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Comparison of the Effectiveness of Different Methods for Constructing Exact Solutions to Nonlinear PDEs. Generalizations and New Solutions

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Abstract: The paper shows that, in looking for exact solutions to nonlinear PDEs, the direct method of functional separation of variables can, in certain cases, be more effective than the method of differential constraints based on the compatibility analysis of PDEs with a single constraint (or the nonclassical method of symmetry reductions based on an invariant surface condition). This fact is illustrated by examples of nonlinear reaction-diffusion and convection-diffusion equations with variable coefficients, and nonlinear Klein–Gordon-type equations. Hydrodynamic boundary layer equations, nonlinear Schrödinger type equations, and a few third-order PDEs are also investigated. Several new exact functional separable solutions are given. A possibility of increasing the efficiency of the Clarkson–Kruskal direct method is discussed. A generalization of the direct method of the functional separation of variables is also described. Note that all nonlinear PDEs considered in the paper include one or several arbitrary functions.

Keywords: functional separation of variables; differential constraints; nonclassical method; symmetry reductions; invariant surface condition; Clarkson–Kruskal direct method; nonlinear Klein–Gordon equations; boundary layer equations; Schrödinger type equations; exact solutions

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Contents

1. Introduction. Discussed Methods

1.1. Preliminary Remarks

The paper deals with nonlinear partial differential equations in which x and t are independent variables, and $u = u(x,t)$ is the unknown function.

In mathematical physics and applied mathematics, the most common solutions that are looked for and used are traveling-wave and self-similar solutions (e.g., see [1–4]). A traveling-wave solution has the form $u = u(z)$ with $z = x - \lambda t$, where λ is a constant and $u(z)$ is a function determined from a single ODE. A self-similar solution has the form $u = x^{\alpha} w(y)$ with $y = t^{\beta} x$, where α and β are constants, and function $w(y)$ is also described by a single ODE. Importantly, the problem of constructing exact solutions for a PDE in these cases is reduced to solving an ODE. The main methods for the integration of ODEs are described, for example, in [5–9].

The present paper makes a comparison of methods for seeking more complex exact solutions to nonlinear PDEs, which are found from the analysis of associated overdetermined systems of coupled differential equations or functional-differential equations. The paper looks at a few of the most effective methods (but not all of them) that are frequently used by scientists and have a wide range of applications (i.e., can be used to construct exact solutions to PDEs of different orders and types).

Nonlinear PDEs that involve one or more arbitrary functions of the unknown and/or independent variables are clearly the most difficult to analyze and find exact solutions. Such equations have significant generality and are of great practical interest for testing various numerical and approximate analytical methods for solving corresponding initial-boundary value problems. It is these equations that will be the focus of the present paper.

It should be emphasized that the focus of the paper is on closed-form exact solutions (or, briefly, *closed-form solutions*) that can be represented by analytical formulas written explicitly or implicitly using a predefined bounded set of allowed functions and mathematical operations. The allowed functions include elementary functions and functions appearing in the equation (this is required when the PDE in question includes arbitrary functions). The allowed mathematical operations are arithmetic operations, a finite number of function composition operations, and the indefinite integral.

1.2. Direct Method for Constructing Functional Separable Solutions in Implicit Form. Splitting Principle

Let us look at nonlinear PDEs of the form

$$F(x, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots) = 0. \tag{1}$$

Equation (1) can be analyzed using a direct method of functional separation of variables based on seeking exact solutions in implicit form [10,11]:

$$\int h(u)\,du = \xi(x)\omega(t) + \eta(x). \tag{2}$$

Functions $h(u)$, $\xi(x)$, $\eta(x)$, and $\omega(t)$ are to be determined in subsequent analysis.

The procedure for constructing such solutions is as follows. First, using Formula (2), one calculates partial derivatives u_x, u_t, u_{xx}, $\ldots$, which are expressed in terms of functions h, ξ, η, ω, and their derivatives. Then, these partial derivatives must be substituted into Equation (1) followed by eliminating the variable t with the help of (2). As a result (with a suitable choice of ω), one arrives at a bilinear functional-differential equation:

$$\sum_{j=1}^{N} \Phi_j[x]\Psi_j[u] = 0. \tag{3}$$

Here, $\Phi_j[x] \equiv \Phi_j(x, \xi, \eta, \xi'_x, \eta'_x, \ldots)$ and $\Psi_j[u] \equiv \Psi_j(u, h, h'_u, \ldots)$ are differential forms (in some cases, functional coefficients) that depend, respectively, on x and u alone. The following statement is true:

The splitting principle (first indicated by Birkhoff [12]). Functional differential equations of Form (3) can have solutions only if forms $\Psi_j[u]$ $(j = 1, \ldots, N)$ are connected by linear relations (see, for example, [3,10,11,13]):

$$\sum_{j=1}^{m_i} k_{ij}\Psi_j[u] = 0, \qquad i = 1, \ldots, n, \tag{4}$$

where k_{ij} are some constants, $1 \leq m_i \leq N-1$, and $1 \leq n \leq N-1$. Degenerate cases must also be treated where, in addition to linear relations, some individual differential forms $\Psi_j[u]$ vanish.

Remark 1. *For more details about the splitting principle (splitting method), its application, and some additional formulas of type* (4)*, see [3].*

The splitting principle is used for the construction of exact solutions to functional differential equations of Form (3) and the corresponding nonlinear PDEs (1). Note that, in the generic case, different linear relations of Form (4) generate different solutions of the PDEs under consideration.

The efficiency of the described direct method was clearly demonstrated in [10,11], where more than 40 functional separable solutions to nonlinear reaction-diffusion and Klein–Gordon equations with variable coefficients and involving arbitrary functions were obtained.

Remark 2. *Solutions* (2) *are a natural generalization of the traveling-wave solutions, which are obtained from Formula* (2) *by substituting $\xi(x) = 1$, $\eta(x) = x$, and $\omega(t) = -\lambda t$, where λ is an arbitrary constant.*

1.3. Method of Differential Constraints

The direct method for constructing functional separable solutions in implicit form based on Formula (2) is closely related to the method of differential constraints, which is based on the compatibility theory of PDEs [14].

To show this, we differentiate Formula (2) with respect to t to obtain

$$u_t = \xi(x)\bar{\omega}(t)\varphi(u), \tag{5}$$

where $\bar{\omega}(t) = \omega'_t(t)$ and $\varphi(u) = 1/h(u)$.

Relation (5) can be treated as a first-order differential constraint, which can be used to find exact solutions of Equation (1) through compatibility analysis of the overdetermined pair of Equations (1) and (5) with single unknown u. Differential Constraint (5) is equivalent to Relation (2); initially, all functions included on the right-hand sides of (2) and (5) are considered arbitrary, and the specific form of these functions is determined in the subsequent analysis.

Differential constraints of the second and higher orders can also be used to construct exact solutions to Equation (1); in the general case, any PDE (or ODE, in degenerate cases) that depends on the same variables as the original equation can be treated as a differential constraint. For a description of the method of differential constraints and its relationship with other methods, as well as a number of specific examples of its application, see, for example, [3,14–22]. Note that exact solutions can be sought using several differential constraints (see, for example, [3,20]).

The construction of exact solutions by the standard variant of the method of differential constraints is based on a compatibility analysis of PDEs and is carried out in several steps, briefly described below.

1. Two PDEs, the original PDE and a differential constraint, are differentiated (sufficiently many times) with respect to x and t, and then the highest-order derivatives are eliminated from the differential relations obtained and PDEs considered. As a result, one arrives at an equation involving powers of lower-order derivatives, for example, u_x.

2. By equating the coefficients of all degrees of derivative u_x with zero in this equation, one obtains compatibility conditions relating the functional coefficients of the PDEs.

3. The compatibility conditions make up a nonlinear system of ODEs for determining functional coefficients. In this step, it is necessary to find a solution to this system in a closed form.

