (96).

6°. The PDE (96) also has multiplicative separable solutions of the form

$$u = r^4U(\varphi, t), \tag{100}$$

where the new desired function $U = U(\varphi, t)$ satisfies the two-dimensional PDE

$$U_{tt} - 12UU_{\varphi\varphi} + 9U_\varphi^2 - 48U^2 = 0. \tag{101}$$

Solution (100) is invariant under a one-parameter group of transformations which is specified by the operator

$$Y = r\frac{\partial}{\partial r} + 4u\frac{\partial}{\partial u},$$

admitted by the PDE (96).

7°. The PDE (101) has a traveling wave solution of the form

$$U = V(\eta), \quad \eta = \varphi - \lambda t,$$

where λ is an arbitrary constant, and the function $V = V(\eta)$ satisfies the autonomous ODE

$$(12V - \lambda^2)V''_{\eta\eta} - 9(V'_\eta)^2 + 48V^2 = 0.$$

The substitution $\Theta(V) = (V'_\eta)^2$ reduces this equation to a first-order linear ODE.

8°. The PDE (101) also has a multiplicative separable solution of the form

$$U = (t + C_1)^{-2}V(\varphi),$$

where C_1 is an arbitrary constant, and the function $V = V(\varphi)$ satisfies the autonomous ODE

$$4VV''_{\varphi\varphi} - 3(V'_\varphi)^2 + 16V^2 - 2V = 0. \tag{102}$$

The substitution $Z(V) = (V'_\varphi)^2$ reduces (102) to the linear first-order ODE

$$2VZ'_V - 3Z + 16V^2 - 2V = 0,$$

the general solution of which is written as follows:

$$Z = C_2 V^{3/2} - 16V^2 - 2V,$$

where C_2 is an arbitrary constant. Integrating further, we obtain the general solution to the ODE (102) in implicit form

$$\int \left(C_2 V^{3/2} - 16V^2 - 2V\right)^{-1/2} dV = \pm\varphi + C_3,$$

where C_3 is an arbitrary constant.

9°. The PDE (101) admits the generalized separable solutions of the form

$$U = F(t) + H(t)[C_1 \cos(4\varphi) + C_2 \sin(4\varphi)], \tag{103}$$

where C_1 and C_2 are arbitrary constants, and the functions $F = F(t)$ and $H = H(t)$ are described by the ODE system

$$\begin{aligned} F''_{tt} &= 48F^2 - 144(C_1^2 + C_2^2)H^2, \\ H''_{tt} &= -96FH. \end{aligned} \tag{104}$$

System (104) allows for particular solutions:

$$F = -\frac{1}{16(t + C_3)^2}, \qquad H = \pm\frac{1}{16\sqrt{C_1^2 + C_2^2}\,(t + C_3)^2}.$$

Remark 17. *To construct exact solutions* (103)*, we used invariant subspaces of the nonlinear differential operator* $F[v] = 12vv_{\varphi\varphi} - 9v_\varphi^2 + 48v^2$*, included in the right side of the PDE* (101) *(for details see [60,73]).*

10°. At the point (x_0, y_0), we introduce the generalized polar coordinates r, φ according to the formulas

$$x = x_0 + ar\cos\varphi, \qquad y = y_0 + br\sin\varphi, \tag{105}$$

where x_0, y_0 are arbitrary constants, and a and b are free positive parameters. As a result, the original PDE (4) takes the form

$$u_{tt} = (ab)^{-2}\left\{r^{-2}u_{rr}(u_{\varphi\varphi} + ru_r) - [(r^{-1}u_\varphi)_r]^2\right\}. \tag{106}$$

Using a simple substitution $u = (ab)^2\bar{u}$ (or $t = ab\,\bar{t}$), this equation is reduced to Equation (96), the exact solutions of which are described in Section 12.

13. Reductions and Exact Solutions in Special Lorentz Coordinates

1°. In special Lorentz coordinates ζ, ψ, which are introduced by the formulas

$$x = x_0 + a\zeta\cosh\psi, \qquad y = y_0 + b\zeta\sinh\psi, \tag{107}$$

where x_0 and y_0 are arbitrary constants, a and b are any non-zero constants, the PDE (4) takes the form

$$u_{tt} = (ab)^{-2}\left\{\zeta^{-2}u_{\zeta\zeta}(u_{\psi\psi} - \zeta u_\zeta) - [(\zeta^{-1}u_\psi)_\zeta]^2\right\}. \tag{108}$$

Remark 18. *Special Lorentz coordinates (107) and Equation (108) can be obtained from generalized polar coordinates (105) and Equation (106) if we set in them as* $r = \zeta$, $\varphi = i\psi$ *and rename* $b \Rightarrow -ib$, *where* $i^2 = -1$.

