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Multi-Parameter Reaction–Diffusion Systems with Quadratic Nonlinearity and Delays: New Exact Solutions in Elementary Functions

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Abstract: The study considers a nonlinear multi-parameter reaction–diffusion system of two Lotka–Volterra-type equations with several delays. It treats both cases of different diffusion coefficients and identical diffusion coefficients. The study describes a few different techniques to solve the system of interest, including (i) reduction to a single second-order linear ODE without delay, (ii) reduction to a system of three second-order ODEs without delay, (iii) reduction to a system of three first-order ODEs with delay, (iv) reduction to a system of two second-order ODEs without delay and a linear Schrödinger-type PDE, and (v) reduction to a system of two first-order ODEs with delay and a linear heat-type PDE. The study presents many new exact solutions to a Lotka–Volterra-type reaction–diffusion system with several arbitrary delay times, including over 50 solutions in terms of elementary functions. All of these are generalized or incomplete separable solutions that involve several free parameters (constants of integration). A special case is studied where a solution contains infinitely many free parameters. Along with that, some new exact solutions are obtained for a simpler nonlinear reaction–diffusion system of PDEs without delays that represents a special case of the original multi-parameter delay system. Several generalizations to systems with variable coefficients, systems with more complex nonlinearities, and hyperbolic type systems with delay are discussed. The solutions obtained can be used to model delay processes in biology, ecology, biochemistry and medicine and test approximate analytical and numerical methods for reaction–diffusion and other nonlinear PDEs with delays.

Keywords: reaction–diffusion equations with delay; Lotka–Volterra-type systems of equations; nonlinear systems of PDEs; reductions; exact solutions; generalized separable solutions; solutions in terms of elementary functions

MSC: 35K57; 35R10; 35C05



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1. Introduction

1.1. Preliminary Remarks. The Lotka–Volterra-Type Diffusion System with Delays

Linear and nonlinear differential equations, ordinary and partial, with delay are not uncommon for mathematical modeling of phenomena and processes in various areas of theoretical physics, mechanics, control theory, biology, biophysics, biochemistry, medicine, ecology, economics and technical applications.

Below are a few factors that lead to the need to introduce delay into mathematical models described by differential equations. In biology and biomechanics, the delay is due to the limited transmission rate of nerve and muscle reactions in living tissues. In medicine, in studying the spread of infectious diseases, the delay time is determined by the incubation period (the time from the moment of infection to the first signs when the disease begins to manifest itself). In the dynamics of populations, a delay arises because individuals participate in reproduction only after reaching a certain age. In control theory, delays are usually associated with the finite speed of signal propagation and the limited speed of technological processes.

The presence of delay in mathematical models and differential equations is a complicating factor, which, as a rule, leads to a narrowing of the stability region of the solutions obtained. The study and solution of ordinary differential equations (ODEs) with a delay are comparable in complexity to the study and solution of partial differential equations (PDE) without delays. Importantly, unlike simpler PDEs without delay, which often have self-similar solutions, PDEs with delay do not admit self-similar solutions.

The present article deals with a Lotka–Volterra-type reaction–diffusion system with two components and four constant delays described by two coupled PDEs with quadratic nonlinearities:

$$\begin{aligned} u_t &= a_1 u_{xx} + u(b_1 + p_1 \bar{u}_1 + q_1 \bar{v}_2), \\ v_t &= a_2 v_{xx} + v(b_2 + p_2 \bar{u}_3 + q_2 \bar{v}_4), \end{aligned} \quad (1)$$

where $u = u(x, t)$ and $v = v(x, t)$ are the unknown functions; $\bar{u}_i = u(x, t - \tau_i)$, $i = 1, 3$; $\bar{v}_j = v(x, t - \tau_j)$, $j = 2, 4$; $\tau_i \geq 0$ and $\tau_j \geq 0$ are delay times; $a_1 > 0$ and $a_2 > 0$ are diffusion-type coefficients; and b_1, b_2, p_1, p_2, q_1 , and q_2 are some constants.

Systems of the form (1) are used in the mathematical modeling of various processes in biology, ecology, biochemistry, medicine, etc. For example, in population theory, the delays τ_1 and τ_4 characterize the mean reproductive age of species. At the same time, τ_2 and τ_3 are responsible for the time required for changes in the population size of one species to cause changes in the other. All the delays are non-negative and can be zero in some models. The terms with nonzero coefficients q_1 and p_2 make the model different from a single equation, while the coefficients are responsible for the interaction between individuals of the two populations. In the case of cooperation, when one species persists in the absence of the other or when species mutually increase each other's growth rate, the coefficients q_1 and p_2 are positive. In the case of competition, an increase in one population leads to a decrease in the other (for example, an increase in the number of predators leads to a decrease in the prey population), and the coefficients q_1 and q_2 are negative. Lotka–Volterra cooperative models with delays, described by PDEs (1), were treated in [1,2]; for competitive models, see [3–5].

The special case of the system of delay PDEs (1) with $b_2 = q_2 = 0$, $\tau_1 = \tau_3 = 0$, and $\tau_2 > 0$ describes a Belousov–Zhabotinsky oscillating chemical reaction, where u and v are the bromic acid and bromide ion concentrations (see [6]).

The system of delay PDEs (1) is a generalization of simpler equations and systems outlined below.

1°. In the simplest case of no diffusion or delays, with $a_1 = a_2 = 0$ and $\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$, the system of PDEs (1) degenerates into a system of ODEs [7,8], which is known as the classical Lotka–Volterra system.

2°. With no diffusion, when $a_1 = a_2 = 0$, system (1) degenerates into a simpler system of delay ODEs. For qualitative features and the dynamics of such delay systems, see, for example, [9–14].

3°. If $a_1 a_2 \neq 0$, the diffusion system of PDEs (1) simplifies significantly when there are no delays, $\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$. Numerous studies are devoted to qualitative analysis of such systems (e.g., see [15–19] and references in [18,19]).

The system of PDEs (1) without delays admits simple traveling wave solutions

$$u = u(z), \quad v = v(z), \quad z = x + \lambda t, \quad (2)$$

with the functions $u(z)$ and $v(z)$ described by a system of ODEs. The studies [20–22] obtained some exact solutions to this system, of the form (2), which are expressible in terms of elementary functions.

The studies [23–25] considered nonclassical symmetries and exact solutions to system (1) and related Lotka–Volterra-type systems in the absence of delays (see also [26,27]).

Remark 1. For exact solutions to various single nonlinear reaction–diffusion-type equations without delays, see, for example, [26,28–44]. The books [26,32,37,40,43,44] describe solution methods for such equations.

4°. In general, the presence of delays in system (1) complicates its analysis dramatically, since the majority of analytical methods that are effective in constructing exact solutions to nonlinear PDEs without delay are either inapplicable for solving delay PDEs at all or extremely limited for this purpose.

The system of delay PDEs (1) admits simple traveling wave solutions (2) (e.g., see [1,5,45]). At present, there is only one publication [46] where two other exact solutions to system (1) were obtained and some reductions of related nonlinear reaction–diffusion systems of PDEs were described.

Remark 2. For exact solutions to single nonlinear reaction–diffusion-type delay equations, see [47–53] (see also [54], where time-fractional reaction–diffusion PDEs with delay were considered). The studies [55,56] describe some exact solutions to nonlinear Klein–Gordon-type PDEs with delay.

1.2. The System of Delay PDEs in Question. Exact Solutions and Generalized Separable Solutions

1°. In general, the multi-parameter nonlinear Lotka–Volterra-type diffusion system of delay PDEs (1) contains 12 free parameters, four of which can be removed by scaling the dependent and independent variables. We will consider a special case of system (1) while imposing one constraint, $p_1/p_2 = q_1/q_2$, on the free parameters. With this in mind, we set $p_1 = c_1k_1, q_1 = c_1k_2, p_2 = c_2k_1, \text{ and } q_2 = c_2k_2$ to obtain

$$\begin{aligned} u_t &= a_1u_{xx} + u[b_1 + c_1(k_1\bar{u}_1 + k_2\bar{v}_2)], \\ v_t &= a_2v_{xx} + v[b_2 + c_2(k_1\bar{u}_3 + k_2\bar{v}_4)], \end{aligned} \tag{3}$$

where $\bar{u}_i = u(x, t - \tau_i), \bar{v}_j = v(x, t - \tau_j)$ ($i = 1, 3, j = 2, 4$); $a_1 > 0, a_2 > 0, b_1, b_2, k_1, k_2, \tau_i \geq 0$, and $\tau_j \geq 0$ are free parameters; $c_1 \neq 0$ and $c_2 \neq 0$ are some constants to be determined later. The present study does not deal with the degenerate cases of $k_1 = 0$ or $k_2 = 0$, when system (3) becomes semi-coupled, meaning that one equation of the system is isolated, while the other one is linear in the unknown.

2°. A solution is called exact if it satisfies the equation in question exactly (i.e., when substituted into the equation, the solution turns it into an identity). At the same time, any approximations or simplifications of the equation are not allowed, and no a priori assumptions are used.

The term ‘exact solution’ for nonlinear PDEs with delay is normally used in cases where the solution is expressed:

- (i) In terms of elementary functions;
- (ii) In terms of elementary and special functions and indefinite integrals;
- (iii) In terms of solutions to ODEs without or with delay or systems of such equations.

Combinations of solutions from items (i)–(iii) are allowed. In cases (i) and (ii), an exact solution can be represented in an explicit, implicit or parametric form.

Exact solutions play an essential role as standard reference results that can be used to verify the consistency and estimate errors of various numerical, asymptotic, and approximate analytical methods for solving nonlinear PDEs with delay. These solutions can also serve as a basis for perfecting and testing computer algebra software packages such as Maple and Mathematica.

The simplest and most preferred exact solutions for testing are solutions in terms of elementary functions (see Item (i)). Currently, the vast majority of the obtained exact solutions of such equations are more complex and relate to Item (iii).

3°. We will be looking for exact generalized separable solutions to the system of delay PDEs (3) in the form

$$u = \zeta(t)\theta(x) + \varphi(x), \quad v = \eta(t)\theta(x) + \psi(x), \tag{4}$$

or in an alternative form that can be obtained from (4) by replacing the functions $\varphi(x)$ and $\psi(x)$ with $\varphi(t)$ and $\psi(t)$, respectively. In some cases, we will look for more complicated solutions involving functions with two arguments. Special attention will be given to the construction of exact solutions expressible in terms of elementary functions.

1.3. Simplest Solutions (Stationary Points)

Constants (stationary points) are the simplest solutions to system (3):

$$u = u^\circ = \text{const}, \quad v = v^\circ = \text{const}. \tag{5}$$

These constants are determined by the formulas

$$\begin{aligned} (a) \quad u^\circ = v^\circ = 0; \quad (b) \quad u^\circ = -\frac{b_1}{c_1k_1}, \quad v^\circ = 0; \quad (c) \quad u^\circ = 0, \quad v^\circ = -\frac{b_2}{c_2k_2}; \\ (d) \quad u^\circ = -\frac{k_2}{k_1}v^\circ - \frac{b_1}{c_1k_1}, \quad v^\circ \text{ is any } \left(\text{a special case where } \frac{b_1}{c_1} = \frac{b_2}{c_2} \right). \end{aligned} \tag{6}$$

The stationary points (6) will further be used to construct more complicated solutions to system (3).

2. Reductions and Exact Solutions of the System of Delay PDEs with Different Diffusion Coefficients ($a_1 \neq a_2$)

2.1. Reduction of the System with Three Delays to a Single Second-Order Linear ODE without Delay

The four delay times in system (3) are assumed to be related by the single constraint

$$\tau_2 - \tau_1 = \tau_4 - \tau_3. \tag{7}$$

This suggests that any three of these delays can be set arbitrarily. Notably, relation (7) holds, for example, in the following three important special cases:

$$\begin{aligned} \tau_2 = \tau_1, \quad \tau_4 = \tau_3, \quad \tau_1 \text{ and } \tau_3 \text{ are arbitrary;} \\ \tau_3 = \tau_1, \quad \tau_4 = \tau_2, \quad \tau_1 \text{ and } \tau_2 \text{ are arbitrary;} \\ \tau_m = m\tau, \quad m = 1, 2, 3, 4, \quad \tau \text{ is arbitrary.} \end{aligned}$$

We seek a generalized separable solution to system (3) in the form

$$u = -k_2 e^{\lambda(t+\tau_1)} \theta(x) + u^\circ, \quad v = k_1 e^{\lambda(t+\tau_2)} \theta(x) + v^\circ, \tag{8}$$

where u° and v° are the stationary points (5) of system (3). We assume that the function $\theta = \theta(x)$ satisfies the second-order linear ODE

$$\theta''_{xx} = \mu\theta. \tag{9}$$

The constants λ and μ appearing in (8) and (9) are to be determined in a subsequent analysis.