4. The obtained coefficients are substituted into the differential constraint, which must then be integrated to find a form (or forms) of unknown function u (in this step, intermediate solutions are obtained that contain undetermined functions).

5. The final form of the unknown function is determined from the original PDE.

In the last three steps of the method of differential constraints, one has to solve different equations (systems of equations). If no solution can be found in at least one of these steps, the procedure fails and no exact solution to the original equation is obtained.

1.4. Nonclassical Method of Symmetry Reductions by Bluman and Cole

The first-order differential Constraint (5) is a special case of an invariant surface condition [23] that characterizes the nonclassical method of symmetry reductions. This method, just like the method of different constraints, is also based on compatibility analysis of PDEs; specific examples of its use can be found, for example, in [3,13,23–32]. For first-order differential constraints, the results obtained with the standard variant of the method of differential constraints (described in Section 1.3) and those with the nonclassical method of symmetry reductions coincide, provided that the differential constraint coincides with the invariant surface condition. Therefore, the nonclassical method of symmetry reductions can be regarded as an important special case of the method of differential constraints.

Remark 3. *Although leading to a more complex (nonlinear) system of PDEs, the nonclassical method of symmetry reductions with splitting by derivatives [23,30] allows one to obtain more closed-form solutions than the classical Lie group analysis of PDEs [1,33,34]. It is noteworthy that solutions of Form* (2) *cannot usually be obtained with classical Lie group analysis of differential equations.*

1.5. Question: Which Method is More Effective?

Although the differential Constraint (5) is equivalent to the functional Relation (2), the subsequent procedure for finding exact solutions by the direct method for constructing functional separable solutions in implicit form and that by the method of differential constraints (the nonclassical method of symmetry reductions) differ significantly. A natural and very important question arises: Do these methods result in the same exact solutions or not?

We show below that the direct method of functional separation of variables based on the implicit representation of Solution (2) can, in certain cases, provide more closed-form solutions than the method of differential constraints with the equivalent differential constraint (or the nonclassical method of symmetry reductions with the equivalent invariant surface condition) (5).

2. Nonlinear Reaction-Diffusion Equations with Variable Coefficients

2.1. Using the Method of Differential Constraints

Let us look at nonlinear reaction-diffusion equations with variable coefficients of form

$$c(x)u_t = [a(x)f(u)u_x]_x + b(x)g(u). \tag{6}$$

Remark 4. *Some exact solutions to nonlinear reaction-diffusion equations of Form (6) were obtained, for example, in [3,13,16,22,35–46]; for exact solutions to more complex, nonlinear delay reaction-diffusion equations, see [32,47–55].*

To construct exact solutions to this equation, we use differential constraint (invariant surface condition)

$$u_t = \theta(x,t)\varphi(u), \tag{7}$$

which is more general than Constraint (5).

We solve Equation (6) for the highest derivative and eliminate u_t with the help of (7) to obtain

$$u_{xx} = -\frac{f'_u}{f}u_x^2 - \frac{a'_x}{a}u_x - \frac{b}{a}\frac{g}{f} + \frac{c\theta}{a}\frac{\varphi}{f}. \tag{8}$$

Differentiating Constraint (7) twice with respect to x and taking into account Relation (8), we get

$$\begin{aligned}
u_t &= \theta\varphi, \quad u_{tx} = \theta\varphi'_u u_x + \theta_x\varphi,\\
u_{txx} &= \theta\varphi'_u u_{xx} + \theta\varphi''_{uu}u_x^2 + 2\theta_x\varphi'_u u_x + \theta_{xx}\varphi\\
&= \theta\Big(\varphi''_u - \frac{f'_u}{f}\varphi'_u\Big)u_x^2 + A_1(x,t,u)u_x + A_0(x,t,u),\\
A_1(x,t,u) &= \Big(2\theta_x - \frac{a'_x}{a}\theta\Big)\varphi'_u,\\
A_0(x,t,u) &= \theta_{xx}\varphi - \frac{b\theta}{a}\frac{g}{f}\varphi'_u + \frac{c\theta^2}{a}\frac{\varphi}{f}\varphi'_u.
\end{aligned} \tag{9}$$

Note that A_1 and A_0 are independent of u_x and are expressed in terms of the functions appearing in PDEs (6) and (7).

Differentiating (8) with respect to t and taking into account the first two relations of (9), we find the mixed derivative in a different way:

$$\begin{aligned}
u_{xxt} &= -\theta\Big[\varphi\Big(\frac{f'_u}{f}\Big)'_u + 2\frac{f'_u}{f}\varphi'_u\Big]u_x^2 + B_1(x,t,u)u_x + B_0(x,t,u),\\
B_1(x,t,u) &= -2\theta_x\varphi\frac{f'_u}{f} - \frac{a_x\theta}{a}\varphi'_u,\\
B_0(x,t,u) &= -\frac{a_x\theta_x}{a}\varphi - \frac{b\theta}{a}\varphi\Big(\frac{g}{f}\Big)'_u + \frac{c\theta_t}{a}\frac{\varphi}{f} + \frac{c\theta^2}{a}\varphi\Big(\frac{\varphi}{f}\Big)'_u.
\end{aligned} \tag{10}$$

By matching up third-order mixed Derivatives (9) and (10), we obtain the following relation, quadratic in u_x:

$$Ku_x^2 + Mu_x + N = 0, \tag{11}$$

where

$$\begin{gathered} K = \theta\Big[\varphi''_u + \varphi'_u \frac{f'_u}{f} + \varphi\Big(\frac{f'_u}{f}\Big)'_u\Big], \quad M = 2\theta_x\Big(\varphi'_u + \varphi\frac{f'_u}{f}\Big), \\ N = \theta_{xx}\varphi + \frac{a_x\theta_x}{a}\varphi - \frac{c\theta_t}{a}\frac{\varphi}{f} + \frac{b\theta}{a}\Big[\varphi\Big(\frac{g}{f}\Big)'_u - \varphi'_u\frac{g}{f}\Big] - \frac{c\theta^2}{a}\Big[\varphi\Big(\frac{\varphi}{f}\Big)'_u - \varphi'_u\frac{\varphi}{f}\Big]. \end{gathered} \tag{12}$$

Functional coefficients K, M, and N depend on a, b, c, f, g, θ, φ, and their derivatives (and are independent of u_x). By equating in Equation (11) functional coefficients K, M, and N to zero (the procedure of splitting by the derivative u_x), one can obtain a determining system of equations. Next, we only need the first equation of this system (corresponding to $K = 0$), which, after dividing by θ, takes the form

$$\varphi''_u + \varphi'_u \frac{f'_u}{f} + \varphi\Big(\frac{f'_u}{f}\Big)'_u = 0. \tag{13}$$

Considering f to be an arbitrary function and φ to be the unknown, we find the general solution of Equation (13):

$$\varphi = \frac{1}{f}\Big(C_1 \int f\,du + C_2\Big), \tag{14}$$

where C_1 and C_2 are arbitrary constants. Thus, the method of differential constraints leads to exact solutions in which functions f and φ (involved in the original equation and the differential constraint) are related by Relation (14).

Using differential Constraint (5) is equivalent to representing the solution in Form (2). Since $\varphi = 1/h$, Solution (14) can be rewritten in terms of f and h as

$$h = f\Big(C_1 \int f\,du + C_2\Big)^{-1}. \tag{15}$$

2.2. *Using Direct Method of Functional Separation of Variables*

The study [10] presents a large number of exact solutions to PDEs of Form (6), obtained using the method described in Section 1.2. In particular, it shows that equation

$$u_t = [a(x)f(u)u_x]_x + \frac{a'_x(x)}{\sqrt{a(x)}}u, \tag{16}$$

that contains two arbitrary functions $a(x) > 0$ and $f(u)$, admits the exact solution in implicit form

$$\int \frac{f(u)}{u}\,du = 4t - 2\int \frac{dx}{\sqrt{a(x)}} + C, \tag{17}$$

where C is an arbitrary constant.

