Exact solutions and reductions of nonlinear Schrödinger equations with delay^{*}

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For the first time, Schrödinger equations with cubic and more complex nonlinearities containing the unknown function with constant delay are analyzed. The physical considerations that can lead to the appearance of a delay in such nonlinear equations and mathematical models are expressed. One-dimensional nonsymmetry reductions are described, which lead the studied partial differential equations with delay to simpler ordinary differential equations and ordinary differential equations with delay. New exact solutions of the nonlinear Schrödinger equation of the general form with delay, which are expressed in quadratures, are found. To construct exact solutions, a combination of methods of generalized separation of variables and the method of functional constraints are used. Special attention is paid to three equations with cubic nonlinearity, which allow simple solutions in elementary functions, as well as more complex exact solutions with generalized separation of variables. Solutions representing a nonlinear superposition of two traveling waves, the amplitude of which varies periodically in time and space, are constructed. Some more complex nonlinear Schrödinger equations of a general form with variable delay are also studied. The results of this work can be useful for the development and improvement of mathematical models described by nonlinear Schrödinger equations with delay and related functional PDEs, and the obtained

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exact solutions can be used as test problems intended to assess the accuracy of numerical methods for integrating nonlinear equations of mathematical physics with delay.

Keywords: nonlinear Schrödinger equations, PDEs with delay, exact solutions, solutions in quadratures, solutions in elementary functions, generalized separable solutions

1 Introduction

1.1 Nonlinear Schrödinger equations and related PDEs

It is well known that the nonlinear Schrödinger equation is one of popular nonlinear partial equation which is used in the many areas of theoretical physics including nonlinear optics, superconductivity and plasma physics. It takes the form [1–8]:

$$i u_t + k u_{xx} + f(|u|)u = 0, (1)$$

where u = u(x, t) is the desired complex-valued function of real variables, the quadrate of the module of which corresponds to the intensity of light, t is the time, x is the spatial variable, f(|u|) is the potential, k is a parameter of equation, $i^2 = -1$.

The classical nonlinear Schrödinger equation with cubic nonlinearity, which is determined by the function

$$f(|u|) = b|u|^2,$$
 (2)

is well known in science. Eq. (1) describes mathematical models for wave propagation in essentially all sections of physics where wave processes are considered. However, this equation became especially popular after the theoretical and experimental substantiation of the application of the nonlinear Schrödinger equation in nonlinear optics [9–12]. When describing the propagation of pulses in an optical fiber, the expression with the second derivative is responsible for the dispersion of the pulse, the function f(|u|) characterizes the interaction of the light pulse with the fiber material and determines the nonlinear dependence of the refractive index of light. For the classical nonlinear Schrödinger equation, function (2) corresponds to the quadratic dependence of the refractive index and is called Kerr nonlinearity. The uniqueness of equation (1)-(2) is explained not only by the fact that this equation is the basic equation for describing the processes of information transmission in optical medium, but also by the fact that it belongs to the class of integrable partial differential equations. The equation has an infinite number of conservation laws, Bäcklund transformations, and passes the Painlevé test [13–17]. Exact solutions of the classical nonlinear Schrödinger equation and related equations of mathematical physics can be found, for example, in reference books [18–20]. The Cauchy problem for equation (1)-(2) with an initial condition of a general form is solved by the method of the inverse scattering problem [4, 5].

Note that some exact solutions of nonlinear Schrödinger equation (1) for arbitrary functions f(|u|) are given in [18, 20]. The main directions of generalizations of the nonlinear Schrödinger equation in describing the propagation of optical pulses in a nonlinear medium are aimed at expanding mathematical models taking into account high-order derivatives and additional nonlinear expressions. The introduction of high-order derivatives for generalizations of the nonlinear Schrödinger equation is dictated by the need to take into account the high-order dispersion during pulse propagation. At the same time, taking into account the non-Kerr nonlinearity in the propagation of optical pulses makes it possible to take into account more complex processes in the propagation of optical solitons, which lead to more complex mathematical models, the analytical study of which leads to additional difficulties. Solutions to related modified and more complex nonlinear Schrödinger-type equations one also can find in [18–43].

1.2 Differential equations with delay

For mathematical modeling of many phenomena and processes exhibiting the properties of heredity (or aftereffect), when the rate of change of the desired value depends not only on its current value, but also on some value (or several values) in the past, differential equations with a delay are used. In biology and biomechanics, lag is associated with a limited rate of transmission of nerve and muscle reactions in living tissues. In medicine, in tasks on the spread of infectious diseases, the time delay is determined by the incubation period (the interval from the moment of infection to the appearance of the first signs of the disease). In the dynamics of populations, the delay is due to the fact that individuals do not participate in reproduction immediately, but only after reaching reproductive age. In the dynamics of populations, the delay is due to the fact that individuals do not participate in reproduction immediately, but only after reaching reproductive age. In control theory, delays occur due to limited signal propagation speeds and limited speeds of technological processes. The most common partial differential equations with a delay, methods for solving them and some applications are described, for example, in books [45, 46].