The functions (8) are selected so as to ensure that under condition (7), the following simple relations hold:

$$k_1 \bar{u}_1 + k_2 \bar{v}_2 = k_1 u^\circ + k_2 v^\circ = \text{const}, \quad k_1 \bar{u}_3 + k_2 \bar{v}_4 = k_1 u^\circ + k_2 v^\circ = \text{const}. \tag{10}$$

Substituting (8) into (3) and taking into account relations (10) and Equation (9) followed by a simple rearrangement of terms, we arrive at the linear algebraic system

$$\begin{aligned} \lambda &= a_1 \mu + b_1 + c_1(k_1 u^\circ + k_2 v^\circ), \\ \lambda &= a_2 \mu + b_2 + c_2(k_1 u^\circ + k_2 v^\circ). \end{aligned} \tag{11}$$

It serves to determine the parameters λ and μ .

If $a_1 \neq a_2$, the solution of system (11) is expressed as

$$\lambda = \frac{a_2 b_1 - a_1 b_2 + (a_2 c_1 - a_1 c_2) \rho^\circ}{a_2 - a_1}, \quad \mu = \frac{b_1 - b_2 + (c_1 - c_2) \rho^\circ}{a_2 - a_1}, \quad \rho^\circ = k_1 u^\circ + k_2 v^\circ. \tag{12}$$

For the first three stationary points in (6), the coefficients (12) are given by

$$\begin{aligned} (a) \quad \lambda &= \frac{a_2 b_1 - a_1 b_2}{a_2 - a_1}, \quad \mu = \frac{b_1 - b_2}{a_2 - a_1}; \\ (b) \quad \lambda &= \frac{a_1(b_1 c_2 - b_2 c_1)}{c_1(a_2 - a_1)}, \quad \mu = \frac{\lambda}{a_1} = \frac{b_1 c_2 - b_2 c_1}{c_1(a_2 - a_1)}; \\ (c) \quad \lambda &= \frac{a_2(b_1 c_2 - b_2 c_1)}{c_2(a_2 - a_1)}, \quad \mu = \frac{\lambda}{a_2} = \frac{b_1 c_2 - b_2 c_1}{c_2(a_2 - a_1)}. \end{aligned} \tag{13}$$

In case (d), the parameters degenerate, $\lambda = \mu = 0$, resulting in solutions of little interest.

The general solution to the linear ODE (9) with $\mu \neq 0$ reads

$$\theta(x) = \begin{cases} C_1 \cos(\sqrt{|\mu|} x) + C_2 \sin(\sqrt{|\mu|} x) & \text{if } \mu < 0, \\ C_1 \exp(-\sqrt{|\mu|} x) + C_2 \exp(\sqrt{|\mu|} x) & \text{if } \mu > 0, \end{cases} \tag{14}$$

where C_1 and C_2 are arbitrary constants.

Formulas (8), (13) and (14) and the first three stationary points (6) determine six nondegenerate solutions, three for $\mu > 0$ and $\mu < 0$ each, to the nonlinear system (3) with $a_1 \neq a_2$ and four constant delays satisfying condition (7). The study [46] presents exact solutions to system (3) corresponding to the stationary point (b) in (6). These are written out below.

Solution for $\mu = \frac{b_1 c_2 - b_2 c_1}{c_1(a_2 - a_1)} < 0$:

$$\begin{aligned} u &= -k_2 e^{a_1 \mu(t+\tau_1)} \left[C_1 \cos(\sqrt{|\mu|} x) + C_2 \sin(\sqrt{|\mu|} x) \right] - b_1 (c_1 k_1)^{-1}, \\ v &= k_1 e^{a_1 \mu(t+\tau_2)} \left[C_1 \cos(\sqrt{|\mu|} x) + C_2 \sin(\sqrt{|\mu|} x) \right], \end{aligned} \tag{15}$$

where C_1 and C_2 are arbitrary constants.

Solution for $\mu = \frac{b_1 c_2 - b_2 c_1}{c_1(a_2 - a_1)} > 0$:

$$\begin{aligned} u &= -k_2 e^{a_1 \mu(t+\tau_1)} \left[C_1 \exp(-\sqrt{\mu} x) + C_2 \exp(\sqrt{\mu} x) \right] - b_1 (c_1 k_1)^{-1}, \\ v &= k_1 e^{a_1 \mu(t+\tau_2)} \left[C_1 \exp(-\sqrt{\mu} x) + C_2 \exp(\sqrt{\mu} x) \right]. \end{aligned} \tag{16}$$

2.2. Reduction of the System of PDEs with Three Delays to a System of Two Second-Order ODEs without Delays

We assume that the four delay times in system (3) are related by the single constraint (7).

Generalized separable solutions exponential in t . We seek a generalized separable solution to system (3) in the form

$$u = -k_2 e^{\lambda(t+\tau_1)} \theta(x) + \varphi(x), \quad v = k_1 e^{\lambda(t+\tau_2)} \theta(x) + \psi(x), \tag{17}$$

with the functions $\theta = \theta(x)$, $\varphi = \varphi(x)$, and $\psi = \psi(x)$ and parameter λ to be determined in the subsequent analysis.

The functions (17) are selected so as to ensure that under condition (7), the following relations hold:

$$k_1 \bar{u}_1 + k_2 \bar{v}_2 = k_1 \varphi(x) + k_2 \psi(x), \quad k_1 \bar{u}_3 + k_2 \bar{v}_4 = k_1 \varphi(x) + k_2 \psi(x). \tag{18}$$

Substituting (17) into (3) and taking into account (18), we obtain

$$\begin{aligned} -k_2 e^{\lambda(t+\tau_1)} [a_1 \theta''_{xx} + (b_1 - \lambda + c_1 \rho) \theta] + a_1 \varphi''_{xx} + (b_1 + c_1 \rho) \varphi &= 0, \\ k_1 e^{\lambda(t+\tau_2)} [a_2 \theta''_{xx} + (b_2 - \lambda + c_2 \rho) \theta] + a_2 \psi''_{xx} + (b_2 + c_2 \rho) \psi &= 0, \end{aligned} \tag{19}$$

with the notation $\rho = k_1 \varphi + k_2 \psi$ used for short.

Relations (19) can be satisfied by setting

$$\begin{aligned} a_1 \varphi''_{xx} + (b_1 + c_1 \rho) \varphi &= 0, \\ a_2 \psi''_{xx} + (b_2 + c_2 \rho) \psi &= 0, \\ a_1 \theta''_{xx} + (b_1 - \lambda + c_1 \rho) \theta &= 0, \\ a_2 \theta''_{xx} + (b_2 - \lambda + c_2 \rho) \theta &= 0. \end{aligned} \tag{20}$$

The first two equations in (20) form a closed system of ODEs for determining the functions φ and ψ , while the other two equations make up an overdetermined system for the single function θ . On requiring that the last two equations in (20) coincide, we find the parameter λ and other constants:

$$\lambda = \frac{a_2 b_1 - a_1 b_2}{a_2 - a_1}, \quad c_1 = a_1, \quad c_2 = a_2 \quad (a_1 \neq a_2). \tag{21}$$

On substituting (21) into (20), we arrive at the following system of stationary second-order ODEs:

$$\begin{aligned} a_1 \varphi''_{xx} + (b_1 + a_1 \rho) \varphi &= 0, \\ a_2 \psi''_{xx} + (b_2 + a_2 \rho) \psi &= 0, \\ \theta''_{xx} + \left(\frac{b_2 - b_1}{a_2 - a_1} + \rho \right) \theta &= 0, \end{aligned} \tag{22}$$

where $\rho = k_1 \varphi + k_2 \psi$.

Statement 1. Let $\varphi = \varphi(x)$, $\psi = \psi(x)$, $\theta = \theta(x)$ be a solution of system (22). Then, the functions

$$\hat{\varphi} = \varphi(\pm x + A), \quad \hat{\psi} = \psi(\pm x + A), \quad \hat{\theta} = B\theta(\pm x + A), \tag{23}$$

where A and B are arbitrary constants ($B \neq 0$) are also a solution of the system.

System (22) simplifies significantly if either φ or ψ is set to zero, in which case the system will only consist of two equations.

Properties and some transformations of the system of ODEs (22) with $\psi = 0$. The autonomous systems of ODEs (22) with $\psi = 0$ can be rewritten in the form

$$\varphi''_{xx} + (b + k\varphi)\varphi = 0, \tag{24}$$

$$\theta''_{xx} + (\beta + k\varphi)\theta = 0. \tag{25}$$

where $b = b_1/a_1$, $k = k_1$, and $\beta = (b_2 - b_1)/(a_2 - a_1)$. We assume that $\varphi(x) \neq \text{const}$ (the special case $\varphi = u^\circ = \text{const}$ was discussed above).

Below are three linear transformations of the system of ODEs (24) and (25), which will be required for the construction of exact solutions.

1°. The linear transformation

$$\varphi = \varphi_1 - \frac{b}{k}, \quad \theta = \theta_1 \quad (\text{transformation } \mathfrak{G}_1), \tag{26}$$

takes system (24) and (25) to the same system with other determining parameters

$$(\varphi_1)''_{xx} + (-b + k\varphi_1)\varphi_1 = 0, \tag{27}$$

$$(\theta_1)''_{xx} + (\beta - b + k\varphi_1)\theta_1 = 0.$$

2°. The linear transformation

$$\varphi = -\varphi_2, \quad \theta = \theta_2, \quad z = ix, \quad i^2 = -1 \quad (\text{transformation } \mathfrak{G}_2), \tag{28}$$

also takes system (24) and (25) to the same system with other parameters

$$(\varphi_2)''_{zz} + (-b + k\varphi_2)\varphi_2 = 0, \tag{29}$$

$$(\theta_2)''_{zz} + (-\beta + k\varphi_2)\theta_2 = 0.$$

We will use (b, β) to refer to systems (24) and (25). Then, transformations \mathfrak{G}_1 and \mathfrak{G}_2 and their composition $\mathfrak{G}_1 \circ \mathfrak{G}_2$ connect this system with three other systems of the same form, which can schematically be depicted as

$$(b, \beta) \xrightarrow{\mathfrak{G}_1} (-b, \beta - b); \quad (b, \beta) \xrightarrow{\mathfrak{G}_2} (-b, -\beta); \quad (b, \beta) \xrightarrow{\mathfrak{G}_1 \circ \mathfrak{G}_2} (b, b - \beta). \quad (30)$$

These transformations allow one to multiply exact solutions to (24) and (25). In addition, the following statement holds true.

Statement 2. *Suppose the functions*

$$\varphi = \Phi(x, b, k), \quad \theta = \Theta(x, b, \beta, k) \quad (31)$$

make up an exact solution to the system of ODEs (24) and (25). Then, exact solutions to the same system with other determining parameters b and β (k remains the same) can be obtained using three pairs of formulas specified in the last three rows of Table 1.

Table 1. Exact solutions to three related systems of ODEs of the form (24) and (25) that can be expressed in terms of solution (b, β) of the original system.

System of ODEs	Function φ	Function θ
(b, β) , original system	$\Phi(x, b, k)$	$\Theta(x, b, \beta, k)$
$(-b, \beta - b)$	$\Phi(x, b, k) + bk^{-1}$	$\Theta(x, b, \beta, k)$
$(-b, -\beta)$	$-\Phi(ix, b, k)$	$\Theta(ix, b, \beta, k)$
$(b, b - \beta)$	$-\Phi(ix, b, k) - bk^{-1}$	$\Theta(ix, b, \beta, k)$

3°. The linear transformation

$$\varphi = -\frac{6}{k}\varphi_3 - \frac{b}{2k}, \quad \theta = \theta_3 \quad (\text{transformation } \mathfrak{H}) \quad (32)$$

reduces (24) and (25) to a system of the special form

$$\begin{aligned} (\varphi_3)''_{xx} &= 6\varphi_3^2 - \frac{1}{24}b^2, \\ (\theta_3)''_{xx} &= (6\varphi_3 + \frac{1}{2}b - \beta)\theta_3. \end{aligned} \quad (33)$$

Exact solutions to the system of ODEs (24) and (25). The general solution to the autonomous ODE (24) can be represented in the implicit form

$$x + C_2 = \pm \int \frac{d\varphi}{\sqrt{C_1 - b\varphi^2 - \frac{2}{3}k\varphi^3}}. \quad (34)$$

where C_1 and C_2 are arbitrary constants. In general, the integral on the right-hand side of (34) is not expressible in terms of elementary functions.

Solutions to ODE (24) are expressible in terms of the Weierstrass elliptic function $\wp = \wp(z, g_2, g_3)$, which is defined explicitly via the elliptic integral [57]:

$$z = \int_{\wp}^{\infty} \frac{dy}{\sqrt{4y^3 - g_2y - g_3}},$$

where g_2 and g_3 are free parameters called invariants. The first ODE in (33) has a solution $\varphi_3 = \wp(x, \frac{1}{12}b^2, \tilde{C}_1)$ (see [57,58]), where $\tilde{C}_1 = g_3$ is an arbitrary constant. By virtue of

transformation \mathfrak{H} (32) and the first property in (23), Equation (24) with $k \neq 0$ admits the solutions

$$\varphi = -\frac{6}{k} \wp\left(\pm x + \tilde{C}_2, \frac{1}{12}b^2, \tilde{C}_1\right) - \frac{b}{2k}, \tag{35}$$

where \tilde{C}_1 and \tilde{C}_2 are arbitrary constants.

Notably, once $\varphi(x)$ is known, the general solution to the linear homogeneous Equation (25) can be written as [58]:

$$\theta = \theta_0 \left(C_3 + C_4 \int \frac{dx}{\theta_0^2} \right), \tag{36}$$

where $\theta_0 = \theta_0(x)$ is any nontrivial particular solution to Equation (25).

