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Research Article

Nonlinear Schrödinger Equations with Delay: Closed-Form and Generalized Separable Solutions

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Abstract: Nonlinear Schrödinger equations with constant delay are considered for the first time. These equations are generalizations of the classical Schrödinger equation with cubic nonlinearity and the more complex nonlinear Schrödinger equation containing functional arbitrariness. From a physical point of view, considerations are formulated about the possible causes of the appearance of a delay in nonlinear equations of mathematical physics. To construct exact solutions, the principle of structural analogy of solutions of related equations was used. New exact solutions of nonlinear Schrödinger equations with delay are obtained, which are expressed in elementary functions or in quadratures. Some more complex solutions with generalized separation of variables are also found, which are described by mixed systems of ordinary differential equations without delay or ordinary differential equations with delay. The results of this work can be useful for the development of new mathematical models described by nonlinear Schrödinger equations with delay, and the given exact solutions can serve as the basis for the formulation of test problems designed to evaluate the accuracy of numerical methods for integrating nonlinear partial differential equations with delay.

Keywords: nonlinear Schrödinger equations, partial differential equations (PDEs) with delay, functional PDEs, exact solutions, solutions in quadratures, solutions in elementary functions

MSC: 35K10, 35Q41, 35Q55, 35Q60, 35R10, 81U15, 81V70

1. Introduction

It is well known that the nonlinear Schrödinger equation with cubic nonlinearity is used in many branches of theoretical physics, including nonlinear optics, superconductivity and plasma physics (see, for example, [1–6]):

$$iu_t + a\,u_{xx} + b|u|^2u = 0, \tag{1}$$

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where $u = u(x,\ t)$ is a complex-value function of real arguments, t is the time, x is the spatial variable, a and b are parameters of the equation, $i^2 = -1$. The square of the modulus of the desired function determines the intensity of light.

There is also a more complex nonlinear Schrödinger equation, generalizing Eq. (1), which has the form

$$iu_t + a\,u_{xx} + f(|u|)u = 0, \tag{2}$$

where $f(|u|)$ is the potential function, which can be an arbitrary real continuous function.

The classical nonlinear Schrödinger equation with cubic nonlinearity (1) is used for mathematical modeling of wave propagation in essentially all branches of physics where wave processes are theoretically studied. This equation became the most popular after the theoretical and experimental substantiation of the application of the nonlinear Schrödinger equation in nonlinear optics [7–10]. The expression with the second derivative is responsible for the dispersion of the pulse at describing the propagation of pulses in an optical fiber, the function $f(|u|)$ characterizes the interaction of the light pulse with the fiber material and determines the nonlinear dependence of the refractive index of light in a nonlinear medium. For the classical nonlinear Schrödinger equation (1) the quadratic multiplier $|u|^2$ characterizes the intensity of the light pulse and is called the Kerr nonlinearity.

The uniqueness of equation (1) is explained not only by the fact that this equation is the basic equation for describing the processes of information transmission in an optical medium, but also by the fact that it belongs to the class of integrable partial differential equations [1]. This equation has an infinite number of conservation laws [11], Backlund transformations, and passes the Painlevé test [12]. The Cauchy problem for equation (1) with an initial condition of a general form is solved by the method of the inverse scattering problem [13]. Note that the exact solutions of the nonlinear Schrödinger equations (1) and (2) (including for an arbitrary function $f(|u|)$) are given in the reference books [14–16]. Related and more complex Schrödinger-type equations that appear in the literature can be found, for example, in [14–16].

A large number of publications to the study of nonlinear mathematical models for describing pulse propagation in an optical medium without delay are currently devoted. Among the most well-known mathematical models, in addition to the classical Schrodinger equation, we will mention the following equations and models here: the Kaup-Newell equation [17], the Triki-Biswas equation [18, 19], the Ginzburg-Landau equation [20, 21], the Schrodinger-Hirota equation [22–28], the Sasa-Satsuma equation [29–32], the Kundu-Mukherjee-Naskar model [33], the Gerjikov-Ivanov equation [34, 35], the Kudryashov model [36–38], the Radhashrishnan-Kundu-Lakshmanan equation [39], the concatenation models with power nonlinearities [40–44] and so on.