Solution (17) is a special case of Solutions (2) with $h = f/u$. This solution is different from Equation (15); consequently, it cannot be obtained by the method of differential constraints using Relation (5), neither can it be obtained using the more general differential Constraint (7).

Solutions of Form (17) are generated by two differential constraints: one of them is (5) and the other (additional) constraint has the form $u_x = p(x)\psi(u)$ (namely, $\sqrt{a}fu_x = -2u$). It is important to note that the latter constraint is determined by the functional coefficients of the original Equation (6) and cannot be obtained from general a priori considerations.

In addition to Solution (17), several other exact solutions of Form (2) were also obtained in [10], which do not satisfy Relation (15) and are omitted here; just as above, these solutions cannot be obtained by the method of differential constraints based on a single constraint (or the nonclassical method of symmetry reductions based on invariant surface Conditions (5) or (7)).

Remark 5. *It can be shown that Solution* (17) *cannot be obtained by the method of differential constraints using a single constraint of the form* $u_t = U(x,t,u)$*, which is even more general than Conditions* (5) *and* (7)*.*

Remark 6. *In applying the nonclassical method of symmetry reductions to Equation* (16)*, the loss of exact Solution* (17) *occurred when Relation* (11) *was split in powers of* u_x*. Instead of splitting in powers of* u_x*, we can consider the compatibility of the complex nonlinear system of three coupled Equations* (7)*,* (8)*, and* (11)*, with three unknown functions* u*,* φ*, and* θ *(this is how weak symmetries are studied [56]). In this case, although Solution* (17) *is not lost, it is technically much more difficult to obtain than with the direct method of functional separation of variables. It is noteworthy that the system of three PDEs,* (7)*,* (8)*, and* (11)*, is not simpler than original Equation* (6)*.*

3. Nonlinear Convection-Diffusion Equations with Variable Coefficients

3.1. Using the Method of Differential Constraints

Let us look at nonlinear convection-diffusion equations of the form

$$c(x)u_t = [a(x)f(u)u_x]_x + b(x)g(u)u_x. \tag{18}$$

Remark 7. *For symmetries, transformations, and some exact solutions to nonlinear convection-diffusion equations with variable coefficients, see, for example, [3,57–62].*

The compatibility analysis of two PDEs, original Equation (18), and differential Constraint (7), is performed in the same fashion as in Section 2.1. As a result, we obtain a relation quadratic in u_x, of Form (11), in which the functional coefficient of u_x^2 coincides with K from (12).

Therefore, the method of differential constraints based on single Constraint (7) for the convection-diffusion equations (18) also results in Relations (14) and (15).

3.2. Using the Direct Method of Functional Separation of Variables

It can be shown that the nonlinear convection-diffusion equation of special form

$$u_t = [a(x)f(u)u_x]_x - \tfrac{1}{2}a'_x(x)f(u)u_x, \tag{19}$$

where $a(x)$ and $f(u)$ are arbitrary functions, admits two exact solutions

$$\int \frac{f(u)}{u}\,du = kt \pm \sqrt{k}\int \frac{dx}{\sqrt{a(x)}} + C, \tag{20}$$

with C and k being arbitrary constants.

Solutions (20) are special cases of solutions of Form (2) with $h = f/u$. These solutions do not satisfy Relation (15) and, therefore, cannot be obtained by the method of differential constraints based on single Constraint (5); however, these solutions can be obtained if two differential constraints are used at once.

4. Nonlinear Klein–Gordon-Type Equations with Variable Coefficients

4.1. Using the Method of Differential Constraints

Now, let us look at the nonlinear Klein–Gordon-type equation with variable coefficients

$$c(x)u_{tt} = [a(x)f(u)u_x]_x + b(x)g(u). \tag{21}$$

Nonlinear Klein–Gordon-type equations play an important role in relativistic quantum mechanics, field theory, and nonlinear optics. Equations of Form (21) describe optical fibers, ultrashort optical pulses, commensurate and incommensurate phase transitions, ferroelectric transitions, crystal growths, dislocations, and others (e.g., see [63–66]).

Remark 8. *For symmetries and some exact solutions to nonlinear Klein–Gordon-type equations with variable coefficients, see, for example, [3,21,67–72]; for exact solutions to more complex, nonlinear delay Klein–Gordon equations, see [73–75].*

To construct exact solutions to this equation, we also used a more general differential Constraint (7) than (5). Differentiating Constraint (7) with respect to t gives

$$u_t = \theta\varphi \quad \Longrightarrow \quad u_{tt} = \theta\varphi'_u u_t + \theta_t\varphi = \theta^2\varphi\varphi'_u + \theta_t\varphi. \tag{22}$$

We solve Equation (21) for u_{xx} and then eliminate u_{tt} with the help of Relation (22) to obtain

$$u_{xx} = -\frac{f'_u}{f}u_x^2 - \frac{a'_x}{a}u_x - \frac{b}{a}\frac{g}{f} + \frac{c}{af}(\theta^2\varphi\varphi'_u + \theta_t\varphi). \tag{23}$$

Differentiating Constraint (7) with respect to x twice and taking into account Relation (23), we find u_{txx}. Differentiating (23) with respect to t and taking into account the first two relations of (9), we determine mixed derivative u_{xxt}. By matching up the two third-order mixed derivatives, $u_{txx} = u_{xxt}$, we arrive at a relation quadratic in u_x, of Form (11), in which the functional coefficient of u_x^2 coincides with K from (12). Using the same reasoning as in Section 2.1, we obtain Relation (15) between functions f and h appearing in Equation (21) and differential Constraint (7).

4.2. Using the Direct Method of Functional Separation of Variables

The study [11] presents a large number of exact solutions to PDEs of Form (21) obtained using the method described in Section 1.2.

Let us look at the nonlinear Klein–Gordon-type equation of special form

$$u_{tt} = [a(x)f(u)u_x]_x + \frac{x^2}{a(x)}g(u), \tag{24}$$

where $a(x)$ is an arbitrary function; functions $f(u)$ and $g(u)$ are expressed in terms of the arbitrary function $h = h(u)$ as

$$f(u) = \frac{h'_u}{h^2}, \quad g(u) = -\frac{1}{h}\left(\frac{h'_u}{h^3}\right)'_u. \tag{25}$$

It was shown in [11], using the direct method of functional separation of variables, that Equation (24) with functions $f(u)$ and $g(u)$ defined by Formulas (25) admits the following exact solution in implicit form:

$$\int h(u)\,du = t - \int \frac{x\,dx}{a(x)} + C. \tag{26}$$

It follows from the first relation of (25) and Solution (26) that Relation (15) is not satisfied, and hence, Solution (26) cannot be obtained by the method of differential constraints with single Constrain (5).

In addition to Solution (26), several other exact solutions of Form (2) were also obtained in [11] that do not satisfy Relation (15) and are omitted here; just as above, these solutions also cannot be obtained by the method of differential constraints based on a single constraint (or the nonclassical method of symmetry reductions based on invariant surface Condition (5) or (7)).

5. Clarkson–Kruskal Direct Method. Axisymmetric Boundary Layer Equations

5.1. Note on the Clarkson–Kruskal Direct Method

Let us now briefly discuss the Clarkson–Kruskal direct method [76] (see also [3,17,30,31,77,78]), which is based on looking for exact solutions in Form $u = U(x,t,w(\xi))$ with $\xi = \xi(x,t)$. Functions $U(x,t,w)$ and $\xi(x,t)$ should be chosen so as to obtain ultimately a single ordinary differential equation for $w = w(\xi)$. The requirement that function w must satisfy a single ODE greatly limits the capabilities of this method and does not allow it to be effectively used to find exact solutions such as presented in this note.