If $\beta = b$, Equation (25) with $\theta = \varphi$ coincides with Equation (24). Therefore, the function $\theta_0 = \varphi$ is a particular solution to Equation (25) in this case. On substituting $\theta_0 = \varphi$ into (36), we obtain the general solution to Equation (25) with $\beta = b$; the representation of the function φ in terms of the Weierstrass elliptic function (35) can also be used.

2.3. Exact Solutions in Terms of Elementary Functions to the Reduced System of Two Second-Order ODEs

Consider a few special cases where solutions to the system of ODEs (24) and (25) are expressible in terms of elementary functions.

1°. For $C_1 = 0$ and $b = 0$, the function φ in (34) can be represented in explicit form as

$$\varphi = -\frac{6}{k(x + C_2)^2}. \tag{37}$$

Substituting (37) into (25) and multiplying by $(x + C_2)^2$, we obtain the equation

$$\xi^2 \theta''_{\xi\xi} + (\beta \xi^2 - 6)\theta = 0, \quad \xi = x + C_2. \tag{38}$$

Up to renaming, it coincides with a special case of an ODE discussed in the book [58] (see Equation (132) with $a = 0$ and $n = 2$ on page 535).

For $\beta = 0$, the general solution of Equation (38) is

$$\theta(x) = C_3(x + C_2)^{-2} + C_4(x + C_2)^3,$$

where C_3 and C_4 are arbitrary constants.

For $\beta \neq 0$, the general solution of Equation (38) can be represented as

$$\theta = \begin{cases} \xi^{1/2} [C_3 J_{5/2}(\sqrt{\beta} \xi) + C_4 Y_{5/2}(\sqrt{\beta} \xi)] & \text{if } \beta > 0, \\ \xi^{1/2} [C_3 I_{5/2}(\sqrt{-\beta} \xi) + C_4 K_{5/2}(\sqrt{-\beta} \xi)] & \text{if } \beta < 0, \end{cases} \tag{39}$$

where C_3 and C_4 are arbitrary constants, $J_{5/2}(\zeta)$ and $Y_{5/2}(\zeta)$ are Bessel functions of the first and second kind, and $I_{5/2}(\zeta)$ and $K_{5/2}(\zeta)$ are modified Bessel functions of the first and second kind. These can be expressed in terms of elementary functions as [58]:

$$\begin{aligned} J_{5/2}(\zeta) &= \sqrt{\frac{2}{\pi}} \frac{3 \sin \zeta - 3 \zeta \cos \zeta - \zeta^2 \sin \zeta}{\zeta^{5/2}}, \\ Y_{5/2}(\zeta) &= -\sqrt{\frac{2}{\pi}} \frac{3 \cos \zeta + 3 \zeta \sin \zeta - \zeta^2 \cos \zeta}{\zeta^{5/2}}, \\ I_{5/2}(\zeta) &= \sqrt{\frac{2}{\pi}} \frac{3 \sinh \zeta - 3 \zeta \cosh \zeta + \zeta^2 \sinh \zeta}{\zeta^{5/2}}, \quad K_{5/2}(\zeta) = \sqrt{\frac{\pi}{2}} \frac{3 + 3 \zeta + \zeta^2}{e^\zeta \zeta^{5/2}}. \end{aligned}$$

2°. For $C_1 = 0$ and $b \neq 0$, the integral in (34) can be expressed in terms of elementary functions, and the function φ can be written explicitly as

$$\varphi(x) = \begin{cases} -\frac{3b}{2k} + \frac{3b}{2k} \tanh^2\left(\frac{1}{2}\sqrt{-b}x + A_1\right), & b < 0, \\ -\frac{3b}{2k} - \frac{3b}{2k} \tan^2\left(\frac{1}{2}\sqrt{b}x + A_1\right), & b > 0, \end{cases} \tag{40}$$

where $A_1 = \frac{1}{2}\sqrt{\mp b} C_2$ is an arbitrary constant.

For $\beta = 0$, the general solutions of Equation (25) corresponding to solutions (40) are

$$\theta(x) = \begin{cases} A_2(3S^2 - 1) + A_3[3S + (3S^2 - 1) \operatorname{artanh}\left(\frac{1}{2}\sqrt{-b}x + A_1\right)], & b < 0, \\ A_2(3T^2 + 1) + A_3[3T + (3T^2 + 1) \arctan\left(\frac{1}{2}\sqrt{b}x + A_1\right)], & b > 0, \end{cases}$$

where

$$S = \tanh\left(\frac{1}{2}\sqrt{-b}x + A_1\right), \quad T = \tan\left(\frac{1}{2}\sqrt{b}x + A_1\right),$$

and A_2 and A_3 are arbitrary constants.

For $\beta \neq 0$ and $b < 0$, Equation (25) admits particular solutions of the form [58]:

$$\theta_0 = \sinh^m\left(\frac{1}{2}\sqrt{-b}x + A_1\right) \cosh^n\left(\frac{1}{2}\sqrt{-b}x + A_1\right).$$

Substituting θ_0 into Equation (25) and in view of the first solution (40), we find the following allowed values of the parameters:

$$\begin{aligned} m = 0, \quad n = -2, \quad \beta = b, \\ m = 0, \quad n = 3, \quad \beta = \frac{9}{4}b, \\ m = 1, \quad n = -2, \quad \beta = \frac{1}{4}b, \\ m = 1, \quad n = 3, \quad \beta = 4b. \end{aligned}$$

As a result, taking into account (36), we obtain four general solutions for different β :

$$\begin{aligned} \beta = b, \quad \theta &= A_2 \cosh^{-2} \zeta + A_3 [(3 + 2 \cosh^2 \zeta) \tanh \zeta + 3 \zeta \cosh^{-2} \zeta]; \\ \beta = \frac{9}{4}b, \quad \theta &= A_2 \cosh^3 \zeta + A_3 \sinh \zeta (4 + 3 \cosh^{-2} \zeta + 8 \cosh^2 \zeta); \\ \beta = \frac{1}{4}b, \quad \theta &= A_2 \sinh \zeta \cosh^{-2} \zeta + A_3 \cosh^{-1} \zeta (3 \zeta \tanh \zeta - 3 + \cosh^2 \zeta); \\ \beta = 4b, \quad \theta &= A_2 \sinh \zeta \cosh^3 \zeta + A_3 (2 + \cosh^{-2} \zeta + 8 \cosh^2 \zeta - 16 \cosh^4 \zeta), \end{aligned} \tag{41}$$

where $\zeta = \frac{1}{2}\sqrt{-b}x + A_1$.

For $\beta \neq 0$ and $b > 0$, Equation (25) admits the following particular solutions [58]:

$$\theta_0 = \begin{cases} \cos^{-2}\left(\frac{1}{2}\sqrt{b}x + A_1\right), & \beta = b, \\ \cos^3\left(\frac{1}{2}\sqrt{b}x + A_1\right), & \beta = \frac{9}{4}b, \\ \sin\left(\frac{1}{2}\sqrt{b}x + A_1\right) \cos^{-2}\left(\frac{1}{2}\sqrt{b}x + A_1\right), & \beta = \frac{1}{4}b, \\ \sin\left(\frac{1}{2}\sqrt{b}x + A_1\right) \cos^3\left(\frac{1}{2}\sqrt{b}x + A_1\right), & \beta = 4b. \end{cases}$$

In view of (36), we obtain four general solutions for different β :

$$\begin{aligned} \beta = b, \quad \theta &= A_2 \cos^{-2} \zeta + A_3 [(3 + 2 \cos^2 \zeta) \tan \zeta + 3 \zeta \cos^{-2} \zeta]; \\ \beta = \frac{9}{4}b, \quad \theta &= A_2 \cos^3 \zeta + A_3 \sin \zeta (4 + 3 \cos^{-2} \zeta + 8 \cos^2 \zeta); \\ \beta = \frac{1}{4}b, \quad \theta &= A_2 \sin \zeta \cos^{-2} \zeta + A_3 \cos^{-1} \zeta (3 \zeta \tan \zeta + 3 - \cos^2 \zeta); \\ \beta = 4b, \quad \theta &= A_2 \sin \zeta \cos^3 \zeta + A_3 (2 + \cos^{-2} \zeta + 8 \cos^2 \zeta - 16 \cos^4 \zeta), \end{aligned} \tag{42}$$

where $\zeta = \frac{1}{2}\sqrt{b}x + A_1$.

3°. For $C_1 = \frac{1}{3}b^3/k^2$, $b \neq 0$, the radicand in (34) can be factorized to $(\frac{1}{3}b - \frac{2}{3}k\varphi) \times (\frac{b}{k} + \varphi)^2$. In this case, the integral can be expressed in terms of elementary functions, and the function φ can be represented in explicit form as

$$\varphi(x) = \begin{cases} \frac{b}{2k} + \frac{3b}{2k} \tan^2\left(\frac{1}{2}\sqrt{-b}x + A_1\right), & b < 0, \\ \frac{b}{2k} - \frac{3b}{2k} \tanh^2\left(\frac{1}{2}\sqrt{b}x + A_1\right), & b > 0. \end{cases} \tag{43}$$

For $\beta = 0$, the general solutions of Equation (25) corresponding to solutions (43) are

$$\theta(x) = \begin{cases} A_2(1 + T^2) + A_3 \left[\frac{2T}{1 + T^2} + 3T + 3(1 + T^2)\left(\frac{1}{2}\sqrt{-b}x + A_1\right) \right], & b < 0, \\ A_2(1 - S^2) + A_3 \left[\frac{2S}{1 - S^2} + 3S + 3(1 - S^2)\left(\frac{1}{2}\sqrt{b}x + A_1\right) \right], & b > 0, \end{cases}$$

where

$$T = \tan\left(\frac{1}{2}\sqrt{-b}x + A_1\right), \quad S = \tanh\left(\frac{1}{2}\sqrt{b}x + A_1\right).$$

For $\beta \neq 0$ and $b < 0$, Equation (25) admits the following exact solutions [58]:

$$\theta_0 = \begin{cases} 1 + 3 \tan^2\left(\frac{1}{2}\sqrt{-b}x + A_1\right), & \beta = b, \\ \cos^3\left(\frac{1}{2}\sqrt{-b}x + A_1\right), & \beta = -\frac{5}{4}b, \\ \sin\left(\frac{1}{2}\sqrt{-b}x + A_1\right) \cos^{-2}\left(\frac{1}{2}\sqrt{-b}x + A_1\right), & \beta = \frac{3}{4}b, \\ \sin\left(\frac{1}{2}\sqrt{-b}x + A_1\right) \cos^3\left(\frac{1}{2}\sqrt{-b}x + A_1\right), & \beta = -3b. \end{cases}$$

The respective general solutions of Equation (25) are

$$\begin{aligned} \beta = b, \quad \theta &= A_2(1 + 3 \tan^2 \xi) + A_3 [\xi(1 + 3 \tan^2 \xi) + 3 \tan \xi]; \\ \beta = -\frac{5}{4}b, \quad \theta &= A_2 \cos^3 \xi + A_3 \sin \xi (4 + 3 \cos^{-2} \xi + 8 \cos^2 \xi); \\ \beta = \frac{3}{4}b, \quad \theta &= A_2 \sin \xi \cos^{-2} \xi + A_3 \cos^{-1} \xi (3 \xi \tan \xi + 3 - \cos^2 \xi); \\ \beta = -3b, \quad \theta &= A_2 \sin \xi \cos^3 \xi + A_3 (2 + \cos^{-2} \xi + 8 \cos^2 \xi - 16 \cos^4 \xi), \end{aligned} \tag{44}$$

where $\xi = \frac{1}{2}\sqrt{-b}x + A_1$.

For $\beta \neq 0$ and $b > 0$, Equation (25) has the following exact solutions [58]:

$$\theta_0 = \begin{cases} 1 - 3 \tanh^2\left(\frac{1}{2}\sqrt{b}x + A_1\right), & \beta = b, \\ \cosh^3\left(\frac{1}{2}\sqrt{b}x + A_1\right), & \beta = -\frac{5}{4}b, \\ \sinh\left(\frac{1}{2}\sqrt{b}x + A_1\right) \cosh^{-2}\left(\frac{1}{2}\sqrt{b}x + A_1\right), & \beta = \frac{3}{4}b, \\ \sinh\left(\frac{1}{2}\sqrt{b}x + A_1\right) \cosh^3\left(\frac{1}{2}\sqrt{b}x + A_1\right), & \beta = -3b. \end{cases}$$

The respective general solutions of Equation (25) are

$$\begin{aligned} \beta = b, \quad \theta &= A_2(1 - 3 \tanh^2 \xi) + A_3 [\xi(1 - 3 \tanh^2 \xi) + 3 \tanh \xi]; \\ \beta = -\frac{5}{4}b, \quad \theta &= A_2 \cosh^3 \xi + A_3 \sinh \xi (4 + 3 \cosh^{-2} \xi + 8 \cosh^2 \xi); \\ \beta = \frac{3}{4}b, \quad \theta &= A_2 \sinh \xi \cosh^{-2} \xi + A_3 \cosh^{-1} \xi (3 \xi \tanh \xi - 3 + \cosh^2 \xi); \\ \beta = -3b, \quad \theta &= A_2 \sinh \xi \cosh^3 \xi + A_3 (2 + \cosh^{-2} \xi + 8 \cosh^2 \xi - 16 \cosh^4 \xi), \end{aligned} \tag{45}$$

where $\xi = \frac{1}{2}\sqrt{b}x + A_1$.