However, it is known [1, 2] that a mathematical model for describing the propagation of waves in optical fibers is obtained from Maxwell's equations taking into account dispersion and nonlinearity, which this system is reduced to one equation for the electromagnetic field strength and polarization of the optical medium. In this case, both the linear part of the polarization of the medium and the nonlinear part are expressed in the form of expressions containing the susceptibility tensors of the medium of the first and third orders. One of the main assumptions in the derivation of the classical nonlinear Schrodinger equation is that the assumption of an instantaneous response of the time dependence of the susceptibility tensor in the form of a Dirac delta function is used, which is generally not the case. As a next step, it is natural to assume that the time response of the susceptibility of the medium has a delay, which is taken into account in this work.

The purpose of this article is to study the nonlinear Schrödinger equations with delay, which are a natural generalization of nonlinear PDEs (1) and (2). Various classes of exact solutions of such equations will be described.

2. Nonlinear Schrödinger equations with delay

Preliminary remarks. The electro-magnetic wave propagate through an optical fiber has a huge speed but the reaction of the optical fiber material has some inertia, which can lead to a delay. This inertia is especially evident in the propagation of ultrashort optical solitons for femtosecond pulses of less than 1 ps. Taking into account forced Raman scattering when describing ultrashort pulses in an optical fiber, led to the discovery of a new phenomenon called soliton frequency self-shift [1], which is directly related to the inertia of scattering and was explained by its occurrence. It is established that this phenomenon generates a continuous shift in the carrier frequency of the optical soliton, in which its spectrum becomes so wide that the high-frequency components begin to transfer their energy to the low-frequency components. The above leads to the expediency of taking into account the delay in the expressions for the potential in various generalizations and further modifications of the nonlinear Schrödinger equations.

The considered nonlinear Schrödinger equations with delay. In this article, we will consider a one-dimensional equation Schrödinger with cubic nonlinearity and delay

$$iu_t + au_{xx} + b|\bar{u}|^2 u = 0, \quad \bar{u} = u(x,\ t-\tau), \tag{3}$$

and as well as a more complex nonlinear Schrödinger equation with delay

$$iu_t + au_{xx} + f(|\bar{u}|)u = 0, \quad \bar{u} = u(x,\ t-\tau), \tag{4}$$

where $u = u(x,\ t)$ is a complex-value function of real variables, a and b are parameter of equations, $i^2 = -1$, $f(z)$ is the potential and $\tau > 0$ is time delay. We assume here and later that $f(z)$ is an arbitrary real continuous function and τ is a constant.

Note that the nonlinear Scrödinger equations with delay (3) and (4) are natural generalization of nonlinear Schrödinger equations without delay (1) and (2); in the limiting case at $\tau \to 0$ Eqs. (3) and (4) coincide with Eqs. (1) and (2).

Equations (3) and (4) have a simple physical interpretation: the process of propagation of an electromagnetic wave through an optical fiber due to imperfection ("resistance") the material exhibits inertial properties, i.e. the system does not react to the impact instantly, as in the classical ideal case, but later for the time delay τ.

Remark 1 More simple nonlinear reaction-diffusion with constant delay and methods their solutions and some applications have been considered in books [45, 46]. Exact solutions of these equations can be found, for examples, in [15, 46–51]. Exact solutions of more complex the reaction-diffusion equations with variable delay have been described in [15, 46, 52, 53]. Some exact solutions of nonlinear wave equations with delay are given in [15, 46, 54, 55].

Remark 2 To construct exact solutions of nonlinear PDEs with delay (3) and (4), we will use the principle of structural analogy of solutions, which is formulated as follows: exact solutions of simpler equations can serve as a basis for constructing solutions of more complex related equations (see, for example, [46, 52]). Namely, to construct exact solutions of equations with delay (3) and (4) in this paper we will use the structure of known exact solutions of simpler related equations without delay (1) and (2), which are given, for example, in [14, 15].

Transformation of complex-valued PDEs with delay into systems of two real PDEs with delay. In order to find exact solutions to nonlinear Schrödinger equations with delay (3) and (4), which include the complex-valued function u, we convert these equations to systems of two real PDEs with delay. To do this, let us present the desired function in an exponential form

$$u = re^{i\varphi}, \quad r = |u|, \tag{5}$$

where $r = r(x,\ t) \geq 0$ and $\varphi = \varphi(x,\ t)$ are real functions. Substitute (5) into (4), and then divide all terms of the resulting expression by $e^{i\varphi}$. Further equating the real and imaginary parts to zero, we arrive at the following system of partial differential equations with delay:

$$
\begin{aligned}
&-r\varphi_t + ar_{xx} - ar\varphi_x^2 + rf(\bar{r}) = 0, \quad \bar{r} = r(x,\, t-\tau),\\
&r_t + 2ar_x\varphi_x + ar\varphi_{xx} = 0.
\end{aligned} \tag{6}
$$

For the equation with cubic nonlinearity (3) in the system (6) one should set $f(\bar{r}) = b\bar{r}^2$.