The nonclassical method of symmetry reductions is more general than the Clarkson–Kruskal direct method [26,27].

The effectiveness of the Clarkson–Kruskal direct method would greatly increase if one assumed that function w could satisfy an overdetermined system of several ODEs (or, in other words, could satisfy several differential constraints); see Sections 5.2 and 5.3 below.

5.2. Axisymmetric Boundary Layer. Functional Separable Solutions in Explicit Form

The system of equations of a laminar unsteady axisymmetric boundary layer on a body of revolution [79] can be reduced through the introduction of stream function u (and a suitable new independent variable z) to a single nonlinear third-order PDE with variable coefficients [80]:

$$u_{tz} + u_z u_{xz} - u_x u_{zz} = \nu r^2(x) u_{zzz} + F(x,t), \tag{27}$$

where $r = r(x)$ describes the shape of the body (this function is considered arbitrary here), while $F(x,t)$ defines the pressure gradient, and ν is the kinematic viscosity.

Exact solutions to Equation (27) can be sought using the method of functional separation of variables in explicit form [80]

$$u = fw(\xi) + gz + h, \qquad \xi = \varphi z + \psi, \tag{28}$$

with functions $f = f(x,t)$, $g = g(x,t)$, $h = h(x,t)$, $\varphi = \varphi(x,t)$, $\psi = \psi(x,t)$, and $w = w(\xi)$ to be determined. Substituting (28) into Equation (27), and replacing z with $(\xi - \psi)/\varphi$ yields functional differential equation

$$\sum_{n=1}^{6} \Phi_n[x,t]\Psi_n[\xi] = \Phi_7[x,t]\Psi_7[\xi]. \tag{29}$$

Here, $\Phi_n = \Phi_n[x,t]$ are differential forms dependent on the functional coefficients (and their derivatives) involved in (28) and (27) (all Φ_n are independent of w),

$$\begin{aligned} &\Phi_1 = g_t + gg_x - F, \quad \Phi_2 = (f\varphi)_t + (fg\varphi)_x, \quad \Phi_3 = f\varphi(f\varphi)_x, \\ &\Phi_4 = f(\varphi\psi_t + g\varphi\psi_x + g_x\varphi\psi - \psi\varphi_t - g\varphi_x\psi - h_x\varphi^2), \\ &\Phi_5 = f(\varphi_t + g\varphi_x - g_x\varphi), \quad \Phi_6 = -ff_x\varphi^2, \quad \Phi_7 = \nu r^2 f\varphi^3. \end{aligned} \tag{30}$$

Forms $\Psi_n = \Psi_n[\xi]$ are only dependent on function w (and its derivatives) and are expressed as [80]:

$$\begin{aligned} &\Psi_1 = 1, \quad \Psi_2 = w'_\xi, \quad \Psi_3 = (w'_\xi)^2, \quad \Psi_4 = w''_{\xi\xi}, \\ &\Psi_5 = \xi w''_{\xi\xi}, \quad \Psi_6 = w w''_{\xi\xi}, \quad \Psi_7 = w'''_{\xi\xi\xi}. \end{aligned} \tag{31}$$

The variables in Equation (29) can be separated if we assume that $\Phi_n[x,t]$ on the left-hand side of Equation (29) are all proportional to $\Phi_7[x,t]$. This leads to an overdetermined system of PDEs:

$$\Phi_n[x,t] = a_n \Phi_7[x,t], \quad n = 1, \ldots, 6 \quad (a_n = \text{const}), \tag{32}$$

and a single nonlinear ODE for $w = w(\xi)$,

$$\sum_{n=1}^{6} a_n \Psi_n = \Psi_7. \tag{33}$$

If, for some a_n, one succeeds in finding a particular solution to nonlinear System (32), then the corresponding solution to Equation (33) generates an exact solution to Equation (27). These solutions correspond to using the Clarkson–Kruskal direct method.

5.3. Axisymmetric Boundary Layer. Using Multiple Differential Constraints

It can be shown that the most interesting solutions of Form (28), those involving several arbitrary functions, may be obtained if one uses two or three differential relations that are linear combinations of forms Ψ_n defined in (31).

Table 1 lists a number of functions $w = w(\xi)$ that generate two or three linear differential constraints among differential Forms (31). The differential constraints shown in the first ten rows were described in [80]; the last four rows show new differential constraints, which generate new exact solutions of Form (28) to Equation (27) (we do not discuss these solutions in this paper).

Table 1. Generating functions w and corresponding linear relations among Ψ_n.

No.	Generating Functions u	Linear Constraints between Ψ_n
1	$w = \xi^2$	$\Psi_4 = 2\Psi_1,\ \Psi_5 = \Psi_2,\ \Psi_6 = \frac{1}{2}\Psi_3$
2	$w = \xi^3$	$\Psi_5 = 2\Psi_2,\ \Psi_6 = \frac{2}{3}\Psi_3,\ \Psi_7 = 6\Psi_1$
3	$w = \xi^4$	$\Psi_5 = 3\Psi_2,\ \Psi_6 = \frac{3}{4}\Psi_3$
4	$w = \xi^{-1}$	$\Psi_5 = -2\Psi_2,\ \Psi_6 = 2\Psi_3,\ \Psi_7 = -6\Psi_3$
5	$w = \xi^n$	$\Psi_5 = (n-1)\Psi_2,\ \Psi_6 = \frac{n-1}{n}\Psi_3\ \ (n \neq -1, 0, 1, 2, 3)$
6	$w = \exp\xi$	$\Psi_2 = \Psi_4 = \Psi_7,\ \Psi_6 = \Psi_3$
7	$w = \cosh\xi$	$\Psi_6 = \Psi_1 + \Psi_3,\ \Psi_7 = \Psi_2$
8	$w = \sinh\xi$	$\Psi_6 = \Psi_3 - \Psi_1,\ \Psi_7 = \Psi_2$
9	$w = \cos\xi$	$\Psi_6 = \Psi_3 - \Psi_1,\ \Psi_7 = -\Psi_2$
10	$w = \sin\xi$	$\Psi_6 = \Psi_3 - \Psi_1,\ \Psi_7 = -\Psi_2$
11	$w = \tanh\xi$	$\Psi_6 = -2\Psi_2 + 2\Psi_3,\ \Psi_7 = -2\Psi_2 - 3\Psi_6$
12	$w = \coth\xi$	$\Psi_6 = -2\Psi_2 + 2\Psi_3,\ \Psi_7 = -2\Psi_2 - 3\Psi_6$
13	$w = \tan\xi$	$\Psi_6 = -2\Psi_2 + 2\Psi_3,\ \Psi_7 = 2\Psi_2 + 3\Psi_6$
14	$w = \cot\xi$	$\Psi_6 = 2\Psi_2 + 2\Psi_3,\ \Psi_7 = 2\Psi_2 - 3\Psi_6$

It is important that the differential constraints specified in Table 1 are not known in advance. They arise in the course of the analysis and result from the representation of solutions to Equation (27) in explicit Form (28) and while using Equation (33).

Similar exact solutions based on several differential constraints for other hydrodynamic boundary layer equations are obtained in [81,82].

6. Functional Separable Solutions of Other Nonlinear PDEs

6.1. Functional Separable Solutions of Nonlinear PDEs with Two or More Space Variables

The method described in Section 1.2 allows one to construct exact solutions of nonlinear PDEs with two or more space variables. In this case, instead of Formula (2), a functional separable solution should be sought in form

$$\int h(u)\,du = \xi(\mathbf{x})\omega(t) + \eta(\mathbf{x}), \tag{34}$$

where $\mathbf{x} = \{x_1, \ldots, x_n\}$ and $x_1, \ldots, x_n$ are spatial variables included in the equation in question. We illustrate this with a specific example.