4°. Apart from the above exact solutions for $\varphi(x)$, Equation (24) admits the solutions

$$\varphi(x) = \begin{cases} -\frac{3b}{2k} + \frac{3b}{2k} \coth^2\left(\frac{1}{2}\sqrt{-b}x + A_1\right), & b < 0, \\ \frac{b}{2k} + \frac{3b}{2k} \cot^2\left(\frac{1}{2}\sqrt{-b}x + A_1\right), & b < 0, \\ \frac{b}{2k} - \frac{3b}{2k} \coth^2\left(\frac{1}{2}\sqrt{b}x + A_1\right), & b > 0, \\ -\frac{3b}{2k} - \frac{3b}{2k} \cot^2\left(\frac{1}{2}\sqrt{b}x + A_1\right), & b > 0. \end{cases} \tag{46}$$

The following replacements should be made in the respective solutions for $\theta(x)$: $\{\sin, \cos, \tan, \arctan\}$ with $\{\cos, \sin, \cot, \operatorname{arccot}\}$ and $\{\sinh, \cosh, \tanh, \operatorname{artanh}\}$ with $\{\cosh, \sinh, \coth, \operatorname{arcoth}\}$, respectively.

In particular, besides the first solution in (40) with $b < 0$ and the first solution in (41) with $\beta = b$, there are solutions

$$\begin{aligned} \varphi(x) &= -\frac{3b}{2k} + \frac{3b}{2k} \coth^2 \zeta, \quad \zeta = \frac{1}{2}\sqrt{-b}x + A_1, \\ \theta(x) &= A_2 \sinh^{-2} \zeta + A_3 [(3 + 2 \sinh^2 \zeta) \coth \zeta + 3 \zeta \sinh^{-2} \zeta]. \end{aligned}$$

Moreover, apart from the first solution in (43) with $b < 0$ and the first solution in (44) with $\beta = -\frac{5}{4}b$, there are solutions

$$\begin{aligned} \varphi(x) &= \frac{b}{2k} + \frac{3b}{2k} \cot^2 \zeta, \quad \zeta = \frac{1}{2}\sqrt{-b}x + A_1, \\ \theta(x) &= A_2 \sin^3 \zeta + A_3 \cos \zeta (4 + 3 \sin^{-2} \zeta + 8 \sin^2 \zeta). \end{aligned}$$

Table 2 summarizes the exact solutions in terms of elementary functions to the system of ODEs (24) and (25) for various values of the parameters b and β . The space variable x can everywhere be replaced with $x + A_1$, where A_1 is an arbitrary constant.

The exact solutions to the original Lotka–Volterra system of delay PDEs (3) for $c_1 = a_1$ and $c_2 = a_2$ and under condition (7) are determined by substituting the functions φ and θ from Table 2, where $b = b_1/a_1$, $k = k_1$, and $\beta = (b_2 - b_1)/(a_2 - a_1)$, into Formula (17) with $\lambda = \frac{a_2 b_1 - a_1 b_2}{a_2 - a_1}$ and $\psi = 0$.

Remark 3. The value $\beta = b$ in Table 2 corresponds to the stationary solutions (17) with $\psi = 0$ to the system of PDEs (3) with parameters (21), where $\lambda = 0$.

Remark 4. By setting $\varphi = 0$ in system (22) and denoting $a_2 = a$, $b_2 = ba$, $k_2 = k$, and $\beta = (b_2 - b_1)/(a_2 - a_1)$, one can obtain exact solutions for the pair of functions $\psi(x)$ and $\theta(x)$ analogous to those described above for the pair of functions $\varphi(x)$ and $\theta(x)$ for $\psi(x) = 0$.

Table 2. Solutions of systems (24) and (25) for various values of the parameters b and β . Notation: $J_{5/2}(\xi)$ and $Y_{5/2}(\xi)$ are the Bessel functions of the first and second kind, $I_{5/2}(\xi)$ and $K_{5/2}(\xi)$ are the modified Bessel functions of the first and second kind, and $A_1, A_2, A_3, C_2, C_3,$ and C_4 are arbitrary constants.

No.	Parameters b and β	Function φ , Equation (24)	Function θ , Equation (25)	Function $\xi = \xi(x)$
1	$b = 0, \beta = 0$	$-\frac{6}{k}\tau^{-2}$	$C_3\tau^{-2} + C_4\tau^3$	$\xi = x + C_2$
2	$b = 0, \beta > 0$	$-\frac{6}{k}\tau^{-2}$	$\sqrt{\xi} [C_3J_{5/2}(\sqrt{\beta}\xi) + C_4Y_{5/2}(\sqrt{\beta}\xi)]$	$\xi = x + C_2$
3	$b = 0, \beta < 0$	$-\frac{6}{k}\tau^{-2}$	$\sqrt{\xi} [C_3I_{5/2}(\sqrt{-\beta}\xi) + C_4K_{5/2}(\sqrt{-\beta}\xi)]$	$\xi = x + C_2$
4	$b < 0, \beta = 0$	$-\frac{3b}{2k} + \frac{3b}{2k} \tanh^2 \xi$	$A_2(3 \tanh^2 \xi - 1) + A_3 [3 \tanh \xi + (3 \tanh^2 \xi - 1) \operatorname{artanh} \xi]$	$\frac{1}{2}\sqrt{-b}x + A_1$
5	$b < 0, \beta = 0$	$-\frac{3b}{2k} + \frac{3b}{2k} \coth^2 \xi$	$A_2(3 \coth^2 \xi - 1) + A_3 [3 \coth \xi + (3 \coth^2 \xi - 1) \operatorname{arcoth} \xi]$	$\frac{1}{2}\sqrt{-b}x + A_1$
6	$b > 0, \beta = 0$	$-\frac{3b}{2k} - \frac{3b}{2k} \tan^2 \xi$	$A_2(3 \tan^2 \xi + 1) + A_3 [3 \tan \xi + (3 \tan^2 \xi + 1) \operatorname{arctan} \xi]$	$\frac{1}{2}\sqrt{b}x + A_1$
7	$b > 0, \beta = 0$	$-\frac{3b}{2k} - \frac{3b}{2k} \cot^2 \xi$	$A_2(3 \cot^2 \xi + 1) + A_3 [3 \cot \xi + (3 \cot^2 \xi + 1) \operatorname{arccot} \xi]$	$\frac{1}{2}\sqrt{b}x + A_1$
8	$b < 0, \beta = 0$	$\frac{b}{2k} + \frac{3b}{2k} \tan^2 \xi$	$A_2 \cos^{-2} \xi + A_3 [\sin(2\xi) + 3 \tan \xi + 3 \xi \cos^{-2} \xi]$	$\frac{1}{2}\sqrt{-b}x + A_1$
9	$b < 0, \beta = 0$	$\frac{b}{2k} + \frac{3b}{2k} \cot^2 \xi$	$A_2 \sin^{-2} \xi + A_3 [\cos(2\xi) + 3 \cot \xi + 3 \xi \sin^{-2} \xi]$	$\frac{1}{2}\sqrt{-b}x + A_1$
10	$b > 0, \beta = 0$	$\frac{b}{2k} - \frac{3b}{2k} \tanh^2 \xi$	$A_2 \cosh^{-2} \xi + A_3 [\sinh(2\xi) + 3 \tanh \xi + 3 \xi \cosh^{-2} \xi]$	$\frac{1}{2}\sqrt{b}x + A_1$
11	$b > 0, \beta = 0$	$\frac{b}{2k} - \frac{3b}{2k} \coth^2 \xi$	$A_2 \sinh^{-2} \xi + A_3 [\cosh(2\xi) + 3 \coth \xi + 3 \xi \sinh^{-2} \xi]$	$\frac{1}{2}\sqrt{b}x + A_1$
12	$b < 0, \beta = -3b$	$\frac{b}{2k} + \frac{3b}{2k} \tan^2 \xi$	$A_2 \sin \xi \cos^3 \xi + A_3 (2 + \cos^{-2} \xi + 8 \cos^2 \xi - 16 \cos^4 \xi)$	$\frac{1}{2}\sqrt{-b}x + A_1$
13	$b < 0, \beta = -3b$	$\frac{b}{2k} + \frac{3b}{2k} \cot^2 \xi$	$A_2 \cos \xi \sin^3 \xi + A_3 (2 + \sin^{-2} \xi + 8 \sin^2 \xi - 16 \sin^4 \xi)$	$\frac{1}{2}\sqrt{-b}x + A_1$
14	$b > 0, \beta = -3b$	$\frac{b}{2k} - \frac{3b}{2k} \tanh^2 \xi$	$A_2 \sinh \xi \cosh^3 \xi + A_3 (2 + \cosh^{-2} \xi + 8 \cosh^2 \xi - 16 \cosh^4 \xi)$	$\frac{1}{2}\sqrt{b}x + A_1$
15	$b > 0, \beta = -3b$	$\frac{b}{2k} - \frac{3b}{2k} \coth^2 \xi$	$A_2 \cosh \xi \sinh^3 \xi + A_3 (2 + \sinh^{-2} \xi + 8 \sinh^2 \xi - 16 \sinh^4 \xi)$	$\frac{1}{2}\sqrt{b}x + A_1$
16	$b < 0, \beta = -\frac{5}{4}b$	$\frac{b}{2k} + \frac{3b}{2k} \tan^2 \xi$	$A_2 \cos^3 \xi + A_3 \sin \xi (4 + 3 \cos^{-2} \xi + 8 \cos^2 \xi)$	$\frac{1}{2}\sqrt{-b}x + A_1$
17	$b < 0, \beta = -\frac{5}{4}b$	$\frac{b}{2k} + \frac{3b}{2k} \cot^2 \xi$	$A_2 \sin^3 \xi + A_3 \cos \xi (4 + 3 \sin^{-2} \xi + 8 \sin^2 \xi)$	$\frac{1}{2}\sqrt{-b}x + A_1$
18	$b > 0, \beta = -\frac{5}{4}b$	$\frac{b}{2k} - \frac{3b}{2k} \tanh^2 \xi$	$A_2 \cosh^3 \xi + A_3 \sinh \xi (4 + 3 \cosh^{-2} \xi + 8 \cosh^2 \xi)$	$\frac{1}{2}\sqrt{b}x + A_1$
19	$b > 0, \beta = -\frac{5}{4}b$	$\frac{b}{2k} - \frac{3b}{2k} \coth^2 \xi$	$A_2 \sinh^3 \xi + A_3 \cosh \xi (4 + 3 \sinh^{-2} \xi + 8 \sinh^2 \xi)$	$\frac{1}{2}\sqrt{b}x + A_1$
20	$b < 0, \beta = \frac{1}{4}b$	$-\frac{3b}{2k} + \frac{3b}{2k} \tanh^2 \xi$	$A_2 \sinh \xi \cosh^{-2} \xi + A_3 \cosh^{-1} \xi (3 \xi \tanh \xi - 3 + \cosh^2 \xi)$	$\frac{1}{2}\sqrt{-b}x + A_1$
21	$b < 0, \beta = \frac{1}{4}b$	$-\frac{3b}{2k} + \frac{3b}{2k} \coth^2 \xi$	$A_2 \cosh \xi \sinh^{-2} \xi + A_3 \sinh^{-1} \xi (3 \xi \coth \xi - 3 + \sinh^2 \xi)$	$\frac{1}{2}\sqrt{-b}x + A_1$
22	$b > 0, \beta = \frac{1}{4}b$	$-\frac{3b}{2k} - \frac{3b}{2k} \tan^2 \xi$	$A_2 \sin \xi \cos^{-2} \xi + A_3 \cos^{-1} \xi (3 \xi \tan \xi + 3 - \cos^2 \xi)$	$\frac{1}{2}\sqrt{b}x + A_1$
23	$b > 0, \beta = \frac{1}{4}b$	$-\frac{3b}{2k} - \frac{3b}{2k} \cot^2 \xi$	$A_2 \cos \xi \sin^{-2} \xi + A_3 \sin^{-1} \xi (3 \xi \cot \xi + 3 - \sin^2 \xi)$	$\frac{1}{2}\sqrt{b}x + A_1$
24	$b < 0, \beta = \frac{3}{4}b$	$\frac{b}{2k} + \frac{3b}{2k} \tan^2 \xi$	$A_2 \sin \xi \cos^{-2} \xi + A_3 \cos^{-1} \xi (3 \xi \tan \xi + 3 - \cos^2 \xi)$	$\frac{1}{2}\sqrt{-b}x + A_1$
25	$b < 0, \beta = \frac{3}{4}b$	$\frac{b}{2k} + \frac{3b}{2k} \cot^2 \xi$	$A_2 \cos \xi \sin^{-2} \xi + A_3 \sin^{-1} \xi (3 \xi \cot \xi + 3 - \sin^2 \xi)$	$\frac{1}{2}\sqrt{-b}x + A_1$
26	$b > 0, \beta = \frac{3}{4}b$	$\frac{b}{2k} - \frac{3b}{2k} \tanh^2 \xi$	$A_2 \sinh \xi \cosh^{-2} \xi + A_3 \cosh^{-1} \xi (3 \xi \tanh \xi - 3 + \cosh^2 \xi)$	$\frac{1}{2}\sqrt{b}x + A_1$
27	$b > 0, \beta = \frac{3}{4}b$	$\frac{b}{2k} - \frac{3b}{2k} \coth^2 \xi$	$A_2 \cosh \xi \sinh^{-2} \xi + A_3 \sinh^{-1} \xi (3 \xi \coth \xi - 3 + \sinh^2 \xi)$	$\frac{1}{2}\sqrt{b}x + A_1$
28	$b < 0, \beta = b$	$\frac{b}{2k} + \frac{3b}{2k} \tan^2 \xi$	$A_2(1 + 3 \tan^2 \xi) + A_3 [\xi(1 + 3 \tan^2 \xi) + 3 \tan \xi]$	$\frac{1}{2}\sqrt{-b}x + A_1$
29	$b < 0, \beta = b$	$\frac{b}{2k} + \frac{3b}{2k} \cot^2 \xi$	$A_2(1 + 3 \cot^2 \xi) + A_3 [\xi(1 + 3 \cot^2 \xi) + 3 \cot \xi]$	$\frac{1}{2}\sqrt{-b}x + A_1$
30	$b > 0, \beta = b$	$\frac{b}{2k} - \frac{3b}{2k} \tanh^2 \xi$	$A_2(1 - 3 \tanh^2 \xi) + A_3 [\xi(1 - 3 \tanh^2 \xi) + 3 \tanh \xi]$	$\frac{1}{2}\sqrt{b}x + A_1$
31	$b > 0, \beta = b$	$\frac{b}{2k} - \frac{3b}{2k} \coth^2 \xi$	$A_2(1 - 3 \coth^2 \xi) + A_3 [\xi(1 - 3 \coth^2 \xi) + 3 \coth \xi]$	$\frac{1}{2}\sqrt{b}x + A_1$
32	$b < 0, \beta = b$	$-\frac{3b}{2k} + \frac{3b}{2k} \tanh^2 \xi$	$A_2 \cosh^{-2} \xi + A_3 [(3 + 2 \cosh^2 \xi) \tanh \xi + 3 \xi \cosh^{-2} \xi]$	$\frac{1}{2}\sqrt{-b}x + A_1$
33	$b < 0, \beta = b$	$-\frac{3b}{2k} + \frac{3b}{2k} \coth^2 \xi$	$A_2 \sinh^{-2} \xi + A_3 [(3 + 2 \sinh^2 \xi) \coth \xi + 3 \xi \sinh^{-2} \xi]$	$\frac{1}{2}\sqrt{-b}x + A_1$
34	$b > 0, \beta = b$	$-\frac{3b}{2k} - \frac{3b}{2k} \tan^2 \xi$	$A_2 \cos^{-2} \xi + A_3 [(3 + 2 \cos^2 \xi) \tan \xi + 3 \xi \cos^{-2} \xi]$	$\frac{1}{2}\sqrt{b}x + A_1$
35	$b > 0, \beta = b$	$-\frac{3b}{2k} - \frac{3b}{2k} \cot^2 \xi$	$A_2 \sin^{-2} \xi + A_3 [(3 + 2 \sin^2 \xi) \cot \xi + 3 \xi \sin^{-2} \xi]$	$\frac{1}{2}\sqrt{b}x + A_1$
36	$b < 0, \beta = \frac{9}{4}b$	$-\frac{3b}{2k} + \frac{3b}{2k} \tanh^2 \xi$	$A_2 \cosh^3 \xi + A_3 \sinh \xi (4 + 3 \cosh^{-2} \xi + 8 \cosh^2 \xi)$	$\frac{1}{2}\sqrt{-b}x + A_1$
37	$b < 0, \beta = \frac{9}{4}b$	$-\frac{3b}{2k} + \frac{3b}{2k} \coth^2 \xi$	$A_2 \sinh^3 \xi + A_3 \cosh \xi (4 + 3 \sinh^{-2} \xi + 8 \sinh^2 \xi)$	$\frac{1}{2}\sqrt{-b}x + A_1$
38	$b > 0, \beta = \frac{9}{4}b$	$-\frac{3b}{2k} - \frac{3b}{2k} \tan^2 \xi$	$A_2 \cos^3 \xi + A_3 \sin \xi (4 + 3 \cos^{-2} \xi + 8 \cos^2 \xi)$	$\frac{1}{2}\sqrt{b}x + A_1$
39	$b > 0, \beta = \frac{9}{4}b$	$-\frac{3b}{2k} - \frac{3b}{2k} \cot^2 \xi$	$A_2 \sin^3 \xi + A_3 \cos \xi (4 + 3 \sin^{-2} \xi + 8 \sin^2 \xi)$	$\frac{1}{2}\sqrt{b}x + A_1$
40	$b < 0, \beta = 4b$	$-\frac{3b}{2k} + \frac{3b}{2k} \tanh^2 \xi$	$A_2 \sinh \xi \cosh^3 \xi + A_3 (2 + \cosh^{-2} \xi + 8 \cosh^2 \xi - 16 \cosh^4 \xi)$	$\frac{1}{2}\sqrt{-b}x + A_1$
41	$b < 0, \beta = 4b$	$-\frac{3b}{2k} + \frac{3b}{2k} \coth^2 \xi$	$A_2 \cosh \xi \sinh^3 \xi + A_3 (2 + \sinh^{-2} \xi + 8 \sinh^2 \xi - 16 \sinh^4 \xi)$	$\frac{1}{2}\sqrt{-b}x + A_1$
42	$b > 0, \beta = 4b$	$-\frac{3b}{2k} - \frac{3b}{2k} \tan^2 \xi$	$A_2 \sin \xi \cos^3 \xi + A_3 (2 + \cos^{-2} \xi + 8 \cos^2 \xi - 16 \cos^4 \xi)$	$\frac{1}{2}\sqrt{b}x + A_1$
43	$b > 0, \beta = 4b$	$-\frac{3b}{2k} - \frac{3b}{2k} \cot^2 \xi$	$A_2 \cos \xi \sin^3 \xi + A_3 (2 + \sin^{-2} \xi + 8 \sin^2 \xi - 16 \sin^4 \xi)$	$\frac{1}{2}\sqrt{b}x + A_1$