To formulate the simplest problems with aftereffect, ordinary differential equations (ODEs) are used, depending on the time t, which, in addition to the desired function u(t), also contain the function $\bar{u} = u(t - \tau)$, where $\tau > 0$ is the constant time delay. Partial differential equations of the reaction-diffusion type are often used to describe related, more complex, spatially inhomogeneous problems with delay [45, 46]:

$$u_t = k u_{xx} + F(u, \bar{u}), \quad \bar{u} = u(x, t - \tau),$$
(3)

where k > 0 is the diffusion coefficient, t is the time, x is the spatial variable, $F(u, \bar{u})$ is the kinetic function, and $\tau > 0$ is the time delay (further, τ is considered a constant unless otherwise specified). The equation with delay (3) is a natural generalization of the usual nonlinear reaction-diffusion equation without delay with the function $F(u, \bar{u}) = f(u)$. A special case of Eq. (3) with $F(u, \bar{u}) = f(\bar{u})$ admits simple physical interpretation: the transfer of matter in a locally nonequilibrium medium exhibits inertial properties, i.e. the system does not react to the impact instantly, as in the classical locally equilibrium case, but with a time delay τ .

Many exact solutions of reaction-diffusion equations of the form (3) with constant delay and related nonlinear parabolic equations with delay can be found in [20, 46–51]. In [20, 46, 52, 53] some exact solutions of more complicated reactiondiffusion equations with variable delay of various types are described. Exact solutions of nonlinear wave equations with constant and variable delay, which are formally obtained from (3) by replacing the first time derivative u_t with the second derivative u_{tt} , are presented in [20, 46, 54, 55]. Similar reasoning can be extended to mathematical models described by nonlinear Schrödinger equations. Although the speed of propagation of an electromagnetic wave through an optical fiber has a huge speed, the reaction of the optical fiber material has some inertia, which can lead to a delay. This inertia is especially evident in the propagation of ultrashort optical solitons for femtosecond pulses of less than 1ps. In addition, as noted in [1], taking into account forced Raman scattering when describing ultrashort pulses in an optical fiber led to the discovery of a new phenomenon called soliton frequency self-shift, which is directly related to the inertia of scattering and was explained by its occurrence. It is established that this phenomenon generates a continuous shift in the carrier frequency of the optical soliton, in which its spectrum becomes so wide that the high-frequency components begin to transfer their energy to the low-frequency components. The above leads to the expediency of taking into account the delay in the expressions for the potential in various generalizations and further modifications of the nonlinear Schrödinger equations.

1.3 Terminology: what we mean when we say about exact solutions of nonlinear PDEs with delay

In this paper, exact solutions of nonlinear partial differential equations with delay are understood as the following solutions [46]:

(a) Solutions that are expressed in terms of elementary functions.

(b) Solutions that are expressed in quadratures, i.e. through elementary functions, functions included in the equation (this is necessary if the equation contains arbitrary or special functions) and indefinite integrals.

(c) Solutions that are expressed through solutions of ordinary differential equations or systems of such equations.

(d) Solutions that are expressed in terms of solutions of ordinary differential equations with delay or systems of such equations.

Various combinations of solutions described in (a)-(d) are also allowed. In cases (a) and (b), the exact solution can be presented in explicit, implicit, or parametric form. It is important to note that the presence of a delay in the equations of mathematical physics significantly complicates the analysis of such

equations and corresponding initial boundary value problems [45,46]. In particular, PDEs with constant delay do not allow self-similar solutions [46], which often have simpler PDEs without delay.

Note that exact solutions are mathematical standards that are often used as test problems to check the adequacy and assess the accuracy of numerical methods for integrating nonlinear PDEs and PDEs with delay. The most preferable for these purposes are simple solutions from Items (a) and (b). It is these exact solutions that are described further in this article.

Remark 1. Of great interest are also solutions of nonlinear PDEs with delay, which are expressed through solutions of linear PDEs without delay (examples of such nonlinear PDEs with delay can be found in [20, 46, 56]). According to the terminology introduced in [57], such equations can be called conditionally integrable nonlinear PDEs (perhaps more accurately, they can be called partially linearizable PDEs).

It is important to note that for complex nonlinear PDEs depending on one or more arbitrary functions (and it is precisely such nonlinear PDE that is considered in this article), the vast majority of existing analytical methods for constructing exact solutions are either not applicable at all or are weakly effective. Statistical processing of reference data [18, 20] showed that at present the majority of exact solutions of such PDEs without delay were obtained by methods of generalized and functional separation of variables. Significantly fewer exact solutions of such equations were obtained by nonclassical methods of symmetry reductions and the method of differential constraints, which are much more difficult to use in practice. In general, very few nonlinear PDEs without delay depending on arbitrary functions that admit exact closed-form solutions are known today. For nonlinear PDEs with delay that depend on arbitrary functions, the most effective method for constructing exact solutions is the method of functional constraints [46, 48].