The system PDEs (6) together with expression (5) will be used further to construct exact solutions of the nonlinear Schrodinger equations with delay (3) and (4).

3. Exact solutions of the Schrödinger equation with the Kerr nonlinearity and delay

Let us present some exact solutions of the nonlinear Schrödinger equation with cubic (Kerr) nonlinearity and delay (3). To find exact solutions we use the principle of structural analogy of solutions (see Remark 2). Note that the same solutions can also be obtained by combining the methods of generalized separation of variables (see, for example, [14, 56, 57]) and the method of functional constraints [46, 48].

Below we first indicate the general structure of the solutions, and then present the main intermediate ordinary differential equations (ODEs) or delay ODEs and final formulas. All results are easily verified by direct substitution of the exact solutions into the delay PDE under consideration or the system of PDEs (6).

Solution 1. In the case $f(r) = br^2$ the system of PDEs (6) has the simple exact solution of the form

$$
r = C_1, \quad \varphi = C_2x + C_3 + Bt, \quad B = bC_1^2 - aC_2^2, \tag{7}
$$

where C_1, C_2, and C_3 are arbitrary real constants ($C_1 > 0$). Substituting (7) into (5), we get the traveling wave solution of the nonlinear Schrödinger equation (3):

$$
u = C_1 e^{i(C_2x + C_3 + Bt)}, \quad B = bC_1^2 - aC_2^2.
$$

We can see that this solution does not depend on the time delay τ, it is periodic in both independent variables x and t and has a constant amplitude.

Solution 2. For $f(r) = br^2$, the system of PDEs (6) admits a more complex than (7), periodic in time t, but independent of the time delay τ, the exact solution with variable amplitude

$$
r = r(x), \quad \varphi = C_1t + \theta(x), \tag{8}
$$

where C_1 is an arbitrary constant, and the functions $r = r(x)$ and $\theta = \theta(x)$ are described by system of ODEs of the form

$$
\begin{aligned}
&ar''_{xx} - ar(\theta'_x)^2 - C_1r + br^3 = 0,\\
&2r'_x\theta'_x + r\theta''_{xx} = 0.
\end{aligned} \tag{9}
$$

Integrating the second ODE of the system (9) twice, we successively find

$$\theta'_x = C_2 r^{-2}, \quad \theta = C_2 \int r^{-2} dx + C_3, \tag{10}$$

where C_2 and C_3 are arbitrary constants. Substituting (10) into the first equation (9), we obtain the second-order nonlinear ODE

$$ar''_{xx} - aC_2^2 r^{-3} - C_1 r + br^3 = 0. \tag{11}$$

This equation is autonomous, its general solution can be represented in the implicit form

$$\int \left(\frac{C_1}{a} r^2 - C_2^2 r^{-2} - \frac{b}{2a} r^4 + C_4 \right)^{-1/2} dr = C_5 \pm x, \tag{12}$$

where C_4 and C_5 are arbitrary constants. Note that the left-hand side of Eq. (12) can be expressed in terms of elliptic functions.

Solution 3. Let us prove that for $f(r) = br^2$ the system of PDEs (6) admits an exact generalized separable solution of the form

$$r = r(t), \quad \varphi = \alpha(t)x^2 + \beta(t)x + \gamma(t). \tag{13}$$

With this aim we substitute (13) into (6). As a result, the first equation of the system is reduced to a quadratic equation with respect to x, the coefficients of which depend on time. Equating the functional coefficients of this quadratic equation to zero and adding the second equation of the system, which in this case depends only on t, we obtain a mixed system consisting of three ODEs without delay and one ODE with delay:

$$\begin{aligned} \alpha'_t &= -4a\alpha^2, \\ \beta'_t &= -4a\alpha\beta, \\ \gamma'_t &= -a\beta^2 + b\bar{r}^2, \\ r'_t &= -2a\alpha r. \end{aligned} \tag{14}$$

Here the first three equations were divided by r and the notation $\bar{r} = r(t - \tau)$ was used.