2.4. Reduction of a System of PDEs with Three Delays to a System of Three Second-Order ODEs without Delays

System (3) with four delays that satisfy the single relation (7) also admits generalized separable solutions linear in time t :

$$u = -k_2(t + \tau_1)\theta(x) + \varphi(x), \quad v = k_1(t + \tau_2)\theta(x) + \psi(x). \tag{47}$$

An analysis similar to that above results in the following parameters of Equation (3):

$$b_1 = \sigma a_1, \quad b_2 = \sigma a_2, \quad c_1 = a_1, \quad c_2 = a_2, \tag{48}$$

where σ is an arbitrary constant. In this case, the functions $\theta = \theta(x)$, $\varphi = \varphi(x)$, and $\psi = \psi(x)$ are described by the stationary system of ODEs

$$\begin{aligned} \varphi''_{xx} + (\sigma + \rho)\varphi + (k_2/a_1)\theta &= 0, \\ \psi''_{xx} + (\sigma + \rho)\psi - (k_1/a_2)\theta &= 0, \\ \theta''_{xx} + (\sigma + \rho)\theta &= 0, \quad \rho = k_1\varphi + k_2\psi. \end{aligned} \tag{49}$$

Let us look at the special case $a_1 = a_2 = a$. By adding up the first two equations of (49) multiplied by k_1 and k_2 , we obtain an isolated ODE for $\rho = k_1\varphi + k_2\psi$. We rewrite this equation, together with the third and first ODEs of (49) and the algebraic relation for ρ , as the mixed algebraic-differential system of equations

$$\begin{aligned} \rho''_{xx} + (\sigma + \rho)\rho &= 0, \\ \theta''_{xx} + (\sigma + \rho)\theta &= 0, \\ \varphi''_{xx} + (\sigma + \rho)\varphi + (k_2/a)\theta &= 0, \\ \rho &= k_1\varphi + k_2\psi, \end{aligned} \tag{50}$$

which contains three second-order ODEs.

2.5. Exact Solutions in Terms of Elementary Functions to the Reduced System of Three Second-Order ODEs

First, let us describe two simple classes of exact solutions to system (50) with $\rho = \text{const}$.
 1°. Exact solution to system (50) with $\rho = -\sigma$:

$$\begin{aligned} \rho &= -\sigma, \quad \varphi = C_1x + C_2 - \frac{k_2}{a} \left(\frac{1}{6}C_3x^3 + \frac{1}{2}C_4x^2 \right), \\ \psi &= -\frac{1}{k_2}(k_1\varphi + \sigma), \quad \theta = C_3x + C_4, \end{aligned}$$

where C_1, \dots, C_4 are arbitrary constants.

2°. Exact solution to system (50) with $\rho = 0$:

$$\begin{aligned} \rho &= 0, \quad \varphi = \varphi_p(x) + \begin{cases} C_1 \cos(\sqrt{\sigma} x) + C_2 \sin(\sqrt{\sigma} x) & \text{if } \sigma > 0, \\ C_1 \cosh(\sqrt{-\sigma} x) + C_2 \sinh(\sqrt{-\sigma} x) & \text{if } \sigma < 0, \end{cases} \\ \psi &= -\frac{k_1}{k_2}\varphi, \quad \theta = \begin{cases} C_3 \cos(\sqrt{\sigma} x) + C_4 \sin(\sqrt{\sigma} x) & \text{if } \sigma > 0, \\ C_3 \cosh(\sqrt{-\sigma} x) + C_4 \sinh(\sqrt{-\sigma} x) & \text{if } \sigma < 0, \end{cases} \\ \varphi_p(x) &= \begin{cases} \frac{k_2x}{2a\sqrt{\sigma}} [C_4 \cos(\sqrt{\sigma} x) - C_3 \sin(\sqrt{\sigma} x)] - \frac{C_3k_2}{2a\sigma} \cos(\sqrt{\sigma} x) & \text{if } \sigma > 0, \\ \frac{-k_2x}{2a\sqrt{-\sigma}} [C_4 \cosh(\sqrt{-\sigma} x) + C_3 \sinh(\sqrt{-\sigma} x)] - \frac{C_3k_2}{2a\sigma} \cosh(\sqrt{-\sigma} x) & \text{if } \sigma < 0. \end{cases} \end{aligned}$$

Table 3. The functions $\rho(x)$ and $\omega(x)$ determining exact solutions (51) to the system of ODEs (50).

No.	Parameter σ	Function ρ	Function ω	Function ξ
1	$\sigma = 0$	$-\frac{6}{k}x^{-2}$	x^3	—
2	$\sigma < 0$	$\frac{\sigma}{2} + \frac{3\sigma}{2} \tan^2 \xi$	$\xi(1 + 3 \tan^2 \xi) + 3 \tan \xi$	$\frac{1}{2} \sqrt{-\sigma} x$
3	$\sigma < 0$	$\frac{\sigma}{2} + \frac{3\sigma}{2} \cot^2 \xi$	$\xi(1 + 3 \cot^2 \xi) + 3 \cot \xi$	$\frac{1}{2} \sqrt{-\sigma} x$
4	$\sigma > 0$	$\frac{\sigma}{2} - \frac{3\sigma}{2} \tanh^2 \xi$	$\xi(1 - 3 \tanh^2 \xi) + 3 \tanh \xi$	$\frac{1}{2} \sqrt{\sigma} x$
5	$\sigma > 0$	$\frac{\sigma}{2} - \frac{3\sigma}{2} \coth^2 \xi$	$\xi(1 - 3 \coth^2 \xi) + 3 \coth \xi$	$\frac{1}{2} \sqrt{\sigma} x$
6	$\sigma < 0$	$-\frac{3\sigma}{2} + \frac{3\sigma}{2} \tanh^2 \xi$	$(3 + 2 \cosh^2 \xi) \tanh \xi + 3 \xi \cosh^{-2} \xi$	$\frac{1}{2} \sqrt{-\sigma} x$
7	$\sigma < 0$	$-\frac{3\sigma}{2} + \frac{3\sigma}{2} \coth^2 \xi$	$(3 + 2 \sinh^2 \xi) \coth \xi + 3 \xi \sinh^{-2} \xi$	$\frac{1}{2} \sqrt{-\sigma} x$
8	$\sigma > 0$	$-\frac{3\sigma}{2} - \frac{3\sigma}{2} \tan^2 \xi$	$(3 + 2 \cos^2 \xi) \tan \xi + 3 \xi \cos^{-2} \xi$	$\frac{1}{2} \sqrt{\sigma} x$
9	$\sigma > 0$	$-\frac{3\sigma}{2} - \frac{3\sigma}{2} \cot^2 \xi$	$(3 + 2 \sin^2 \xi) \cot \xi + 3 \xi \sin^{-2} \xi$	$\frac{1}{2} \sqrt{\sigma} x$

3°. The first two ODEs of system (50) coincide, up to renaming, with system (24) and (25) at $k = 1$ and $\beta = b$, whose exact solutions were described above. For known $\rho = \rho(x)$, the second equation in (50) for θ is a second-order linear homogeneous ODE, which has a particular solution $\theta_0 = \rho(x)$. Hence, the general solution of this equation is given by Formula (36). Once θ is found, the third equation in (50) is a second-order linear nonhomogeneous ODE for $\varphi = \varphi(x)$, with the particular solution $\varphi_0 = \rho(x)$ of the homogeneous ODE already known. Considering the aforesaid and using relevant formulas from [58], one can find the general solution to the ODE for φ . As a result, we obtain the solution to system (50) in the form

$$\begin{aligned} \rho &= \rho(x), \quad \theta = C_1\rho(x) + C_2\omega(x), \quad \varphi = C_3\rho(x) + C_4\omega(x) + \varphi_p(x), \\ \psi &= \frac{1}{k_2}[(1 - C_3k_1)\rho(x) - C_4k_1\omega(x) - k_1\varphi_p(x)], \quad \omega(x) = \rho(x) \int \frac{dx}{\rho^2(x)}, \end{aligned} \tag{51}$$

where C_1, \dots, C_4 are arbitrary constants, and $\varphi_p(x)$ is a particular solution to the third equation of system (50), which is defined as

$$\varphi_p(x) = -\frac{k_2}{a} \left[\omega(x) \int \rho(x)\theta(x) \frac{dx}{W(x)} - \rho(x) \int \omega(x)\theta(x) \frac{dx}{W(x)} \right]. \tag{52}$$

Here, the function $W(x) = \rho\omega'_x - \omega\rho'_x$ is the Wronskian determinant. Simple computations show that $W(x) = 1$.