In this article, we will consider a nonlinear Schrödinger equation with delay that is much more complex than the nonlinear Schrödinger equation without delay (1). In this equation, instead of the potential f(u), we include a potential of the general form $F(|u|, |\bar{u}|)$, where F is an arbitrary function of two arguments and $\bar{u} = u(x, t - \tau)$ is an unknown function with delay. The presence of an arbitrary function of two arguments in the equation under consideration, as well as the delay, are highly complicating factors, since for such equations the vast majority of existing analytical methods for constructing exact solutions either do not work at all or do not work effectively enough.

2 Nonlinear Schrödinger equation with delay. Special cases and transformations

2.1 Nonlinear Schrödinger equation of general form with delay

Let us consider the one-dimensional nonlinear Schrödinger equation of the general form with delay

$$iu_t + ku_{xx} + F(|u|, |\bar{u}|)u = 0, \quad \bar{u} = u(x, t - \tau),$$
(4)

where u = u(x, t) is the desired complex-valued function of real variables, k > 0is a parameter, $F(z_1, z_2)$ is an arbitrary real continuous function of two variables, $\tau > 0$ is the time delay, and $i^2 = -1$. The nonlinear Schrödinger equation with delay (4) is generalization of the usual nonlinear Schrödinger equation without delay (1), which is determined by formula $F(|u|, |\bar{u}|) = f(|u|)$.

We will specifically highlight three functions

$$F(|u|, |\bar{u}|) = b|\bar{u}|^2, \quad F(|u|, |\bar{u}|) = b|u||\bar{u}|, \quad F(|u|, |\bar{u}|) = b_1|u|^2 + b_2|\bar{u}|^2, \quad (5)$$

that determine the potentials of equations of the form (4) with cubic nonlinearity, which in the absence of delay (i.e., at $\tau = 0$) lead to the classical nonlinear Schrödinger equation (1)–(2). The nonlinear Schrödinger equations with delay with quadratic potentials $F(|u|, |\bar{u}|) = b_1|u|^2 + b_2|u||\bar{u}| + b_3|\bar{u}|^2$ will also be considered in Section 5.

2.2 Transformations of the nonlinear Schrödinger equation with delay

Let us consider the transformations that will be used further to analyze the nonlinear Schrödinger equation with delay.

1°. The transformation

$$x = C_1 X + C_2, \quad t = C_3 T + C_4, \quad u = U(X, T) \exp[i(C_5 T + C_6)], \quad (6)$$

where C_1, \ldots, C_6 are arbitrary real constants $(C_1, C_3 \neq 0)$, leads Eq. (4) to an equation of the similar form

$$iU_T + kC_3C_1^{-2}U_{XX} + [C_3F(|U|, |\bar{U}|) - C_5]U = 0,$$

$$\bar{U} = U(X, T - \tau_*), \quad \tau_* = \tau/C_3.$$
(7)

 $2^\circ.$ Let us represent the desired complex-valued function in exponential form

$$u = r e^{i\varphi}, \quad r = |u|, \tag{8}$$

where $r = r(x, t) \ge 0$ and $\varphi = \varphi(x, t)$ are real functions.

Differentiating (8), we find the derivatives:

$$u_{t} = (r_{t} + ir\varphi_{t})e^{i\varphi},$$

$$u_{x} = (r_{x} + ir\varphi_{x})e^{i\varphi},$$

$$u_{xx} = [r_{xx} - r\varphi_{x}^{2} + i(2r_{x}\varphi_{x} + r\varphi_{xx})]e^{i\varphi}.$$
(9)

Substitute (9) into (4), and then divide all terms by $e^{i\varphi}$. Having further equated the real and imaginary parts of the obtained relation to zero, we arrive at the following system of two real PDEs with delay:

$$-r\varphi_t + kr_{xx} - kr\varphi_x^2 + F(r,\bar{r})r = 0, \quad \bar{r} = r(x,t-\tau),$$

$$r_t + 2kr_x\varphi_x + kr\varphi_{xx} = 0.$$
 (10)

The system of functional PDEs (10) together with expression (8) will be used further to construct exact solutions of the nonlinear Schrödinger equation with delay (4).

3 Exact solutions of the general nonlinear Schrödinger equation with delay

Below we describe exact solutions of the nonlinear Schrödinger equation with delay (4) with a potential of the general form, which is given by an arbitrary function of two variables $F(|u|, |\bar{u}|)$. In order to construct exact solutions we use a combination of the methods of generalized separation of variables (see, for example, [18,58,59]) and the method of functional constraints [46,48]. Note that this paper uses exactly the same functional constraints as in work [44].