First we integrate the first equation of system (14), then the second and fourth, and finally the third. As a result, we have

$$r=\frac{C_3}{\sqrt{t+C_1}},\quad \alpha=\frac{1}{4a(t+C_1)},\quad \beta=\frac{C_2}{2a(t+C_1)},$$
$$\gamma=\frac{C_2^2}{4a(t+C_1)}+bC_3^2\ln(t-\tau+C_1)+C_4, \tag{15}$$

where C_1, C_2, C_3, and C_4 are arbitrary constants. Substituting expressions (15) into (13), we obtain

$$r=\frac{C_3}{\sqrt{t+C_1}},\quad \varphi=\frac{(x+C_2)^2}{4a(t+C_1)}+bC_3^2\ln(t-\tau+C_1)+C_4. \tag{16}$$

Solution 4. For $f(r)=br^2$, the system of PDEs (6) admits an exact solution of the form

$$r=r(z),\quad \varphi=C_1t+C_2x+\theta(z),\quad z=x+\lambda t, \tag{17}$$

where C_1, C_2, and λ are arbitrary constants. Solution (17) is the generalization of solution (8). The special case $C_1=C_2=0$ in (17) corresponds to a traveling wave solution.

Substituting (17) into (6), we obtain the following mixed nonlinear system consisting of an ODE with delay and an ODE without delay:

$$-r(C_1+\lambda\theta'_z)+ar''_{zz}-ar(C_2+\theta'_z)^2+br\bar{r}^2=0,\quad \bar{r}=r(z-\lambda\tau),$$
$$\lambda r'_z+2ar'_z(C_2+\theta'_z)+ar\theta''_{zz}=0. \tag{18}$$

The substitution $\xi=\theta'_z$ allows us to lower the order of this system by one.

Solution 5. The nonlinear Schrödinger equation with cubic nonlinearity and constant delay (3) allows exact generalized separable solutions of the form

$$u(x,\,t)=(\lambda x+\mu)\exp[i(\alpha x^2+\beta x+\gamma)], \tag{19}$$

where fife defining functions $\lambda=\lambda(t)$, $\mu=\mu(t)$, $\alpha=\alpha(t)$, $\beta=\beta(t)$, and $\gamma=\gamma(t)$ are described by the mixed system of equations containing ODEs without delay and ODEs with delay

In variables (5) solution (19) is reduced to the system of PDEs (6), in which one should set

$$r=\lambda x+\mu,\quad \varphi=\alpha x^2+\beta x+\gamma. \tag{20}$$

Let us substitute the functions (20) into (6). Separating the variables in the resulting equations, we arrive at the following system for the defining functions:

$$\lambda_t' = -6a\alpha\lambda,$$

$$\mu_t' = -2a\beta\lambda - 2a\alpha\mu,$$

$$\alpha_t' = -4a\alpha^2 + b\bar{\lambda}^2, \tag{21}$$

$$\beta_t' = -4a\alpha\beta + 2b\bar{\lambda}\bar{\mu},$$

$$\gamma_t' = -a\beta^2 + b\bar{\mu}^2,$$

where $\bar{\lambda} = \lambda(t-\tau)$ and $\bar{\mu} = \mu(t-\tau)$.

The mixed system (21), consisting of two ODEs without delay and three ODEs with delay, is significantly simpler than the considered nonlinear Schrödinger equation with delay (3). This system can be integrated by numerical methods described, for example, in [46].

In the special case of $\lambda = 0$, the system (21) is completely integrated, since it is reduced to the system (14) with $r = \mu$.

4. Exact solutions of the general nonlinear Schrödinger equation with delay

Below we describe some exact solutions of the general nonlinear Schrödinger equation with an arbitrary potential function $f(z)$ and constant delay (4), which by substitution (5) reduces to the system of real PDEs with delay (6). To find exact solutions, as in Section 3, we use the principle of structural analogy of solutions (see Remark 2). Note that these exact solutions have the same structure as the above solutions 1-4 of the simpler Schrödinger equation with cubic nonlinearity and delay (3). All results are easily verified by direct substitution of the exact solutions into the delay PDE under consideration or the system of PDEs (6).

Solution 6. In the general case, the nonlinear Schrödinger equation with constant delay (4) has a simple exact solution of the traveling wave type

$$u = C_1 e^{i(C_2 x + C_3 + Bt)}, \quad B = f(C_1) - aC_2^2.$$

This solution is periodic in space and time with constant amplitude; it does not depend on the time delay τ.