Exact solutions to system (50) can be obtained by substituting the functions $\rho(x)$ and $\omega(x)$ from Table 3 into formulas (51) and (52).

4°. For arbitrary σ , the function $\rho(x)$ in solution (51) can be taken to be the Weierstrass elliptic function:

$$\rho(x) = -6\wp(x, \frac{1}{12}\sigma^2, C_5) - \frac{1}{2}\sigma,$$

where C_5 is an arbitrary constant.

2.6. Reduction of the System of PDEs with a Single Delay to a Unsteady System of Two First-Order Delay ODEs

Let the four delays in system (3) be identical:

$$\tau_1 = \tau_2 = \tau_3 = \tau_4 = \tau. \tag{53}$$

We look for generalized separable solutions to system (3) in the form

$$u = -k_2\xi(t)\theta(x) + \varphi(t), \quad v = k_1\xi(t)\theta(x) + \psi(t), \tag{54}$$

with the functions $\theta = \theta(x)$, $\zeta = \zeta(t)$, $\varphi = \varphi(t)$, and $\psi = \psi(t)$ to be determined in the subsequent analysis.

The functions (54) are selected to ensure that the composite arguments of $f(\dots)$ are only dependent on t . We impose the following additional condition (a linear differential constraint) on the function $\theta = \theta(x)$:

$$\theta''_{xx} = \mu\theta + \varepsilon, \tag{55}$$

where the constants μ and ε are to be found in the subsequent analysis. Notably, without loss of generality, one can set $\varepsilon = 0$ in (55) in the nondegenerate case of $\mu \neq 0$, since the translation of θ by a constant only leads, by virtue of representation (54), to new definitions of $\varphi(t)$ and $\psi(t)$.

Substituting (54) into (3) and taking into account (55), we get

$$\begin{aligned} -k_2[(a_1\mu + b_1 + c_1\bar{f})\zeta - \zeta'_t]\theta + a_1k_2\varepsilon\zeta + b_1\varphi + c_1\varphi\bar{f} - \varphi'_t &= 0, \\ k_1[(a_2\mu + b_2 + c_2\bar{f})\zeta - \zeta'_t]\theta + a_2k_1\varepsilon\zeta + b_2\psi + c_2\psi\bar{f} - \psi'_t &= 0, \end{aligned} \tag{56}$$

where the notations $\bar{f} = k_1\bar{\varphi} + k_2\bar{\psi}$, $\bar{\varphi} = \varphi(t - \tau)$, and $\bar{\psi} = \psi(t - \tau)$ are used for short.

Relation (56) can be satisfied if we set

$$\begin{aligned} \varphi'_t &= a_1k_2\varepsilon\zeta + b_1\varphi + c_1\varphi\bar{f}, \\ \psi'_t &= a_2k_1\varepsilon\zeta + b_2\psi + c_2\psi\bar{f}, \\ \zeta'_t &= (a_1\mu + b_1 + c_1\bar{f})\zeta, \\ \zeta'_t &= (a_2\mu + b_2 + c_2\bar{f})\zeta. \end{aligned} \tag{57}$$

The system of ODEs (57) is overdetermined because it consists of four equations for three functions ζ , φ , and ψ . On requiring that the last two equations of system (57) coincide, we find the parameter μ and other constants:

$$\mu = \frac{b_1 - b_2}{a_2 - a_1}, \quad c_1 = c_2 = 1 \quad (a_1 \neq a_2). \tag{58}$$

For $b_1 \neq b_2$, it follows from (58) that $\mu \neq 0$, and hence, we can set $\varepsilon = 0$ in Equations (55) and (57). The general solution of Equation (55) with $\varepsilon = 0$ is given by Formula (14), while the first two equations of (57) are independent of $\zeta = \zeta(t)$ and make up a closed nonlinear system of first-order delay ODEs for $\varphi = \varphi(t)$ and $\psi = \psi(t)$:

$$\begin{aligned} \varphi'_t &= b_1\varphi + \varphi(k_1\bar{\varphi} + k_2\bar{\psi}), \\ \psi'_t &= b_2\psi + \psi(k_1\bar{\varphi} + k_2\bar{\psi}), \end{aligned} \tag{59}$$

where $\bar{\varphi} = \varphi(t - \tau)$ and $\bar{\psi} = \psi(t - \tau)$. Integrating the last equation in Equation (57) and using relations (58), we find that the function $\zeta = \zeta(t)$ is expressed in terms of $\varphi = \varphi(t)$ and $\psi = \psi(t)$ as

$$\zeta = A \exp \left[\frac{a_2b_1 - a_1b_2}{a_2 - a_1} t + \int (k_1\bar{\varphi} + k_2\bar{\psi}) dt \right], \tag{60}$$

where A is an arbitrary constant.

It is noteworthy that the system of delay ODEs (59) generally admits one-component solutions $\varphi = \varphi(t)$, $\psi = 0$ and $\varphi = 0$, $\psi = \psi(t)$. In the absence of delay, these solutions can be expressed in terms of elementary functions.

2.7. The First Integral and Exact Solutions of the Reduced System of Two First-Order Delay ODEs

On eliminating the combination $k_1\bar{\varphi} + k_2\bar{\psi}$ from the system of delay ODEs (59), we get

$$\frac{\varphi'_t}{\varphi} - b_1 = \frac{\psi'_t}{\psi} - b_2.$$

Integrating yields the first integral

$$\psi = C_3 e^{(b_2 - b_1)t} \varphi, \tag{61}$$

where C_3 is an arbitrary constant, which may depend on τ . Substituting (61) into the first equation of system (59), we arrive at one first-order delay ODE

$$\varphi'_t = b_1\varphi + (k_1 + k_2 C_3 e^{(b_2 - b_1)(t - \tau)}) \varphi \bar{\varphi}, \quad \bar{\varphi} = \varphi(t - \tau). \tag{62}$$

Example 1. ODE (62) for the Lotka–Volterra system of PDEs without delay (3), where $\tau = 0$ is a Bernoulli equation. The integration of it gives, in view of (61), an exact solution to system (59):

$$\begin{aligned} \varphi &= \left[-\frac{C_3 k_2}{b_2} e^{(b_2 - b_1)t} + C_4 e^{-b_1 t} - \frac{k_1}{b_1} \right]^{-1}, \\ \psi &= C_3 e^{(b_2 - b_1)t} \left[-\frac{C_3 k_2}{b_2} e^{(b_2 - b_1)t} + C_4 e^{-b_1 t} - \frac{k_1}{b_1} \right]^{-1}, \end{aligned} \tag{63}$$

where C_4 is an arbitrary constant. The respective function $\xi(t)$ is defined by Formula (60) with $\tau = 0$. It can be expressed in terms of elementary functions if, for example,

$$b_1 = 0; \quad b_2 = 0; \quad b_1 = b_2; \quad C_3 = 0; \quad C_4 = 0.$$

In particular, by setting $\tau = 0$ and $C_3 = 0$ in (60) and (63), we get

$$\varphi = \frac{b_1 e^{b_1 t}}{b_1 C_4 - k_1 e^{b_1 t}}, \quad \psi = 0, \quad \xi = \frac{A}{|b_1 C_4 - k_1 e^{b_1 t}|} \exp\left(\frac{a_2 b_1 - a_1 b_2}{a_2 - a_1} t\right). \tag{64}$$

Formulas (14), (54), (58) and (64) describe a new solution to the Lotka–Volterra system of PDEs without delay (3).

Similarly, if $b_1 = b_2 = b$ and $\tau = 0$, we get

$$\varphi = \frac{b e^{bt}}{C_4 b - (C_3 k_2 + k_1) e^{bt}}, \quad \psi = \frac{b C_3 e^{bt}}{C_4 b - (C_3 k_2 + k_1) e^{bt}}, \quad \xi = \frac{A}{b} \varphi. \tag{65}$$

In general, the function φ for the delay ODE (62) must be defined on the interval $[-\tau, 0]$:

$$\varphi = \varphi_0(t) \quad \text{at} \quad -\tau \leq t \leq 0. \tag{66}$$

The Cauchy-type problem for the delay ODE (62) subject to the initial condition (66) can be solved using the method of steps [59,60]. To this end, we split time t into segments of length τ and denote

$$\varphi(t) = \varphi_m(t) \quad \text{at} \quad t_{m-1} \leq t \leq t_m, \tag{67}$$

where $t_m = m\tau$ and $m = 0, 1, 2, \dots$. Then, on integrating Equation (62) from t_{m-1} to t on the interval $[t_{m-1}, t_m]$, we get

$$\begin{aligned} \varphi_m(t) &= \varphi_m^\circ \exp\left[b_1(t - t_{m-1}) + \int_{t_{m-1}}^t (k_1 + k_2 C_3 e^{(b_2 - b_1)(t - \tau)}) \varphi_{m-1}(t) dt \right], \\ \varphi_m^\circ &= \varphi_m(t_{m-1}) = \varphi_{m-1}(t_m). \end{aligned} \tag{68}$$

The function $\varphi_m(t)$ on the left-hand side of Formula (68) is sought on the interval $[t_{m-1}, t_m]$, whereas $\varphi_{m-1}(t)$ on the right-hand side is defined on the preceding interval $[t_{m-2}, t_{m-1}]$. The computations are carried out successively starting from $m = 1$, when the right-hand side is a known function defined on the initial interval (66). This step results in $\varphi_1(t)$. Then, one sets $m = 2$ and finds the function $\varphi_2(t)$ using (68) via the already known function $\varphi_1(t)$. The procedure continues further in a similar fashion.

For $\varepsilon \neq 0$, in order to determine $\zeta = \zeta(t)$, $\varphi = \varphi(t)$, and $\psi = \psi(t)$, one has to solve the nonlinear system consisting of the last three equations in (57) with $b_1 = b_2$ and $\mu = 0$.

3. Reductions and Exact Solutions of the System of Delay PDEs with Identical Diffusion Coefficients ($a_1 = a_2$)

3.1. Reduction of the System of PDEs with Three Delays to a System of Two Second-Order ODEs without Delay and One Linear PDE

We assume that the four delay times in system (3) are linked by the single relation (7).

On setting $a_1 = a_2 = a$, $b_1 = b_2 = b$, and $c_1 = c_2 = 1$, we look for solutions to system (3) in the form

$$u = -k_2 e^{\lambda(t+\tau_1)} \theta(x, t) + \varphi(x), \quad v = k_1 e^{\lambda(t+\tau_2)} \theta(x, t) + \psi(x), \tag{69}$$

where λ is a free parameter, while $\theta = \theta(x, t)$, $\varphi = \varphi(x)$, and $\psi = \psi(x)$ are functions to be determined in the subsequent analysis.

Solutions of the form (69), where θ is a function of two independent variables and λ is an arbitrary parameter, generalize substantially solutions (17), in which θ is independent of t , and λ is expressed in terms of the system constants. We will refer to such solutions as incomplete separable solutions.

We assume the function $\theta = \theta(x, t)$ to satisfy the additional periodicity condition

$$\theta(x, t + \tau_2 - \tau_1) = \theta(x, t). \tag{70}$$

One can verify that relations (18) remain valid under conditions (7) and (70). Considering the above and substituting (69) into (3), we obtain a closed system for φ and ψ consisting of two second-order ODEs without delay:

$$a\varphi''_{xx} + (b + k_1\varphi + k_2\psi)\varphi = 0, \tag{71}$$

$$a\psi''_{xx} + (b + k_1\varphi + k_2\psi)\psi = 0. \tag{72}$$

The function $\theta = \theta(x, t)$ satisfies the nonstationary Schrödinger equation

$$\theta_t = a\theta''_{xx} + (b - \lambda + k_1\varphi + k_2\psi)\theta \tag{73}$$

and the periodic condition (70). Importantly, the coefficients of Equation (73) depend on the space variable x alone.