Remark 19. *For* $x_0 = y_0 = 0$ *and* $a = b = 1$, *the quantity* $\zeta^2 = x^2 - y^2$ *in (107) is an invariant of the Lorentz transformation*

$$\bar{x} = \frac{x - \sigma y}{\sqrt{1-\sigma^2}}, \quad \bar{y} = \frac{y - \sigma x}{\sqrt{1-\sigma^2}}, \tag{109}$$

where the free parameter σ *satisfies the condition* $0 \le |\sigma| < 1$. *Note that the one-parameter transformation (109), which is a special case of transformation (8), preserves the form the original Monge–Ampère-type PDE (4) (this follows from Corollary 2 with* $\sigma = \tanh\beta$ *from Section 2) as well as the form of the linear wave equation* $u_{xx} = u_{yy}$.

Equation (108) differs from Equation (106) only in the sign of the second term on the right-hand side. Therefore, its exact solutions can be sought in the same form as in Section 12.

2°. Equation (108) admits solutions independent of the pseudo-angular variable ψ, which are described by the two-dimensional PDE

$$u_{tt} = -(ab)^{-2}\zeta^{-1}u_\zeta u_{\zeta\zeta},$$

which, after substituting $u = -(ab)^2 W$ and renaming ζ to ρ, reduces to Equation (91). This circumstance allows us to obtain five exact solutions, described earlier in Items 3°–7° from Section 11.

3°. Equation (108), using new mixed-type variables

$$u = t^{-4\alpha-2}U(\xi,\eta), \quad \xi = \zeta t^\alpha, \quad \eta = \psi + \beta\ln t, \tag{110}$$

where α and β are arbitrary constants, is reduced to a two-dimensional Monge–Ampère-type PDE for the function U, which is omitted here.

The values of $\alpha = \beta = 0$ in the PDE (110) correspond to a multiplicative separable solution of the form $u = t^{-2}U(\zeta,\psi)$.

For $U = U(\xi)$ in (110), we have a self-similar solution.

4°. The PDE (108), using other mixed-type variables

$$u = e^{-4\gamma t}U(\xi,\eta), \quad \xi = e^{\gamma t}\zeta, \quad \eta = \psi - \lambda t,$$

where γ and λ are arbitrary constants, also reduces to a two-dimensional Monge–Ampère-type PDE, which is not written here.

5°. The PDE (108) also admits multiplicative separable solutions of the form

$$u = \zeta^4 U(\psi,t), \tag{111}$$

where the new desired function $U = U(\psi,t)$ satisfies the two-dimensional PDE

$$U_{tt} = (ab)^{-2}(12UU_{\psi\psi} - 9U_\psi^2 - 48U^2). \tag{112}$$

6°. The PDE (112) has a traveling wave solution of the form

$$U = V(\eta), \quad \eta = \psi - \lambda t,$$

where λ is an arbitrary constant, and the function $V = V(\eta)$ satisfies the autonomous ODE

$$V''_{\eta\eta} = (ab\lambda)^{-2}(12VV''_{\eta\eta} - 9V_\eta^2 - 48V^2).$$

The substitution $\Theta(V) = (V'_\eta)^2$ reduces this equation to a first-order linear ODE.

7°. The PDE (112) also admits a multiplicative separable solution of the form

$$U = (t + C_1)^{-2}V(\psi),$$

where C_1 is an arbitrary constant, and the function $V = V(\psi)$ satisfies the autonomous ODE

$$4VV''_{\psi\psi} - 3(V'_\psi)^2 - 16V^2 - 2(ab)^2V = 0. \tag{113}$$

The substitution $Z(V) = (V'_\psi)^2$ leads (113) to a first-order linear ODE

$$2VZ'_V - 3Z - 16V^2 - 2(ab)^2V = 0,$$

the general solution of which is written as follows:

$$Z = C_2V^{3/2} + 16V^2 - 2(ab)^2V,$$

where C_2 is an arbitrary constant. Integrating further, we obtain the general solution to the ODE (113) in implicit form

$$\int \left[C_2V^{3/2} + 16V^2 - 2(ab)^2V\right]^{-1/2} dV = \pm\psi + C_3,$$

where C_3 is an arbitrary constant.