Remark 2. To construct exact solutions of the nonlinear PDE with delay (4), one can also use the principle of structural analogy of solutions, which is formulated as follows: exact solutions of simpler equations can serve as a basis for constructing solutions of more complex related equations (see, for example, [46, 52]). Namely, to construct exact solutions of equation with delay (4) one can use the structure of known exact solutions of the simpler related equation without delay (1) (these auxiliary exact solutions are given, for example, in [18, 20]).

Below we will first indicate the general structure of the solutions of the PDE system with delay (10), and then present the main intermediate ODEs or delay ODEs and final formulas. All results are easily verified by direct substitution of the obtained exact solutions into the delay PDE (4) or system (10).

3.1 Traveling wave solutions with constant amplitude

The system of equations (10) has the following simple exact solutions of the form

$$r = C_1, \quad \varphi = C_2 x + C_3 + Bt, \quad B = F(C_1, C_1) - kC_2^2,$$
 (11)

where C_1 , C_2 , C_3 are arbitrary real constants. Substituting (11) into (8), we obtain a traveling wave solution of the considered nonlinear PDE (4):

$$u = C_1 e^{i(C_2 x + C_3 + Bt)}, \quad B = F(C_1, C_1) - kC_2^2.$$

This solution is periodic in space and time with a constant amplitude; it does not depend on the time delay τ .

3.2 Time-periodic solutions with amplitude depending on spatial variable

The system of equations (10) admits more complex periodic in time t, but independent of the time delay τ , exact solutions of the form

$$r = r(x), \quad \varphi = C_1 t + \theta(x),$$
(12)

where C_1 is an arbitrary constant, and the functions r = r(x) and $\theta = \theta(x)$ are described by an ODE system of the form

$$kr''_{xx} - kr(\theta'_x)^2 - C_1r + F(r,r)r = 0,$$

$$2r'_x\theta'_x + r\theta''_{xx} = 0.$$
(13)

Integrating the second equation (13) twice, we consistently find

$$\theta'_x = C_2 r^{-2}, \quad \theta = C_2 \int r^{-2} dx + C_3,$$
(14)

where C_2 and C_3 are arbitrary constants. Substituting (14) into the first equation (13), we obtain a second-order nonlinear ODE of the autonomous form

$$kr''_{xx} - kC_2^2 r^{-3} - C_1 r + F(r, r)r = 0.$$
 (15)

The general solution of Eq. (15) can be represented in implicit form

$$\int \left[\frac{C_1}{k}r^2 - C_2^2r^{-2} - \frac{2}{k}\int rF(r,r)\,dr + C_4\right]^{-1/2}\,dr = C_5 \pm x,\tag{16}$$

where C_4 and C_5 are arbitrary constants.

Thus, it is shown that the system of PDEs with delay (10) admits the exact solution (12), which can be expressed in quadratures.

Note that for Schrödinger equations with cubic potentials, which are determined by the functions (5), the left part (16) can be expressed in terms of elliptic integrals.

Remark 3. A more complex nonlinear Schrödinger equation with variable delay (4), in which $\tau = \tau(x, t) > 0$ is an arbitrary continuous function, also admits a solution of the form (8), (12), where the function r = r(x) is described by ODE (15), and the function $\theta = \theta(x)$ is found using the second relation (14). Note that exact solutions of nonlinear reaction-diffusion equations with variable delay have been considered in [20, 46, 52, 53].

3.3 Generalized separable solutions with amplitude depending on time

Let us show that the system of PDEs (10) admits exact generalized separable solutions of the form

$$r = r(t), \quad \varphi = a(t)x^2 + b(t)x + c(t).$$
 (17)

To do this we substitute (17) into (10). As a result the first equation of the system of equations is reduced to quadratic equation with respect to x, the coefficients of which depend on time. By equating the functional coefficients of this quadratic equation to zero and adding the second equation of the system, which in this case depends only on t, we obtain the following system of ODEs:

$$a'_{t} = -4ka^{2},$$

 $b'_{t} = -4kab,$
 $c'_{t} = -kb^{2} + F(r, \bar{r}),$
 $r'_{t} = -2kar.$
(18)

Here, the first three equations were divided by r and the notation was introduced $\bar{r} = r(t - \tau)$.