Consider a nonlinear reaction-diffusion equation with n space variables:

$$u_t = a(\mathbf{x})\nabla \cdot [b(\mathbf{x})f(u)\nabla u] + c(\mathbf{x}) + \frac{k(t)}{f(u)}, \tag{35}$$

where $a(\mathbf{x})$, $b(\mathbf{x})$, $c(\mathbf{x})$, $k(t)$, and $f(u)$ are arbitrary functions, ∇ is the gradient operator, and

$$\nabla \cdot [b(\mathbf{x})f(u)\nabla u] = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\Big[b(\mathbf{x})f(u)\frac{\partial u}{\partial x_j}\Big].$$

It is easy to show that Equation (35) admits an exact solution in implicit form

$$\int f(u)\,du = \int k(t)\,dt + \eta(\mathbf{x}), \tag{36}$$

where function $\eta = \eta(\mathbf{x})$ satisfies linear elliptic equation

$$\nabla \cdot [b(\mathbf{x})\nabla \eta] = -c(\mathbf{x})/a(\mathbf{x}). \tag{37}$$

In the special case $b(\mathbf{x}) \equiv 1$, this is the Poisson equation $\Delta\eta = -c(\mathbf{x})/a(\mathbf{x})$ (for exact solutions to the Poisson equation, see, for example, [83]).

6.2. Functional Separable Solutions of Third-Order Nonlinear PDEs

In [3,84], some explicit functional separable solutions to nonlinear KDV type equations are given. The method described in Section 1.2 can be successfully applied to construct exact solutions in implicit form to nonlinear PDEs of the third and higher orders. This is illustrated with two specific examples.

1. Consider a third-order nonlinear PDE of form

$$u_t = a(x)[b(x)f(u)u_x]_{xx} + c(x) + \frac{k}{f(u)}, \tag{38}$$

where $a(x)$, $b(x)$, $c(x)$, and $f(u)$ are fairly arbitrary functions, and $k \neq 0$ is an arbitrary constant.

By the method described in Section 1.2, we can construct an exact solution to Equation (38) in implicit form:

$$\int f(u)\,du = kt + \int \zeta(x)\,dx + C_1, \tag{39}$$

where C_1 is an arbitrary constant, and function $\zeta = \zeta(x)$ is determined by the second-order linear ODE

$$a(x)[b(x)\zeta]''_{xx} + c(x) = 0. \tag{40}$$

Integrating twice, we obtain the general solution of Equation (40):

$$\zeta = \frac{1}{b(x)}\left[C_2 x + C_3 - \int\left(\int \frac{c(x)}{a(x)}\,dx\right)dx\right], \tag{41}$$

where C_2 and C_3 are arbitrary constants.

Thus, Formulas (39) and (41) define an exact functional separable solution to third-order nonlinear Equation (38).

2. Let us look at another nonlinear PDE of the third order

$$u_t = [x^2 a(x) f(u) u_x]_{xx} - a(x)[k + 2f(u)]u_x, \tag{42}$$

where $a(x)$ and $f(u)$ are arbitrary functions. It is not difficult to verify that this equation admits the following functional separable solution in implicit form:

$$\int f(u)\,du = kt - \int \frac{dx}{a(x)} + C_1.$$

6.3. Functional Separable Solutions of the Nonlinear Schrödinger Equation of General Form

Consider the nonlinear Schrödinger equation of general form:

$$i w_t + w_{xx} + f(|w|)w = 0, \tag{43}$$

where w is a complex-valued function of real variables x and t, $f(u)$ is an arbitrary real-valued function of a real variable, and $i^2 = -1$.

Nonlinear Schrödinger-type equations are often used to describe different processes in theoretical physics, including nonlinear optics, superconductivity, and plasma physics [85–88].

Exact solutions of Equation (43) are sought in form

$$w = u(x,t)\exp[iv(x,t)], \tag{44}$$

where real-valued functions $u = u(x,t)$ and $v = v(x,t)$ satisfy the following nonlinear system of coupled PDEs:

$$-uv_t + u_{xx} - uv_x^2 + uf(|u|) = 0, \tag{45}$$

$$u_t + u_x v_x + (uv_x)_x = 0. \tag{46}$$

Some traveling wave solutions (optical solitons) to coupled Equations (45) and (46) with $f(u)$ of special form were obtained, for example, in [89–91]. Below are several functional separable solutions of more general System (45)–(46) with arbitrary function $f(u)$ [3].

1. There is a traveling-wave solution of form

$$u = u(y), \quad v = Ax + Bt + C, \quad y = x - 2At,$$

where A, B, and C are arbitrary real constants, and function $u = u(y)$ determined by the autonomous ODE

$$u''_{yy} + uf(|u|) - (A^2 + B)u = 0.$$

Integrating yields the general solution in implicit form:

$$\int \frac{du}{\sqrt{(A^2+B)u^2 - 2F(u) + C_1}} = C_2 \pm y, \quad F(u) = \int uf(|u|)\,du,$$

where A, B, C, C_1, and C_2 are arbitrary real constants.

2. There is a functional separable solution of form

$$u = u(z), \quad v = Axt - \tfrac{2}{3}A^2t^3 + Bt + C, \quad z = x - At^2,$$

where A, B, and C are arbitrary real constants; the function $u = u(z)$ is determined by the ODE

$$u''_{zz} + uf(|u|) - (Az + B)u = 0.$$

3. There are functional separable solutions of the form

$$u = \frac{1}{C_1\sqrt{t}}, \quad v = \frac{(x + C_2)^2}{4t} + \int f\big(|C_1^2 t|^{-1/2}\big)\,dt + C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants ($C_1 \neq 0$).

4. There are functional separable solutions of form

$$u = u(x), \quad v = C_1 t + C_2 \int \frac{dx}{u^2(x)} + C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants, and the function $u = u(x)$ is determined by the autonomous ODE

$$u''_{xx} - C_1 u - C_2^2 u^{-3} + uf(|u|) = 0,$$

whose general solution can be written in implicit form.

5. There is a functional separable solution of the form

$$u = u(\zeta), \quad v = At + \phi(\zeta), \quad \zeta = kx + \lambda t,$$

where A, k, and λ are arbitrary real constants; functions $u = u(\zeta)$ and $\phi = \phi(\zeta)$ are determined by system of coupled ODEs

$$\begin{aligned} k^2 u\phi''_{\zeta\zeta} + 2k^2 u'_\zeta \phi'_\zeta + \lambda u'_\zeta &= 0, \\ k^2 u''_{\zeta\zeta} - k^2 u(\phi'_\zeta)^2 - \lambda u\phi'_\zeta - Au + uf(|u|) &= 0. \end{aligned}$$

Remark 9. *The handbook [3] presents a number of exact solutions to more complex nonlinear Schrödinger and Ginzburg–Landau type equations with variable coefficients that may depend on t or x.*

7. A Generalization of the Method of Functional Separation of Variables

7.1. Using Nonlocal Transformations

Instead of searching for exact solutions to Equation (1) in Form (2), let us make a nonlocal transformation

$$\vartheta(x,t) = \int h(u)\,du, \tag{47}$$

where $\vartheta = \vartheta(x,t)$ a new unknown function; function $h(u)$ can vary and is determined in the subsequent analysis when constructing exact solutions. The representation of nonlocal transformation (47) as a nonlinear integral is a generalization of Solution (2). Schematically, this can be written as

$$\xi(x)\omega(t) + \eta(x) \implies \vartheta(x,t).$$

To be specific, let us look at nonlinear reaction-diffusion Equation (6). Using (47), we calculate partial derivatives u_x, u_t, and u_{xx}. Then, substituting these into Equation (6), we obtain

$$-c\vartheta_t + (a\vartheta_x)_x f + a\vartheta_x^2\Big(\frac{f}{h}\Big)'_u + bgh = 0. \tag{48}$$

For $h = 1$, Equation (48) coincides with original Equation (6). Therefore, at this stage, no solutions are lost.