We multiply ODEs (71) and (72) by k_1 and k_2 , respectively, and add up to obtain an ODE for $\rho = k_1\varphi + k_2\psi$. We write this equation in conjunction with ODE (71) and PDE (70) as the following mixed algebraic-differential system of equations:

$$a\rho''_{xx} + (b + \rho)\rho = 0, \tag{74}$$

$$a\varphi''_{xx} + (b + \rho)\varphi = 0, \tag{75}$$

$$\rho = k_1\varphi + k_2\psi, \tag{76}$$

$$\theta_t = a\theta''_{xx} + (b - \lambda + \rho)\theta. \tag{77}$$

Systems (74)–(77) can be solved sequentially from one equation to another. We start from the isolated stationary ODE (74). Divided by a , it coincides, up to obvious renaming, with Equation (24). PDE (77) is subjected to the additional conditions (7) and (70), which

hold true, for example, in the stationary case $\theta = \theta(x)$ and the nonstationary case of $\tau_2 = \tau_1$ and $\tau_4 = \tau_3$.

3.2. Exact Solutions to the Reduced System Consisting of Two ODEs and One PDE

We first describe two simple classes of stationary exact solutions to system (74)–(77) with $\rho = \text{const}$.

1°. Stationary exact solution to system (74)–(77) with $\rho = 0$:

$$\rho = 0, \quad \varphi = \begin{cases} C_1 \cos(\sqrt{b}x) + C_2 \sin(\sqrt{b}x) & \text{if } b > 0, \\ C_1 \cosh(\sqrt{-b}x) + C_2 \sinh(\sqrt{-b}x) & \text{if } b < 0; \end{cases} \quad (78)$$

$$\psi = \begin{cases} -\frac{k_1}{k_2} [C_1 \cos(\sqrt{b}x) + C_2 \sin(\sqrt{b}x)] & \text{if } b > 0, \\ -\frac{k_1}{k_2} [C_1 \cosh(\sqrt{-b}x) + C_2 \sinh(\sqrt{-b}x)] & \text{if } b < 0; \end{cases} \quad (79)$$

$$\theta = \begin{cases} C_3 \cos[\sqrt{(b-\lambda)/a}x] + C_4 \sin[\sqrt{(b-\lambda)/a}x] & \text{if } b > \lambda, \\ C_3 \cosh[\sqrt{(\lambda-b)/a}x] + C_4 \sinh[\sqrt{(\lambda-b)/a}x] & \text{if } \lambda > b, \\ C_3x + C_4 & \text{if } \lambda = b, \end{cases} \quad (80)$$

where C_1, \dots, C_4 are arbitrary constants.

2°. Stationary exact solution to system (74)–(77) with $\rho = -b$:

$$\rho = -b, \quad \varphi = C_1x + C_2, \quad \psi = -\frac{k_1}{k_2}(C_1x + C_2) - \frac{b}{k_2}, \quad (81)$$

$$\theta = \begin{cases} C_3 \cosh(\sqrt{\lambda/a}x) + C_4 \sinh(\sqrt{\lambda/a}x) & \text{if } \lambda > 0, \\ C_3 \cos(\sqrt{-\lambda/a}x) + C_4 \sin(\sqrt{-\lambda/a}x) & \text{if } \lambda < 0. \end{cases} \quad (82)$$

3°. More complicated stationary solutions to system (74)–(77) with $\rho = \rho(x) \neq \text{const}$ and $\theta = \theta(x)$ can be constructed in a few steps as shown below (see Table 4).

- (i) The isolated subsystem consisting of two ODEs (74) and (75) for ρ and φ coincides, up to notation, with the system of ODEs (24) and (25) with $\beta = b$. It follows that exact solutions to Equations (74) and (75) can be obtained using Table 2 and formulas with $\beta = b$, in which the functions and determining parameters must be renamed as follows: $\varphi \Rightarrow \rho, \theta \Rightarrow \varphi, b \Rightarrow b/a, \beta \Rightarrow b/a$, and $k \Rightarrow 1/a$.
- (ii) The function ψ is determined by substituting the functions ρ and φ obtained in step (i) into (76), which results in

$$\psi = (\rho - k_1\varphi)/k_2.$$

- (iii) The isolated subsystem of two ODEs, (74) and (77), for ρ and θ with $\theta_t = 0$ coincides, up to notation, with systems (24) and (25). Hence, exact solutions to Equations (74) and (77) can be obtained using the formulas from Table 2, where φ and the determining parameters must be renamed as follows: $\varphi \Rightarrow \rho, b \Rightarrow b/a, \beta \Rightarrow (b-\lambda)/a$, and $k \Rightarrow 1/a$.

Table 4. Stationary solutions to systems (74)–(77). Notations: $\xi = \frac{1}{2} \sqrt{|b/a|} x$; C_1, \dots, C_4 are arbitrary constants. The function ψ is defined by $\psi = (\rho - k_1\varphi)/k_2$.

No.	Parameters b and λ	Function ρ , Equation (74)	Function φ , Equation (75)	Function θ , Equation (77)
1	$\lambda a < 0, b = 0$	$-6ax^{-2}$	$C_1x^{-2} + C_2x^3$	$\sqrt{x} [C_3J_{5/2}(\sqrt{-\lambda/a}x) + C_4Y_{5/2}(\sqrt{-\lambda/a}x)]$
2	$\lambda a > 0, b = 0$	$-6ax^{-2}$	$C_1x^{-2} + C_2x^3$	$\sqrt{x} [C_3I_{5/2}(\sqrt{\lambda/a}x) + C_4K_{5/2}(\sqrt{\lambda/a}x)]$
3	$\lambda = -3b, b < 0$	$\frac{3b}{2}(\tanh^2 \xi - 1)$	$C_1 \cosh^{-2} \xi + C_2 [(3 + 2 \cosh^2 \xi) \tanh \xi + 3 \xi \cosh^{-2} \xi]$	$C_3 \sinh \xi \cosh^3 \xi + C_4 (2 + \cosh^{-2} \xi + 8 \cosh^2 \xi - 16 \cosh^4 \xi)$
4	$\lambda = -3b, b < 0$	$\frac{3b}{2}(\coth^2 \xi - 1)$	$C_1 \sinh^{-2} \xi + C_2 [(3 + 2 \sinh^2 \xi) \coth \xi + 3 \xi \sinh^{-2} \xi]$	$C_3 \cosh \xi \sinh^3 \xi + C_4 (2 + \sinh^{-2} \xi + 8 \sinh^2 \xi - 16 \sinh^4 \xi)$
5	$\lambda = -3b, b > 0$	$-\frac{3b}{2}(\tanh^2 \xi + 1)$	$C_1 \cos^{-2} \xi + C_2 [(3 + 2 \cos^2 \xi) \tan \xi + 3 \xi \cos^{-2} \xi]$	$C_3 \sin \xi \cos^3 \xi + C_4 (2 + \cos^{-2} \xi + 8 \cos^2 \xi - 16 \cos^4 \xi)$
6	$\lambda = -3b, b > 0$	$-\frac{3b}{2}(\coth^2 \xi + 1)$	$C_1 \sin^{-2} \xi + C_2 [(3 + 2 \sin^2 \xi) \cot \xi + 3 \xi \sin^{-2} \xi]$	$C_3 \cos \xi \sin^3 \xi + C_4 (2 + \sin^{-2} \xi + 8 \sin^2 \xi - 16 \sin^4 \xi)$
7	$\lambda = -\frac{5}{4}b, b < 0$	$\frac{3b}{2}(\tanh^2 \xi - 1)$	$C_1 \cosh^{-2} \xi + C_2 [(3 + 2 \cosh^2 \xi) \tanh \xi + 3 \xi \cosh^{-2} \xi]$	$C_3 \cosh^3 \xi + C_4 \sinh \xi (4 + 3 \cosh^{-2} \xi + 8 \cosh^2 \xi)$
8	$\lambda = -\frac{5}{4}b, b < 0$	$\frac{3b}{2}(\coth^2 \xi - 1)$	$C_1 \sinh^{-2} \xi + C_2 [(3 + 2 \sinh^2 \xi) \coth \xi + 3 \xi \sinh^{-2} \xi]$	$C_3 \sinh^3 \xi + C_4 \cosh \xi (4 + 3 \sinh^{-2} \xi + 8 \sinh^2 \xi)$
9	$\lambda = -\frac{5}{4}b, b > 0$	$-\frac{3b}{2}(\tanh^2 \xi + 1)$	$C_1 \cos^{-2} \xi + C_2 [(3 + 2 \cos^2 \xi) \tan \xi + 3 \xi \cos^{-2} \xi]$	$C_3 \cos^3 \xi + C_4 \sin \xi (4 + 3 \cos^{-2} \xi + 8 \cos^2 \xi)$
10	$\lambda = -\frac{5}{4}b, b > 0$	$-\frac{3b}{2}(\coth^2 \xi + 1)$	$C_1 \sin^{-2} \xi + C_2 [(3 + 2 \sin^2 \xi) \cot \xi + 3 \xi \sin^{-2} \xi]$	$C_3 \sin^3 \xi + C_4 \cos \xi (4 + 3 \sin^{-2} \xi + 8 \sin^2 \xi)$
11	$\lambda = \frac{1}{4}b, b < 0$	$\frac{b}{2}(1 + 3 \tan^2 \xi)$	$C_1(1 + 3 \tan^2 \xi) + C_2[\xi(1 + 3 \tan^2 \xi) + 3 \tan \xi]$	$C_3 \sin \xi \cos^{-2} \xi + C_4 \cos^{-1} \xi (3 \xi \tan \xi + 3 - \cos^2 \xi)$
12	$\lambda = \frac{1}{4}b, b < 0$	$\frac{b}{2}(1 + 3 \cot^2 \xi)$	$C_1(1 + 3 \cot^2 \xi) + C_2[\xi(1 + 3 \cot^2 \xi) + 3 \cot \xi]$	$C_3 \cos \xi \sin^{-2} \xi + C_4 \sin^{-1} \xi (3 \xi \tan \xi + 3 - \sin^2 \xi)$
13	$\lambda = \frac{1}{4}b, b > 0$	$\frac{b}{2}(1 - 3 \tanh^2 \xi)$	$C_1(1 - 3 \tanh^2 \xi) + C_2[\xi(1 - 3 \tanh^2 \xi) + 3 \tanh \xi]$	$C_3 \sinh \xi \cosh^{-2} \xi + C_4 \cosh^{-1} \xi (3 \xi \tanh \xi - 3 + \cosh^2 \xi)$
14	$\lambda = \frac{1}{4}b, b > 0$	$\frac{b}{2}(1 - 3 \coth^2 \xi)$	$C_1(1 - 3 \coth^2 \xi) + C_2[\xi(1 - 3 \coth^2 \xi) + 3 \coth \xi]$	$C_3 \cosh \xi \sinh^{-2} \xi + C_4 \sinh^{-1} \xi (3 \xi \coth \xi - 3 + \sinh^2 \xi)$
15	$\lambda = \frac{3}{4}b, b < 0$	$\frac{3b}{2}(\tanh^2 \xi - 1)$	$C_1 \cosh^{-2} \xi + C_2 [(3 + 2 \cosh^2 \xi) \tanh \xi + 3 \xi \cosh^{-2} \xi]$	$C_3 \sinh \xi \cosh^{-2} \xi + C_4 \cosh^{-1} \xi (3 \xi \tanh \xi - 3 + \cosh^2 \xi)$
16	$\lambda = \frac{3}{4}b, b < 0$	$\frac{3b}{2}(\coth^2 \xi - 1)$	$C_1 \sinh^{-2} \xi + C_2 [(3 + 2 \sinh^2 \xi) \coth \xi + 3 \xi \sinh^{-2} \xi]$	$C_3 \cosh \xi \sinh^{-2} \xi + C_4 \sinh^{-1} \xi (3 \xi \coth \xi - 3 + \sinh^2 \xi)$
17	$\lambda = \frac{3}{4}b, b > 0$	$-\frac{3b}{2}(\tanh^2 \xi + 1)$	$C_1 \cos^{-2} \xi + C_2 [(3 + 2 \cos^2 \xi) \tan \xi + 3 \xi \cos^{-2} \xi]$	$C_3 \sin \xi \cos^{-2} \xi + C_4 \cos^{-1} \xi (3 \xi \tan \xi + 3 - \cos^2 \xi)$
18	$\lambda = \frac{3}{4}b, b > 0$	$-\frac{3b}{2}(\coth^2 \xi + 1)$	$C_1 \sin^{-2} \xi + C_2 [(3 + 2 \sin^2 \xi) \cot \xi + 3 \xi \sin^{-2} \xi]$	$C_3 \cos \xi \sin^{-2} \xi + C_4 \sin^{-1} \xi (3 \xi \cot \xi + 3 - \sin^2 \xi)$
19	$\lambda = b, b < 0$	$\frac{3b}{2}(\tanh^2 \xi - 1)$	$C_1 \cosh^{-2} \xi + C_2 [(3 + 2 \cosh^2 \xi) \tanh \xi + 3 \xi \cosh^{-2} \xi]$	$C_3(3 \tanh^2 \xi - 1) + C_4 [3 \tanh \xi + (3 \tanh^2 \xi - 1) \operatorname{artanh} \xi]$
20	$\lambda = b, b < 0$	$\frac{3b}{2}(\coth^2 \xi - 1)$	$C_1 \sinh^{-2} \xi + C_2 [(3 + 2 \sinh^2 \xi) \coth \xi + 3 \xi \sinh^{-2} \xi]$	$C_3(3 \coth^2 \xi - 1) + C_4 [3 \coth \xi + (3 \coth^2 \xi - 1) \operatorname{arcoth} \xi]$
21	$\lambda = b, b < 0$	$\frac{b}{2}(1 + 3 \tan^2 \xi)$	$C_1(1 + 3 \tan^2 \xi) + C_2[\xi(1 + 3 \tan^2 \xi) + 3 \tan \xi]$	$C_3 \cos^{-2} \xi + C_4 [\sin(2\xi) + 3 \tan \xi + 3 \xi \cos^{-2} \xi]$
22	$\lambda = b, b < 0$	$\frac{b}{2}(1 + 3 \cot^2 \xi)$	$C_1(1 + 3 \cot^2 \xi) + C_2[\xi(1 + 3 \cot^2 \xi) + 3 \cot \xi]$	$C_3 \sin^{-2} \xi + C_4 [\cos(2\xi) + 3 \tan \xi + 3 \xi \sin^{-2} \xi]$
23	$\lambda = b, b > 0$	$-\frac{3b}{2}(\tanh^2 \xi + 1)$	$C_1 \cos^{-2} \xi + C_2 [(3 + 2 \cos^2 \xi) \tan \xi + 3 \xi \cos^{-2} \xi]$	$C_3(3 \tan^2 \xi + 1) + C_4 [3 \tan \xi + (3 \tan^2 \xi + 1) \operatorname{arctan} \xi]$
24	$\lambda = b, b > 0$	$-\frac{3b}{2}(\coth^2 \xi + 1)$	$C_1 \sin^{-2} \xi + C_2 [(3 + 2 \sin^2 \xi) \cot \xi + 3 \xi \sin^{-2} \xi]$	$C_3(3 \cot^2 \xi + 1) + C_4 [3 \cot \xi + (3 \cot^2 \xi + 1) \operatorname{arccot} \xi]$
25	$\lambda = b, b > 0$	$\frac{b}{2}(1 - 3 \tanh^2 \xi)$	$C_1(1 - 3 \tanh^2 \xi) + C_2[\xi(1 - 3 \tanh^2 \xi) + 3 \tanh \xi]$	$C_3 \cosh^{-2} \xi + C_4 [\sinh(2\xi) + 3 \tanh \xi + 3 \xi \cosh^{-2} \xi]$
26	$\lambda = b, b > 0$	$\frac{b}{2}(1 - 3 \coth^2 \xi)$	$C_1(1 - 3 \coth^2 \xi) + C_2[\xi(1 - 3 \coth^2 \xi) + 3 \coth \xi]$	$C_3 \sinh^{-2} \xi + C_4 [\cosh(2\xi) + 3 \coth \xi + 3 \xi \sinh^{-2} \xi]$
27	$\lambda = \frac{9}{4}b, b < 0$	$\frac{b}{2}(1 + 3 \tan^2 \xi)$	$C_1(1 + 3 \tan^2 \xi) + C_2[\xi(1 + 3 \tan^2 \xi) + 3 \tan \xi]$	$C_3 \cos^3 \xi + C_4 \sin \xi (4 + 3 \cos^{-2} \xi + 8 \cos^2 \xi)$
28	$\lambda = \frac{9}{4}b, b < 0$	$\frac{b}{2}(1 + 3 \cot^2 \xi)$	$C_1(1 + 3 \cot^2 \xi) + C_2[\xi(1 + 3 \cot^2 \xi) + 3 \cot \xi]$	$C_3 \sin^3 \xi + C_4 \cos \xi (4 + 3 \sin^{-2} \xi + 8 \sin^2 \xi)$
29	$\lambda = \frac{9}{4}b, b > 0$	$\frac{b}{2}(1 - 3 \tanh^2 \xi)$	$C_1(1 - 3 \tanh^2 \xi) + C_2[\xi(1 - 3 \tanh^2 \xi) + 3 \tanh \xi]$	$C_3 \cosh^3 \xi + C_4 \sinh \xi (4 + 3 \cosh^{-2} \xi + 8 \cosh^2 \xi)$
30	$\lambda = \frac{9}{4}b, b > 0$	$\frac{b}{2}(1 - 3 \coth^2 \xi)$	$C_1(1 - 3 \coth^2 \xi) + C_2[\xi(1 - 3 \coth^2 \xi) + 3 \coth \xi]$	$C_3 \sinh^3 \xi + C_4 \cosh \xi (4 + 3 \sinh^{-2} \xi + 8 \sinh^2 \xi)$
31	$\lambda = 4b, b < 0$	$\frac{b}{2}(1 + 3 \tan^2 \xi)$	$C_1(1 + 3 \tan^2 \xi) + C_2[\xi(1 + 3 \tan^2 \xi) + 3 \tan \xi]$	$C_3 \sin \xi \cos^3 \xi + C_4 (2 + \cos^{-2} \xi + 8 \cos^2 \xi - 16 \cos^4 \xi)$
32	$\lambda = 4b, b < 0$	$\frac{b}{2}(1 + 3 \cot^2 \xi)$	$C_1(1 + 3 \cot^2 \xi) + C_2[\xi(1 + 3 \cot^2 \xi) + 3 \cot \xi]$	$C_3 \cos \xi \sin^3 \xi + C_4 (2 + \sin^{-2} \xi + 8 \sin^2 \xi - 16 \sin^4 \xi)$
33	$\lambda = 4b, b > 0$	$\frac{b}{2}(1 - 3 \tanh^2 \xi)$	$C_1(1 - 3 \tanh^2 \xi) + C_2[\xi(1 - 3 \tanh^2 \xi) + 3 \tanh \xi]$	$C_3 \sinh \xi \cosh^3 \xi + C_4 (2 + \cosh^{-2} \xi + 8 \cosh^2 \xi - 16 \cosh^4 \xi)$
34	$\lambda = 4b, b > 0$	$\frac{b}{2}(1 - 3 \coth^2 \xi)$	$C_1(1 - 3 \coth^2 \xi) + C_2[\xi(1 - 3 \coth^2 \xi) + 3 \coth \xi]$	$C_3 \cosh \xi \sinh^3 \xi + C_4 (2 + \sinh^{-2} \xi + 8 \sinh^2 \xi - 16 \sinh^4 \xi)$