First we integrate the first equation of system (18), then the second and fourth, and finally the third. As a result we have

$$r = \frac{C_3}{\sqrt{t+C_1}}, \quad a = \frac{1}{4k(t+C_1)}, \quad b = \frac{C_2}{2k(t+C_1)},$$

$$c = \frac{C_2^2}{4k(t+C_1)} + \int F\left(\frac{C_3}{\sqrt{t+C_1}}, \frac{C_3}{\sqrt{t-\tau+C_1}}\right) dt + C_4,$$
(19)

where C_1 , C_2 , C_3 , and C_4 are arbitrary constants. Substituting the expressions (19) into (17), we obtain

$$r = \frac{C_3}{\sqrt{t+C_1}}, \quad \varphi = \frac{(x+C_2)^2}{4k(t+C_1)} + \int F\left(\frac{C_3}{\sqrt{t+C_1}}, \frac{C_3}{\sqrt{t-\tau+C_1}}\right) dt + C_4. \tag{20}$$

Note that for the nonlinear Schrödinger equations with cubic potentials, which are defined by functions (5), the integral on the right-hand side of the second expression (20) is expressed in term of elementary functions. In particular, for the first function (5) the solutions (20) take the form

$$r = \frac{C_3}{\sqrt{t + C_1}}, \quad \varphi = \frac{(x + C_2)^2}{4k(t + C_1)} + bC_3^2 \ln(t - \tau + C_1) + C_4.$$

Remark 4. A more complex nonlinear Schrödinger equation (4) with variable delay, where $\tau = \tau(t) > 0$ is an arbitrary continuous function, also admits a solution of the form (8), (12), in which the functions r = r(t) and $\theta = \theta(x, t)$ are found by formulas (20) with $\tau = \tau(t)$. For comparison, it is important to note that exact solutions (other than traveling wave solutions) of the general nonlinear reaction-diffusion equation with constant delay (3) are currently unknown. Even more so are the solutions of this nonlinear reaction-diffusion equation with variable delay.

3.4 Solutions that are nonlinear superpositions of traveling waves

 1° . The system of equations (10) admits exact solutions of the form

$$r = r(z), \quad \varphi = C_1 t + C_2 x + \theta(z), \quad z = x + \lambda t, \tag{21}$$

where C_1 , C_2 , and λ are arbitrary constants, which generalizes solution (12). The special case $C_1 = C_2 = 0$ in (21) defines the traveling wave solution.

Substituting (21) into (10), we obtain a mixed nonlinear system consisting of an ODE with delay and an ODE without delay:

$$-r(C_1 + \lambda \theta'_z) + kr''_{zz} - kr(C_2 + \theta'_z)^2 + F(r, \bar{r})r = 0, \quad \bar{r} = r(z - \lambda \tau),$$

$$\lambda r'_z + 2kr'_z(C_2 + \theta'_z) + kr\theta''_{zz} = 0.$$
(22)

The substitution $\xi = \theta'_z$ allows us to lower the order of this system by one.

Special case. In the particular case $\theta(z) = C_3$, $\lambda = -2kC_2$ for $C_2 < 0$ system (22) is reduced to the single second-order ODE with constant delay

$$kr''_{zz} - (C_1 + C_2^2 k)r + F(r, \bar{r})r = 0, \quad \bar{r} = r(z - \tau_1), \quad \tau_1 = -2kC_2\tau.$$
(23)

Note that solution (21) with $\theta(z) = C_3$ can be interpreted as a nonlinear superposition of two traveling waves (with different velocities in the variables r and φ).

 2° . For the nonlinear Schrödinger equation with delay (4) with a potential of the special form

$$F(|u|, |\bar{u}|) = f(|u|^2 + |\bar{u}|^2),$$

where f(z) is an arbitrary function, in the ODE with delay (23) one should set $F(r, \bar{r}) = f(r^2 + \bar{r}^2)$. In this case, Eq. (23) admits exact periodic solutions

$$r(z) = \beta_n |\sin(\sigma_n z + C_4)|, \quad n = 0, 1, 2, \dots$$
 (24)

Here C_4 is an arbitrary constant, the parameters β_n are found from the algebraic (transcendental) equation

$$f(\beta_n^2) = k\sigma_n^2 + C_1 + C_2^2 k,$$

and the constants σ_n are determined by the formulas

$$\sigma_n = \frac{\pi}{2\tau_1}(1+2n) = \frac{\pi}{2\lambda\tau}(1+2n), \quad n = 0, 1, 2, \dots$$
 (25)

3°. The nonlinear Schrödinger equation with delay (4) with a cubic nonlinearity of the form

$$F(|u|, |\bar{u}|) = b(|u|^2 + |\bar{u}|^2), \quad b = \text{const},$$

has an exact solution of the form (21), where $\theta(z) = C_3$, $\lambda = -2kC_2$ (for $C_2 < 0$), the function r(z) is given in (24), and the constants β_n and σ_n are determined by the formulas

$$\beta_n = \sqrt{\frac{k\sigma_n^2 + C_1 + C_2^2 k}{b}}, \quad \sigma_n = \frac{\pi}{2\tau_1}(1+2n) = \frac{\pi}{2\lambda\tau}(1+2n), \quad n = 0, 1, 2, \dots$$

4 Generalized separable solutions of the Schrödinger equations with cubic nonlinearities and delay

4.1 Structure of exact solutions for the equations under consideration

The nonlinear Schrödinger equations with delay (4) and cubic nonlinearities, which are defined by the functions (5), admit exact generalized separable solutions of the form

$$u(x,t) = (ax+c) \exp[i(\alpha x^2 + \beta x + \gamma)], \qquad (26)$$

where the five defining functions a = a(t), c = c(t), $\alpha = \alpha(t)$, $\beta = \beta(t)$, and $\gamma = \gamma(t)$ are described by mixed systems of equations containing ODEs without delay and ODEs with delay.