Introducing functions

$$\begin{aligned} &\varphi_1 = -c\vartheta_t, \quad &&\varphi_2 = (a\vartheta_x)_x, \quad &&\varphi_3 = a\vartheta_x^2, \quad &&\varphi_4 = b;\\ &\psi_1 = 1, &&\psi_2 = f, &&\psi_3 = (f/h)'_u, &&\psi_4 = gh, \end{aligned} \tag{49}$$

we rewrite Equation (48) as

$$\sum_{j=1}^{4} \varphi_j \psi_j = 0. \tag{50}$$

Equation (50) is similar in form to functional differential Equation (3). However, in Equation (50), functions φ_j and ψ_j depend on the same independent variables x and t, whereas in Equation (3) they depend on different variables. Therefore, in this case, it is not possible to make full use of the splitting principle formulated in Section 1.2, as there may also be other exact solutions. However, one can try to construct exact solutions of Equation (50) by equating several linear combinations of functions φ_j (and ψ_j) with zero; in addition, one can also consider degenerate cases in which, in addition to the linear combinations, individual φ_j or ψ_j vanish. We call this approach to constructing exact solutions the *generalized splitting principle*.

Example 1. *Equation* (50) *holds if we set*

$$\varphi_1 = -A\varphi_3, \quad \varphi_2 = \varphi_4; \quad \psi_3 = A\psi_1, \quad \psi_4 = -\psi_2, \tag{51}$$

where A is an arbitrary constant. Substituting (49) *in* (51) *gives*

$$c\vartheta_t = Aa\vartheta_x^2, \quad (a\vartheta_x)_x = b; \quad (f/h)'_u = A, \quad gh = -f. \tag{52}$$

For $c(x) = 1$, the analysis of System of Equations (52) *leads to Equation* (16) *and its Solution* (17) *(if $A = \frac{1}{4}k$). In addition, there is another equation, which only differs from Equation* (16) *in the sign of the last term, and its solution; these are given in [10].*

7.2. Possible Modifications

In addition to Equation (50), we can consider equivalent differential equations that reduce to Equation (50) on the set of functions satisfying Relation (47). Two examples of such equations are given below:

$$c(-\vartheta_t + \lambda\vartheta) - \lambda cH + (a\vartheta_x)_x f + a\vartheta_x^2\Big(\frac{f}{h}\Big)'_u + bgh = 0, \tag{53}$$

$$-c\vartheta_t e^{\lambda\vartheta} e^{-\lambda H} + (a\vartheta_x)_x f + a\vartheta_x^2\Big(\frac{f}{h}\Big)'_u + bgh = 0, \tag{54}$$

where $H = \int h\,du$ and λ is an arbitrary constant.

Using the generalized splitting principle for Equations (48), (53), and (54) leads to a number of exact solutions. For nonlinear reaction-diffusion Equation (6), the application of the generalized splitting principle to Equations (48), (53), and (54) allows us to find all functional separable solutions of Form (2) with $\xi(x) = 1$, which were obtained previously in [10].

A similar approach can also be used to find exact solutions to other nonlinear PDEs.

8. Brief Conclusions

The efficiency of various methods for constructing exact solutions of nonlinear PDEs was discussed. It was shown that the direct method of functional separation of variables can, in certain cases, be more effective than the method of differential constraints based on the compatibility analysis of PDEs with a single constraint (or the nonclassical method of symmetry reductions based on an invariant surface condition). Nonlinear reaction-diffusion and convection-diffusion equations with variable coefficients, nonlinear Klein–Gordon-type equations, hydrodynamic boundary layer equations, nonlinear Schrödinger-type equations, and some third-order PDEs were considered. Several new exact solutions were given. A possibility of increasing the efficiency of the Clarkson–Kruskal direct method was discussed. A generalization of the direct method of functional separation of variables was suggested.

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References

1. Ovsiannikov, L.V. *Group Analysis of Differential Equations*; Academic Press: Boston, MA, USA, 1982; ISBN 0124318011.
2. Barenblatt, G.I. *Scaling, Self-similarity, and Intermediate Asymptotics: Dimensional Analysis and Intermediate Asymptotics*; Cambridge University Press: Cambridge, UK, 1996; ISBN 0521435226.
3. Polyanin, A.D.; Zaitsev, V.F. *Handbook of Nonlinear Partial Differential Equations*, 2nd ed.; CRC Press: Boca Raton, FL, USA, 2012; ISBN 9781420087239.
4. Griffiths, G.; Schiesser, W. *Traveling Wave Analysis of Partial Differential Equations*; Academic Press: Burlington, NJ, USA, 2012; ISBN 9780123846525.
5. Murphy, G.M. *Ordinary Differential Equations and Their Solutions*; D. Van Nostrand: London, UK, 1960.
6. Zwillinger, D. *Handbook of Differential Equations*, 3rd ed.; Academic Press: New York, NY, USA, 1997.
7. Polyanin, A.D.; Zaitsev, V.F. *Handbook of Exact Solutions for Ordinary Differential Equations*, 2nd ed.; Chapman & Hall/CRC Press: Boca Raton, FL, USA, 2003; ISBN 1584882972.
8. Kudryashov, N.A. Simplest equation method to look for exact solutions of nonlinear differential equations. *Chaos Solitons Fractals* **2005**, *24*, 1217–1231. [CrossRef]
9. Polyanin, A.D.; Zaitsev, V.F. *Handbook of Ordinary Differential Equations: Exact Solutions, Methods, and Problems*; CRC Press: Boca Raton, FL, USA, 2018; ISBN 9781466569379.
10. Polyanin, A.D. Construction of exact solutions in implicit form for PDEs: New functional separable solutions of non-linear reaction-diffusion equations with variable coefficients. *Int. J. Non-Linear Mech.* **2019**, *111*, 95–105. [CrossRef]
11. Polyanin, A.D. Construction of functional separable solutions in implicit form for non-linear Klein–Gordon type equations with variable coefficients. *Int. J. Non-Linear Mech.* **2019**. [CrossRef]
12. Birkhoff, G. *Hydrodynamics*; Princeton University Press: Princeton, NY, USA, 1960.
13. Pucci, E.; Saccomandi, G. Evolution equations, invariant surface conditions and functional separation of variables. *Phys. D* **2000**, *139*, 28–47. [CrossRef]
14. Yanenko, N.N. The compatibility theory and methods of integration of systems of nonlinear partial differential equations. In Proceedings of the 4th All-Union Mathematics Congress, Leningrad, Soviet, 3–12 July 1961; pp. 613–621.
15. Meleshko, S.V. Differential constraints and one-parameter Lie-Bäcklund groups. *Sov. Math. Dokl.* **1983**, *28*, 37–41.