Solution 7. In the general case, the system of PDEs (6) admits a more complex periodic in time t, but independent of the time delay τ, exact solution of the form

$$r = r(x), \quad \varphi = C_1 t + C_2 \int r^{-2} dx + C_3, \tag{22}$$

where C_1, C_2, and C_3 are arbitrary constants and the function $r = r(x)$ is described by the second-order nonlinear ODE

$$ar_{xx}'' - aC_2^2 r^{-3} - C_1 r + rf(r) = 0. \tag{23}$$

The general solution of this equation can be expressed in quadratures in implicit form

$$\int \left[\frac{C_1}{a} r^2 - C_2^2 r^{-2} - \frac{2}{a} \int r f(r)\, dr + C_4 \right]^{-1/2} dr = C_5 \pm x, \tag{24}$$

where C_4 and C_5 are also arbitrary constants.

Solution 8. The system of PDEs (6) admits exact generalized separable solution of the form (13), where the functional coefficients $r = r(t)$, $\alpha = \alpha(t)$, $\beta = \beta(t)$, and $\gamma = \gamma(t)$ are described by the system of three ODEs without delay and one ODE with delay:

$$\begin{aligned} \alpha_t' &= -4a\alpha^2, \\ \beta_t' &= -4a\alpha\beta, \\ \gamma_t' &= -a\beta^2 + f(\bar{r}), \\ r_t' &= -2a\alpha r. \end{aligned} \tag{25}$$

Here the first three equations were divided by r and the notation $\bar{r} = r(t - \tau)$ was used.

First of all, we integrate the first equation of system (25), then the second and fourth, and finally the third. As a result we obtain

$$\begin{aligned} &r = \frac{C_3}{\sqrt{t + C_1}}, \quad \alpha = \frac{1}{4a(t + C_1)}, \quad \beta = \frac{C_2}{2a(t + C_1)}, \\ &\gamma = \frac{C_2^2}{4a(t + C_1)} + \int f\left(\frac{C_3}{\sqrt{t - \tau + C_1}} \right) dt + C_4, \end{aligned} \tag{26}$$

where C_1, C_2, C_3, and C_4 are arbitrary constants. Substituting expressions (26) into (13), we obtain functions

$$r = \frac{C_3}{\sqrt{t + C_1}}, \quad \varphi = \frac{(x + C_2)^2}{4a(t + C_1)} + \int f\left(\frac{C_3}{\sqrt{t - \tau + C_1}} \right) dt + C_4, \tag{27}$$

which give exact solution (5) of nonlinear Schrödinger equation with constant delay (4).

Solution 9. The system of PDEs (6) admits exact solution of the form

$$r = r(z), \quad \varphi = C_1 t + C_2 x + \theta(z), \quad z = x + \lambda t, \tag{28}$$

where C_1, C_2, and λ are arbitrary constants and the functions $r = r(z)$ and $\theta = \theta(z)$ are described by the following mixed nonlinear system consisting of an ODE with delay and an ODE without delay:

$$
\begin{aligned}
&-r(C_1+\lambda\theta'_z)+ar''_{zz}-ar(C_2+\theta'_z)^2+rf(\bar{r})=0, \qquad \bar{r}=r(z-\lambda\tau),\\
&\lambda r'_z+2ar'_z(C_2+\theta'_z)+ar\theta''_{zz}=0.
\end{aligned}
\tag{29}
$$

The substitution $\xi=\theta'_z$ allows us to lower the order of this system by one.

Remark 3 In [45, 58–60] nonlinear Schrödinger equations with distributed delay containing integral terms were considered.

5. Conclusion

In this paper, we study for the first time two nonlinear Schrödinger-type equations with potentials depending on the sought functions with delay. To construct exact solutions, we used the principle of structural analogy of solutions of related PDEs. We have obtained several new exact solutions of such nonlinear functional PDEs with delay, which are expressed in quadratures or elementary functions. Some more complex solutions with generalized separation of variables have also been found, which are described by mixed systems of ODEs without delay and ODEs with delay. The obtained exact solutions can be used to test numerical methods for integrating nonlinear initial-boundary value problems of mathematical physics with delay. It is important to note that these exact solutions are valid for an arbitrary potential function $f(z)$ included in the general nonlinear Schrödinger equation with delay (4), so they can be used for a wide range of problems by specifying a specific form of this function.

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Conflict of interest

The authors declare there is no conflict of interest at any point with reference to research findings.

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