Solution (26) in variables (8) is reduced to system (10), in which one should put

$$r = ax + c, \quad \varphi = \alpha x^2 + \beta x + \gamma.$$
 (27)

Substitute functions (27) into system (10). Using the dependencies (5) and separating the variables in the resulting equations, we arrive at systems for defining functions. These systems for all three dependencies (5) are listed below.

4.2 Mixed systems of equations for the defining functions

1°. In the case $F(|u|, |\bar{u}|) = b|\bar{u}|^2$ the system of equations for the defining functions is written as follows:

$$a'_{t} = -6ka\alpha,$$

$$c'_{t} = -2ka\beta - 2kc\alpha,$$

$$\alpha'_{t} = -4k\alpha^{2} + b\bar{a}^{2},$$

$$\beta'_{t} = -4k\alpha\beta + 2b\bar{a}\bar{c},$$

$$\gamma'_{t} = -k\beta^{2} + b\bar{c}^{2},$$
(28)

where $\bar{a} = a(t - \tau)$ and $\bar{c} = c(t - \tau)$.

2°. For $F(|u|, |\bar{u}|) = bu|\bar{u}|$ the system of equations for the defining functions has the form

$$a'_{t} = -6ka\alpha,$$

$$c'_{t} = -2ka\beta - 2kc\alpha,$$

$$\alpha'_{t} = -4k\alpha^{2} + ba\bar{a},$$

$$\beta'_{t} = -4k\alpha\beta + b(a\bar{c} + \bar{a}c),$$

$$\gamma'_{t} = -k\beta^{2} + bc\bar{c}.$$
(29)

3°. For $F(|u|, |\bar{u}|) = b_1 |u|^2 + b_2 |\bar{u}|^2$ the system of equations for the defining functions is written as follows:

$$a'_{t} = -6ka\alpha,$$

$$c'_{t} = -2ka\beta - 2kc\alpha,$$

$$\alpha'_{t} = -4k\alpha^{2} + b_{1}a^{2} + b_{2}\bar{a}^{2},$$

$$\beta'_{t} = -4k\alpha\beta + 2b_{1}ac + 2b_{2}\bar{a}\bar{c},$$

$$\gamma'_{t} = -k\beta^{2} + b_{1}c^{2} + b_{2}\bar{c}^{2}.$$
(30)

The mixed systems (28)-(30) consisting of ordinary differential equations and ordinary differential equations with delay are significantly simpler than the considered nonlinear Schrödinger equations with delay (4)-(5). These systems can, for example, be integrated by the numerical methods described in [46].

In the special case a = 0, the mixed systems (28)–(30) are completely integrated, since in this case solution (26) coincides, up to obvious re-notations,

with solution (17) of the more general nonlinear Schrödinger equation with delay (4).

Note that the solutions of nonlinear Schrödinger equations with delay (4) and cubic nonlinearities described in this section, which are determined by dependencies (5), are generalized to the case of variable delay of the general form (in these equations and solutions, $\tau = \text{const}$ should be replaced by $\tau = \tau(t)$).

5 Solutions of Schrödinger equations with other nonlinearities and delay

5.1 Weakly nonlinear Schrödinger equations with delay

Let us first consider the weakly nonlinear Schrödinger equations with a delay and two potentials of a special but rather general form that satisfy the condition

$$F(|u|,|u|) = \text{const.} \tag{31}$$

This condition means that at $\tau \to 0$, the nonlinear Schrödinger equation with delay (4) degenerates into a linear Schrödinger equation without delay.

1°. Using the system of equations (10) it is easy to verify that the nonlinear Schrödinger equation with delay (4) and a potential of the form

$$F(|u|, |\bar{u}|) = f(|u| - |\bar{u}|), \qquad (32)$$

where f(z) is an arbitrary function, has solution (8) with defining functions linear in both independent variables

$$r = C_1 x + 2kC_1 C_2 t + C_3, \quad \varphi = -C_2 x + [f(2kC_1 C_2 \tau) - kC_2^2]t + C_4, \quad (33)$$

which include four arbitrary real constants C_1 , C_2 , C_3 , and C_4 . Note that potential (32) satisfies condition (31).