16. Galaktionov, V.A. Quasilinear heat equations with first-order sign-invariants and new explicit solutions. *Nonlinear Anal. Theor. Meth. Appl.* **1994**, *23*, 1595–1621. [CrossRef]
17. Olver, P.J. Direct reduction and differential constraints. *Proc. R. Soc. Lond. Ser. A* **1994**, *444*, 509–523. [CrossRef]
18. Kaptsov, O.V. Determining equations and differential constraints. *Nonlinear Math. Phys.* **1995**, *2*, 283–291. [CrossRef]
19. Kaptsov, O.V. Linear determining equations for differential constraints. *Sb. Math.* **1998**, *189*, 1839–1854. [CrossRef]
20. Sidorov, A.F.; Shapeev, V.P.; Yanenko, N.N. *Method of Differential Constraints and its Applications in Gas Dynamics*; Nauka: Novosibirsk, Soviet, 1984. (In Russian)
21. Andreev, V.K.; Kaptsov, O.V.; Pukhnachov, V.V.; Rodionov, A.A. *Applications of Group-Theoretical Methods in Hydrodynamics*; Kluwer: Dordrecht, The Netherlands, 1998.
22. Kaptsov, O.V.; Verevkin, I.V. Differential constraints and exact solutions of nonlinear diffusion equations. *J. Phys. A Math. Gen.* **2003**, *36*, 1401–1414. [CrossRef]
23. Bluman, G.W.; Cole, J.D. The general similarity solution of the heat equation. *J. Math. Mech.* **1969**, *18*, 1025–1042.
24. Olver, P.J.; Rosenau, P. The construction of special solutions to partial differential equations. *Phys. Lett. A* **1986**, *114*, 107–112. [CrossRef]
25. Levi, D.; Winternitz, P. Nonclassical symmetry reduction: Example of the Boussinesq equation. *J. Phys. A* **1989**, *22*, 2915–2924. [CrossRef]
26. Nucci, M.C.; Clarkson, P.A. The nonclassical method is more general than the direct method for symmetry reductions. An example of the Fitzhugh–Nagumo equation. *Phys. Lett. A* **1992**, *164*, 49–56. [CrossRef]
27. Pucci, E. Similarity reductions of partial differential equations. *J. Phys. A Math. General* **1992**, *25*, 2631–2640. [CrossRef]
28. Clarkson, P.A. Nonclassical symmetry reductions of the Boussinesq equation. *Chaos Solitons Fractals* **1995**, *5*, 2261–2301. [CrossRef]
29. Olver, P.J.; Vorob'ev, E.M. Nonclassical and conditional symmetries. In *CRC Handbook of Lie Group Analysis of Differential Equations*; Ibragimov, N.H., Eds.; CRC Press: Boca Raton, FL, USA, 1996; Volume 3, pp. 291–328.
30. Clarkson, P.A.; Ludlow, D.K.; Priestley, T.J. The classical, direct and nonclassical methods for symmetry reductions of nonlinear partial differential equations. *Methods Appl. Anal.* **1997**, *4*, 173–195. [CrossRef]
31. Saccomandi, G. A personal overview on the reduction methods for partial differential equations. *Note Math.* **2004**, *23*, 217–248.
32. Bradshaw-Hajek, B.H. Nonclassical symmetry solutions for non-autonomous reaction-diffusion equations. *Symmetry* **2019**, *11*, 208. [CrossRef]
33. Olver, P.L. *Applications of Lie Groups to Differential Equations*; Springer: New York, NY, USA, 1986; ISBN 9781468402742.
34. Ibragimov, N.H. *CRC Handbook of Lie Group Analysis of Differential Equations, Volume 1: Symmetries, Exact solutions and Conservation Laws*; CRC Press: Boca Raton, FL, USA, 1994; ISBN 9780849344886.
35. Dorodnitsyn, V.A. On invariant solutions of the equation of non-linear heat conduction with a source. *USSR Comput. Math. Math. Phys.* **1982**, *22*, 115–122. [CrossRef]
36. Kudryashov, N.A. On exact solutions of families of Fisher equations. *Theor. Math. Phys.* **1993**, *94*, 211–218. [CrossRef]
37. Clarkson, P.A.; Mansfield, E.L. Symmetry reductions and exact solutions of a class of nonlinear heat equations. *Phys. D* **1994**, *70*, 250–288. [CrossRef]
38. Doyle, P.W.; Vassiliou, P.J. Separation of variables for the 1-dimensional non-linear diffusion equation. *Int. J. Non-Linear Mech.* **1998**, *33*, 315–326. [CrossRef]
39. Hood, S. On direct, implicit reductions of a nonlinear diffusion equation with an arbitrary function-generalizations of Clarkson's and Kruskal's method. *IMA J. Appl. Math.* **2000**, *64*, 223–244. [CrossRef]
40. Galaktionov, V.A.; Svirshchevskii, S.R. *Exact Solutions and Invariant Subspaces of Nonlinear Partial Differential Equations in Mechanics and Physics*; Chapman & Hall/CRC Press: Boca Raton, FL, USA, 2006; ISBN 9781584886631.

41. Cherniha, R.M.; Pliukhin, O. New conditional symmetries and exact solutions of reaction-diffusion-convection equations with exponential nonlinearities. *J. Math. Anal. Appl.* **2013**, *403*, 23–37. [CrossRef]
42. Cherniha, R.; Serov, M.; Pliukhin, O. *Nonlinear Reaction-Diffusion-Convection Equations: Lie and Conditional Symmetry, Exact Solutions and Their Applications*; Chapman & Hall/CRC Press: Boca Raton, FL, USA, 2018; ISBN 9781498776196.
43. Vaneeva, O.O.; Johnpillai, A.G.; Popovych, R.O.; Sophocleous, C. Extended group analysis of variable coefficient reaction-diffusion equations with power nonlinearities. *J. Math. Anal. Appl.* **2007**, *330*, 1363–1386. [CrossRef]
44. Vaneeva, O.O.; Popovych, R.O.; Sophocleous, C. Enhanced group analysis and exact solutions of variable coefficient semilinear diffusion equations with a power source. *Acta Appl. Math.* **2009**, *106*, 1–46. [CrossRef]
45. Vaneeva, O.O.; Popovych, R.O.; Sophocleous, C. Extended group analysis of variable coefficient reaction-diffusion equations with exponential nonlinearities. *J. Math. Anal. Appl.* **2012**, *396*, 225–242. [CrossRef]
46. Polyanin, A.D. Functional separable solutions of nonlinear reaction-diffusion equations with variable coefficients. *Appl. Math. Comput.* **2019**, *347*, 282–292. [CrossRef]
47. Meleshko, S.V.; Moyo, S. On the complete group classification of the reaction-diffusion equation with a delay. *J. Math. Anal. Appl.* **2008**, *338*, 448–466. [CrossRef]
48. Polyanin, A.D.; Zhurov, A.I. Exact solutions of linear and non-linear differential-difference heat and diffusion equations with finite relaxation time. *Int. J. Non-Linear Mech.* **2013**, *54*, 115–126. [CrossRef]
49. Polyanin, A.D.; Zhurov, A.I. Exact separable solutions of delay reaction-diffusion equations and other nonlinear partial functional-differential equations. *Commun. Nonlinear Sci. Numer. Simul.* **2014**, *19*, 409–416. [CrossRef]
50. Polyanin, A.D.; Zhurov, A.I. Functional constraints method for constructing exact solutions to delay reaction-diffusion equations and more complex nonlinear equations. *Commun. Nonlinear Sci. Numer. Simul.* **2014**, *19*, 417–430. [CrossRef]
51. Polyanin, A.D.; Zhurov, A.I. New generalized and functional separable solutions to non-linear delay reaction-diffusion equations. *Int. J. Non-Linear Mech.* **2014**, *59*, 16–22. [CrossRef]
52. Polyanin, A.D.; Zhurov, A.I. Nonlinear delay reaction-diffusion equations with varying transfer coefficients: Exact methods and new solutions. *Appl. Math. Lett.* **2014**, *37*, 43–48. [CrossRef]
53. Polyanin, A.D.; Zhurov, A.I. The functional constraints method: Application to non-linear delay reaction-diffusion equations with varying transfer coefficients. *Int. J. Non-Linear Mech.* **2014**, *67*, 267–277. [CrossRef]
54. Polyanin, A.D.; Sorokin, V.G. Nonlinear delay reaction-diffusion equations: Traveling-wave solutions in elementary functions. *Appl. Math. Lett.* **2015**, *46*, 38–43. [CrossRef]
55. Polyanin, A.D. Generalized traveling-wave solutions of nonlinear reaction-diffusion equations with delay and variable coefficients. *Appl. Math. Lett.* **2019**, *90*, 49–53. [CrossRef]
56. Pucci, E.; Saccomandi, G. On the weak symmetry groups of partial differential equations. *J. Math. Anal. Appl.* **1992**, *163*, 588–598. [CrossRef]
57. Gandarias, M.L.; Romero, J.L.; Díaz, J.M. Nonclassical symmetry reductions of a porous medium equation with convection. *J. Phys. A Math. Gen.* **1999**, *32*, 1461–1473. [CrossRef]
58. Popovych, R.O.; Ivanova, N.M. New results on group classification of nonlinear diffusion-convection equations. *J. Phys. A Math. Gen.* **2004**, *37*, 7547–7565. [CrossRef]
59. Ivanova, N.M.; Sophocleous, C. On the group classification of variable-coefficient nonlinear diffusion-convection equations. *J. Comput. Appl. Math.* **2006**, *197*, 322–344. [CrossRef]
60. Ivanova, N.M. Exact solutions of diffusion-convection equations. *Dyn. PDE* **2008**, *5*, 139–171. [CrossRef]
61. Vaneeva, O.O.; Popovych, R.O.; Sophocleous, C. Group analysis of variable coefficient diffusion-convection equations, I. Enhanced group classification. *Lobachevskii J. Math.* **2010**, *73*, 100–122.
62. Polyanin, A.D. Functional separable solutions of nonlinear convection-diffusion equations with variable coefficients. *Commun. Nonlinear Sci. Numer. Simul.* **2019**, *73*, 379–390. [CrossRef]
63. Bulbul, B.; Sezer, M.; Greiner, W. *Relativistic Quantum Mechanics—Wave Equations*, 3rd ed.; Springer: Berlin, Germany, 2000; ISBN 9783540995357.