8°. The PDE (112) admits generalized separable solutions of the form

$$U = F(t) + H(t)[C_1\exp(-4\psi) + C_2\exp(4\psi)], \tag{114}$$

where C_1 and C_2 are arbitrary constants, and the functions $F = F(t)$ and $H = H(t)$ are described by the ODE system

$$\begin{aligned} (ab)^2F''_{tt} &= -48F^2 + 576C_1C_2H^2, \\ (ab)^2H''_{tt} &= 96FH. \end{aligned} \tag{115}$$

System (115) allows for particular solutions:

$$F = \frac{(ab)^2}{16(t + C_3)^2}, \quad H = \pm\frac{(ab)^2}{32\sqrt{C_1C_2}\,(t + C_3)^2}.$$

14. Reductions and Exact Solutions Based on a Fractional-Rational Transformation

The special fractional-rational transformation

$$x = \frac{\xi}{\alpha\xi + \beta\eta + 1}, \quad y = \frac{\eta}{\alpha\xi + \beta\eta + 1}, \quad u = \frac{w}{\alpha\xi + \beta\eta + 1}, \tag{116}$$

where α and β are arbitrary constants, brings the PDE (4) to the form

$$w_{\xi\xi}w_{\eta\eta} - w_{\xi\eta}^2 = (\alpha\xi + \beta\eta + 1)^{-5}w_{tt}. \tag{117}$$

Note that the fractional-rational transformation (116) brings the stationary Monge–Ampère equation $u_{xx}u_{yy} - u_{xy}^2 = f(x,y)$ to a similar form with a different right-hand side [7,18].

Setting $\beta = 0$ in (116) and (117), we obtain the PDE

$$w_{\xi\xi}w_{\eta\eta} - w_{\xi\eta}^2 = (1+\alpha\xi)^{-5}w_{tt}. \tag{118}$$

Let us now describe some exact solutions of a more general PDE (118) as follows:

$$w_{\xi\xi}w_{\eta\eta} - w_{\xi\eta}^2 = f(\xi)w_{tt}, \tag{119}$$

where $f = f(\xi)$ is an arbitrary function.

1°. The PDE (119) has generalized separable solutions of the form

$$w = \tfrac{1}{2}(a\xi + b)t^2 + (c\xi + d)t + Z(\xi,\eta),$$

where a, b, c, and d are arbitrary constants, and the function $Z = Z(\xi,\eta)$ satisfies the stationary Monge–Ampère PDE

$$Z_{\xi\xi}Z_{\eta\eta} - Z_{\xi\eta}^2 = f(\xi)(a\xi + b). \tag{120}$$

Equations of this type were considered in [16]. The PDE (120) admits the following generalized separable solutions in closed form:

$$\begin{aligned}
Z &= \pm\eta \int \sqrt{-f(\xi)(a\xi+b)}\,d\xi + \varphi(\xi),\\
Z &= C_1\eta^2 + C_2\xi\eta + \frac{C_2^2}{4C_1}\xi^2 + \frac{1}{2C_1}\int d\xi \int f(\xi)(a\xi+b)\,d\xi + C_3\xi + C_4\eta,\\
Z &= \frac{1}{\xi + C_1}\left(C_2\eta^2 + C_3\eta + \frac{C_3^2}{4C_2}\right) + \frac{1}{2C_2}\int d\xi \int f(\xi)(a\xi+b)\,d\xi + C_4,
\end{aligned}$$

where $\varphi(\xi)$ is an arbitrary function, $C_1, \ldots, C_4$ are arbitrary constants.