Setting $f(z) = bz^2$ in (32), we obtain the nonlinear Schrödinger equation with delay (4) and cubic nonlinearity determined by the potential

$$F(|u|, |\bar{u}|) = b(|u| - |\bar{u}|)^2.$$

 2° . The nonlinear Schrödinger equation with delay (4) and potential (32) has also other exact solutions of the form (8) with functions

$$r = C_1 \sin(C_2 x + \beta_n t + C_3), \quad \varphi = A_n x + B_n t + C_4,$$
 (34)

where C_1 , C_2 , C_3 , and C_4 are arbitrary real constants ($C_2 \neq 0$), and other parameters are defined as follows

$$\beta_n = \frac{2\pi n}{\tau}, \quad A_n = -\frac{\beta_n}{2kC_2}, \quad B_n = -kC_2^2 - \frac{\beta_n^2}{4kC_2^2} + f(0), \quad n = 0, \pm 1, \pm 2, \dots$$
(35)

The solutions which are described by formulas (8), (34), and (35) represent a nonlinear superposition of two traveling waves with periodically varying amplitude.

 3° . The nonlinear Schrödinger equation with delay (4) with a more general than (32) potential of the form

$$F(|u|, |\bar{u}|) = f(z), \quad z = g(|u|) - g(|\bar{u}|), \tag{36}$$

where f(z) and g(w) are arbitrary functions, also admits exact solutions which are determined by formulas (8), (33) and (8), (34), (35).

Setting in (36) f(z) = bz and $g(w) = w^2$, we obtain the nonlinear Schrödinger equation with delay (4) with cubic nonlinearity given by the potential

$$F(|u|, |\bar{u}|) = b(|u|^2 - |\bar{u}|^2).$$

 4° . Let us now consider the nonlinear Schrödinger equation with delay (4) and another potential satisfying the condition (31) and having the form

$$F(|u|, |\bar{u}|) = f(|\bar{u}|/|u|).$$
(37)

We look for an exact solution of equation (4), (37) of the form (8), assuming

$$r = a(x)b(t), (38)$$

where the functions of different arguments a = a(x) and b = b(t) must be found in the course of further analysis. Substituting (38) into system (10), and then divide the resulting equations by r = ab. Taking into account the type of potential (37), as a result we obtain the functional differential equations with delay

$$-\varphi_t + k\frac{a''_{xx}}{a} - k\varphi_x^2 + f\left(\frac{\bar{b}}{b}\right) = 0, \quad \bar{b} = b(t - \tau),$$

$$\frac{b'_t}{b} + 2k\frac{a'_x}{a}\varphi_x + k\varphi_{xx} = 0,$$
(39)

containing functions of different arguments.

The function φ is looked for as the sum of two functions

$$\varphi = c(x) + d(t). \tag{40}$$

Substituting (40) into the system of equations (39) and separating the variables, we come to a mixed system consisting of one ODE with delay and three ODEs without delay:

$$k\frac{a''_{xx}}{a} - k(c'_{x})^{2} = C_{1},$$

$$d'_{t} - f\left(\frac{\bar{b}}{b}\right) - C_{1} = 0,$$

$$2k\frac{a'_{x}}{a}c'_{x} + kc''_{xx} = C_{2},$$

$$\frac{b'_{t}}{b} + C_{2} = 0,$$

(41)

where C_1 and C_2 are arbitrary constants.

Integrating the fourth ODE (41), and then the second equation, we find functions depending on time

$$b = C_3 e^{-C_2 t}, \quad d = [C_1 + f(e^{C_2 \tau})]t + C_4,$$
(42)

where C_3 and C_4 are arbitrary constants. Functions depending on the spatial coordinate are described by the nonlinear system of second-order ODEs

$$k\frac{a''_{xx}}{a} - k(c'_x)^2 = C_1,$$

$$2k\frac{a'_x}{a}c'_x + kc''_{xx} = C_2.$$
(43)

Note that ODEs (43) admit the simple exact solution

$$a = C_5 e^{\beta x}, \quad c = \gamma x + C_6,$$

where C_5 and C_6 are arbitrary constants, and β and γ are the real roots of the algebraic system of equations

$$k\beta^2 - k\gamma^2 = C_1, \quad 2k\beta\gamma = C_2,$$

which is reduced to the biquadratic equation.

In the general case, the system (43) using the transformation

$$\xi = a'_x/a, \quad \eta = c'_x$$

is reduced to an autonomous system of first-order ODEs, which, after eliminating x, is reduced to a single first-order ODE (to the Abel equation of the second kind).