64. Cuevas-Maraver, J.; Kevrekidis, P.G.; Williams, F. (Eds.) *The Sine-Gordon Model and its Applications*; Springer: Heidelberg, Germany, 2014; ISBN 9783319067216.
65. Wazwaz, A.-M. The tanh and the sine-cosine methods for compact and noncompact solutions of the nonlinear Klein–Gordon equation. *Appl. Math. Comput.* **2005**, *167*, 1179–1195. [CrossRef]
66. Abazari, R.; Jamshidzadeh, S. Exact solitary wave solutions of the complex Klein–Gordon equation. *Optik* **2015**, *126*, 1970–1975. [CrossRef]
67. Grundland, A.M.; Infeld, E. A family of non-linear Klein–Gordon equations and their solutions. *J. Math. Phys.* **1992**, *33*, 2498–2503. [CrossRef]
68. Zhdanov, R.Z. Separation of variables in the non-linear wave equation. *J. Phys. A* **1994**, *27*, L291–L297. [CrossRef]
69. Estévez, P.G.; Qu, C.Z. Separation of variables in nonlinear wave equations with variable wave speed. *Theor. Math. Phys.* **2002**, *133*, 1490–1497. [CrossRef]
70. Hu, J.; Qu, C. Functionally separable solutions to nonlinear wave equations by group foliation method. *J. Math. Anal. Appl.* **2007**, *330*, 298–311. [CrossRef]
71. Huang, D.J.; Zhou, S. Group properties of generalized quasi-linear wave equations. *J. Math. Anal. Appl.* **2010**, *366*, 460–472. [CrossRef]
72. Huang, D.J.; Zhu, Y.; Yang, Q. Reduction operators and exact solutions of variable coefficient nonlinear wave equations with power nonlinearities. *Symmetry* **2017**, *9*, 3. [CrossRef]
73. Polyanin, A.D.; Zhurov, A.I. Generalized and functional separable solutions to nonlinear delay Klein–Gordon equations. *Commun. Nonlinear Sci. Numer. Simul.* **2014**, *19*, 2676–2689. [CrossRef]
74. Long, F.-S.; Meleshko, S.V. On the complete group classification of the one-dimensional nonlinear Klein–Gordon equation with a delay. *Math. Meth. Appl. Sci.* **2016**, *39*, 3255–3270. [CrossRef]
75. Long, F.-S.; Meleshko, S.V. Symmetry analysis of the nonlinear two-dimensional Klein–Gordon equation with a time-varying delay. *Math. Meth. Appl. Sci.* **2017**, *40*, 4658–4673. [CrossRef]
76. Clarkson, P.A.; Kruskal, M.D. New similarity reductions of the Boussinesq equation. *J. Math. Phys.* **1989**, *30*, 2201–2213. [CrossRef]
77. Ludlow, D.K.; Clarkson, P.A.; Bassom, A.P. Similarity reductions and exact solutions for the two-dimensional incompressible Navier–Stokes equations. *Stud. Appl. Math.* **1999**, *103*, 183–240. [CrossRef]
78. Ludlow, D.K.; Clarkson, P.A.; Bassom, A.P. New similarity solutions of the unsteady incompressible boundary-layer equations. *Quart. J. Mech. Appl. Math.* **2000**, *53*, 175–206. [CrossRef]
79. Schlichting, H. *Boundary Layer Theory*; McGraw-Hill: New York, NY, USA, 1981.
80. Polyanin, A.D.; Zhurov, A.I. Unsteady axisymmetric boundary-layer equations: Transformations, properties, exact solutions, order reduction and solution method. *Int. J. Non-Linear Mech.* **2015**, *74*, 40–50. [CrossRef]
81. Polyanin, A.D.; Zhurov, A.I. Direct functional separation of variables and new exact solutions to axisymmetric unsteady boundary-layer equations. *Commun. Nonlinear Sci. Numer. Simul.* **2016**, *31*, 11–20. [CrossRef]
82. Polyanin, A.D.; Zhurov, A.I. One-dimensional reductions and functional separable solutions to unsteady plane and axisymmetric boundary-layer equations for non-Newtonian fluids. *Int. J. Non-Linear Mech.* **2016**, *85*, 70–80. [CrossRef]
83. Polyanin, A.D.; Nazaikinskii, V.E. *Handbook of Linear Partial Differential Equations for Engineers and Scientists*, 2nd ed.; CRC Press: Boca Raton, FL, USA, 2016; ISBN 9781466581494.
84. Fushchich, W.I.; Serov, N.I.; Ahmerov, T.K. On the conditional symmetry of the generalized Korteweg-de Vries equation. *Proc. Ukr. Acad. Sci.* **1991**, *A12*, 28–30.
85. Sulem, C.; Sulem, P.-L. *The Nonlinear Schrödinger Equation. Self-Focusing and Wave Collapse*; Springer: New York, NY, USA, 1999; ISBN 9780387986111.
86. Fibich, G. *The Nonlinear Schrödinger Equation*; Springer: Basel, Switzerland, 2015; ISBN 9783319127477.
87. Kevrekidis, P.G.; Frantzeskakis, D.J.; Carretero-Gonzalez, R. *The Defocusing Nonlinear Schrodinger Equation: From Dark Solitons to Vortices and Vortex Rings*; SIAM: Philadelphia, PA, USA, 2015; ISBN 9781611973938.
88. Liu, W.-M.; Kengne, E. *Schrödinger Equations in Nonlinear Systems*; Springer: Singapore, 2019; ISBN 9789811365805.
89. Eslami, M.; Mirzazadeh, M. Optical solitons with Biswas-Milovic equation for power law and dual-power law nonlinearities. *Nonlinear Dyn.* **2016**, *83*, 731–738. [CrossRef]

90. Biswas, A.; Ekici, M.; Sonmezoglu, A.; Belic, M. Optical solitons in birefringent fibers with quadratic-cubic nonlinearity by extended trial function scheme. *Optik* **2019**, *176*, 542–548. [CrossRef]
91. Biswas, A.; Ekici, M.; Sonmezoglu, A.; Belic, M. Optical solitons in birefringent fibers with quadratic-cubic nonlinearity by extended G'/G-expansion scheme. *Optik* **2019**, *178*, 59–65. [CrossRef]