2°. Passing in the PDE (119) to new variables of self-similar type:

$$w = t^{-2k-2}W(\xi,\theta), \quad \theta = \eta t^k, \tag{121}$$

where k is a free parameter, we arrive at the two-dimensional Monge–Ampère-type PDE:

$$W_{\xi\xi}W_{\theta\theta} - W_{\xi\theta}^2 - f(\xi)[k^2\theta^2 W_{\theta\theta} - k(3k+5)\theta W_\theta + (2k+2)(2k+3)W] = 0.$$

Solution (121) is invariant under the transformation group which is specified by the operator

$$Y = k\eta\frac{\partial}{\partial\eta} - t\frac{\partial}{\partial t} + (2k+2)w\frac{\partial}{\partial w},$$

admitted by the PDE (119).

3°. Passing in the PDE (119) to new variables of limit self-similar type:

$$w = \exp(-2\beta t)W(\xi,\theta), \quad \theta = \exp(\beta t)\eta, \tag{122}$$

where β is a free parameter, we arrive at the two-dimensional Monge–Ampère-type PDE

$$W_{\xi\xi}W_{\theta\theta} - W_{\xi\theta}^2 - \beta^2 f(\xi)(\theta^2 W_{\theta\theta} - 3\theta W_\theta + 4W) = 0.$$

Solution (122) is invariant under the transformation group which is specified by the operator

$$Y = \beta\eta\frac{\partial}{\partial\eta} - \frac{\partial}{\partial t} + 2\beta w\frac{\partial}{\partial w},$$

admitted by the PDE (119).

4°. Equation (119) admits a one-dimensional invariant solution

$$w = t^{-2}\eta^2\varphi(\xi), \tag{123}$$

where the function $\varphi = \varphi(\xi)$ satisfies the ODE

$$\varphi\varphi''_{\xi\xi} - 2(\varphi'_\xi)^2 - 3f(\xi)\varphi = 0.$$

Solution (123) is invariant under the transformation group which is specified by the operator

$$Y = \eta\frac{\partial}{\partial\eta} + t\frac{\partial}{\partial t},$$

admitted by the PDE (119).

15. Conclusions

We study a highly nonlinear partial differential equation with three independent variables (4), namely the following:

$$u_{tt} = u_{xx}u_{yy} - u_{xy}^2,$$

which is encountered in geophysical fluid dynamics. To find exact solutions to this nonlinear PDE, classical method of symmetry reduction, methods of generalized and functional separation of variables, the principle of structural analogy of solutions, and as well as various combinations of all of the above methods were used. One-dimensional symmetry reductions leading to invariant solutions that are described by single ODEs are considered. A large number of new non-invariant solutions in closed form were obtained, including more than thirty solutions that are expressed through elementary functions. More than twenty two-dimensional reductions are discussed, when the three-variable PDE under consideration is reduced to a single simpler two-variable PDE or a system of such PDEs. Several classes of solutions have been discovered that can be expressed in terms of solutions of linear wave and heat type PDEs. To construct exact solutions, in addition to Cartesian coordinates, polar, generalized polar, and special Lorentz coordinates were also used. A number of specific examples demonstrate that the type of the mixed, highly nonlinear PDE under consideration, depending on the choice of its specific solutions, can be either hyperbolic or elliptic. All obtained exact solutions were verified using the computer algebra system Maple.

The applied aspect of this paper is that the solutions found in closed form, especially in elementary functions, can be used for direct error estimation and testing of numerical and approximate analytical methods for solving complex problems described by highly nonlinear PDEs (the type of which can vary depending on the choice of solutions). Exact solutions can also be utilized to improve the corresponding sections of computer programs designed for symbolic calculations (in computer algebra systems such as Mathematica, Maple, etc.). Furthermore, the described symmetries, reductions, and solutions can be used to update and expand the reference literature on nonlinear PDEs.

Below, we formulate possible promising directions for further research on highly nonlinear PDE (4) and related equations:

1. To more fully describe the one-dimensional and two-dimensional symmetry reductions in PDE (4), find optimal systems of one-dimensional and two-dimensional subalgebras.
2. Describe the symmetries and find exact solutions to the multidimensional generalization of PDE (4) as well as other related, more complex, highly nonlinear PDEs.
3. Formulate well-posed statements of initial-boundary value (and boundary value) problems and prove existence and uniqueness theorems for them.
4. Obtain and analyze numerical solutions to the initial-boundary value and boundary value problems (taking into account that PDE (4) is of mixed type). Verify the numerical methods used by comparing them with test problems based on exact solutions.

The formulated research directions may be useful for interested readers in choosing topics for further work.

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