 5° . The nonlinear Schrödinger equation with delay (4) with potential (37) has also other exact solutions of the form (8) with functions

$$r = C_1 \sin(C_2 x + \beta_n t + C_3), \quad \varphi = A_n x + B_n t + C_4,$$
 (44)

where C_1 , C_2 , C_3 , and C_4 are arbitrary real constants ($C_2 \neq 0$), and other parameters are determined by the formulas

$$\beta_n = \frac{2\pi n}{\tau}, \quad A_n = -\frac{\beta_n}{2kC_2}, \quad B_n = -kC_2^2 - \frac{\beta_n^2}{4kC_2^2} + f(1), \quad n = 0, \pm 1, \pm 2, \dots$$
(45)

The solutions described by the formulas (8), (44), (45) represent a nonlinear superposition of two traveling waves, periodic in time and space.

 6° . The nonlinear Schrödinger equation with delay (4) with a more general than (37) potential of the form

$$F(|u|, |\bar{u}|) = f(z), \quad z = g(|\bar{u}|)/g(|u|),$$

where f(z) and g(w) are arbitrary constants, also admits exact solutions, which are determined by the formulas given above in 3° and 4° of this section.

5.2 Other nonlinear Schrödinger equations with delay. Some remarks

1°. Using the system of equations (10) it can be shown that the nonlinear Schrödinger equation with delay (4) and the potential

$$F(|u|, |\bar{u}|) = f(|u|^p - c|\bar{u}|^p), \quad c > 0,$$
(46)

has solution (8) with functions

$$r = C_1 \exp(C_2 x + \lambda t), \quad \varphi = Ax + Bt + C_3, \tag{47}$$

where C_1 , C_2 , and C_3 are arbitrary real constants ($C_2 \neq 0$), and other parameters are expressed by the formulas

$$\lambda = \frac{\ln c}{p\tau}, \quad A = -\frac{\lambda}{2kC_2}, \quad B = kC_2^2 - \frac{\lambda^2}{4kC_2^2} + f(0).$$
 (48)

Substituting $f(z) = bz^2$ and p = 1 into (46), we arrive at the Schrödinger equation with delay (4) and cubic nonlinearity determined by a potential of the form

$$F(|u|, |\bar{u}|) = b(|u| - c|\bar{u}|)^2, \quad c > 0.$$

Setting f(z) = bz and p = 2 in (46), we obtain the Schrödinger equation with delay (4) and cubic nonlinearity with the potential

$$F(|u|, |\bar{u}|) = b(|u|^2 - c|\bar{u}|^2), \quad c > 0.$$

 2° . The nonlinear Schrödinger equation with delay (4) with a more general than (46) potential of the form

$$F(|u|, |\bar{u}|) = f(z), \quad z = (|u|^p - c_1 |\bar{u}|^p) g(|u|, |\bar{u}|), \quad c_1 > 0,$$
(49)

where f(z) and g(v, w) are arbitrary functions, admits exact solutions, which for $c_1 \neq 1$ are determined by the formulas (8), (47)–(48), and for $c_1 = 1$ by the formulas (8), (34)–(35).

Setting in (49) f(z) = bz, $g(v, w) = v - c_2 w$, and p = 1, we obtain the nonlinear Schrödinger equation with delay (4) and cubic nonlinearity, given by the potential

$$F(|u|, |\bar{u}|) = b(|u| - c_1 |\bar{u}|)(|u| - c_2 |\bar{u}|).$$
(50)

For $c_1 > 0$, $c_2 > 0$, and $c_1 \neq c_2$ $(c_1, c_2 \neq 1)$, the nonlinear Schrödinger equation with delay (4), (50) admits two exact solutions, which are described by the formulas (8), (47)–(48) for $c = c_1$ and $c = c_2$.

Note that any quadratic potential of the general form $F(|u|, |\bar{u}|) = b_1|u|^2 + b_2|u||\bar{u}| + b_3|\bar{u}|^2$, subject to the condition $b_1^2 - 4b_1b_2 \ge 0$, is reduced to a potential of the form (50).

Remark 5. A linear Schrödinger equation of the special form was studied in [60], where delay was taken into account in space derivatives.

Remark 6. Nonlinear Schrödinger equations with distributed delay containing integral terms were considered in [45, 61–63].

Brief conclusion

In this paper, for the first time, the general nonlinear Schrödinger equation is investigated, the potential of which is set using an arbitrary function of two arguments, depending on the unknown function and on the unknown function with delay. One-dimensional reductions have been described, which lead the equations under consideration to ordinary differential equations and ordinary differential equations with delay. A number of exact solutions of nonlinear Schrödinger equations with delay, which have been expressed in quadratures or elementary functions, are given. Solutions have been constructed, the amplitude of which varies periodically in time and space. All obtained results are new. The exact solutions presented in the article can be used to test numerical methods for integrating nonlinear equations of mathematical physics with delay. It is important to note that the exact solutions obtained are valid for an arbitrary function $f(|u|, |\bar{u}|)$ included in the general nonlinear Schrödinger equation with delay (4), so they can be used for a wide variety of problems by specifying a specific form of this function.

Declaration of competing interest

The authors declare that there is no conflict of interest.

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