4°. If $\tau_2 = \tau_1$ and $\tau_2 = \tau_4$, some nonstationary exact solutions to systems (74)–(77) with $\rho = 0$ are defined by Formulas (78) and (79) for φ and ψ with any expression of the function $\theta = \theta(x, t)$ specified below:

$$\begin{aligned}
 \theta &= [(x^2 + 2at) + C]e^{(b-\lambda)t}, \\
 \theta &= \exp[(a\mu^2 + b - \lambda)t \pm \mu x], \\
 \theta &= \frac{1}{\sqrt{t}} \exp\left[-\frac{x^2}{4at} + (b - \lambda)t\right], \\
 \theta &= \exp[(b - \lambda - a\mu^2)t] \cos(\mu x), \\
 \theta &= \exp[(b - \lambda - a\mu^2)t] \sin(\mu x), \\
 \theta &= \exp[-\mu x + (b - \lambda)t] \cos(\mu x - 2a\mu^2 t), \\
 \theta &= \exp[-\mu x + (b - \lambda)t] \sin(\mu x - 2a\mu^2 t), \\
 \theta &= \exp(-\mu x) \cos(\beta x - 2a\beta\mu t), \quad \beta = \sqrt{\mu^2 + (b - \lambda)/a}, \\
 \theta &= \exp(-\mu x) \sin(\beta x - 2a\beta\mu t), \quad \beta = \sqrt{\mu^2 + (b - \lambda)/a},
 \end{aligned}
 \tag{83}$$

where C and μ are arbitrary constants. Some other nonstationary exact solutions to systems (74)–(77) with $\rho = 0$ can be obtained using Formulas (78) and (79) and the expression $\theta = e^{(b-\lambda)t}\zeta$, where $\zeta = \zeta(x, t)$ is any solution of the standard linear heat equation $\zeta_t = a\zeta_{xx}$.

5°. If $\tau_2 = \tau_1$ and $\tau_2 = \tau_4$, there are nonstationary exact solutions to systems (74)–(77) with $\rho = -b$ that are defined by Formula (81) for φ and ψ and any expression of θ from (83) with $b = 0$.

6°. For $\tau_1 \neq \tau_2$, one can seek exact solutions to PDE (73) that satisfy the periodicity condition (70) and relation (7) in the form

$$\theta_m(x, t) = \zeta_m(x) \cos(\beta_m t) + \eta_m(x) \sin(\beta_m t), \quad \beta_m = \frac{2\pi m}{\tau_2 - \tau_1}, \quad m = 0, 1, 2, \dots, \quad (84)$$

where $m = 0$ corresponds to a stationary solution. Substituting (84) into (71) yields the following linear stationary system of ODEs for $\zeta_m = \zeta_m(x)$ and $\eta_m = \eta_m(x)$:

$$\begin{aligned} a\zeta_m'' + (b - \lambda + k_1\varphi + k_2\psi)\zeta_m - \beta_m\eta_m &= 0, \\ a\eta_m'' + (b - \lambda + k_1\varphi + k_2\psi)\eta_m + \beta_m\zeta_m &= 0. \end{aligned} \quad (85)$$

Since PDE (73) is linear in θ , an arbitrary linear combination of exact solutions (84),

$$\theta = \sum_{n=1}^{\infty} \alpha_n \theta_n(x, t) = \sum_{n=1}^{\infty} \alpha_n [\zeta_n(x) \cos(\beta_n t) + \eta_n(x) \sin(\beta_n t)],$$

where α_m are arbitrary constants, is also an exact solution to this equation.

Example 2. For any of the four simplest solutions (5), the general solution of Equation (73) satisfying the periodicity condition (70) can be represented as

$$\begin{aligned} \theta(x, t) &= \sum_{m=0}^{\infty} \exp(-\mu_m x) [A_m \cos(\beta_m t - \gamma_m x) + B_m \sin(\beta_m t - \gamma_m x)] \\ &+ \sum_{m=1}^{\infty} \exp(\mu_m x) [C_m \cos(\beta_m t + \gamma_m x) + D_m \sin(\beta_m t + \gamma_m x)], \end{aligned} \quad (86)$$

where

$$\begin{aligned} \mu_m &= \left(\frac{\sqrt{d^2 + \beta_m^2} - d}{2a} \right)^{1/2}, \quad \gamma_m = \left(\frac{\sqrt{d^2 + \beta_m^2} + d}{2a} \right)^{1/2}, \\ d &= b - \lambda + k_1\varphi^\circ + k_2\psi^\circ, \quad \beta_m = \frac{2\pi m}{\tau_2 - \tau_1}. \end{aligned}$$

The constants $A_m, B_m, C_m,$ and D_m , otherwise arbitrary, must ensure the convergence of the series (86) and derivatives θ_t and θ''_{xx} ; the convergence is certainly ensured if, for example, we set $A_m = B_m = C_m = D_m = 0$ for all $m > M$, where M is an arbitrary positive integer.

Let us highlight two special cases:

- (i) Time-periodic solutions decaying as $x \rightarrow \infty$ are given by Formula (86) with $A_0 = B_0 = 0$ and $C_m = D_m = 0, m = 1, 2, \dots$;
- (ii) Time-periodic solutions bounded as $x \rightarrow \infty$ are given by Formula (86) with $C_m = D_m = 0, m = 1, 2, \dots$

3.3. Reduction of the System of PDEs with a Single Delay to a System of Two First-Order Delay ODEs and One Linear PDE

We look for incomplete separable solutions to system (3) with $a_1 = a_2 = a, b_1 = b_2 = b,$ and $c_1 = c_2 = 1$ and a common delay time (53) in the form

$$u = -k_2\theta(x, t) + \varphi(t), \quad v = k_1\theta(x, t) + \psi(t), \quad (87)$$

with the functions $\theta = \theta(x, t)$, $\varphi = \varphi(t)$, and $\psi = \psi(t)$ to be determined in the subsequent analysis.

On substituting (87) into the system of PDEs (3), we arrive at a nonlinear system of delay ODEs for φ and ψ ,

$$\begin{aligned} \varphi'_t &= b\varphi + \varphi(k_1\bar{\varphi} + k_2\bar{\psi}), & \bar{\varphi} &= \varphi(t - \tau), \\ \psi'_t &= b\psi + \psi(k_1\bar{\varphi} + k_2\bar{\psi}), & \bar{\psi} &= \psi(t - \tau), \end{aligned} \tag{88}$$

and a linear parabolic PDE with variable coefficients for θ :

$$\theta_t = a\theta_{xx} + (b + k_1\bar{\varphi} + k_2\bar{\psi})\theta. \tag{89}$$

The system of delay ODEs (88) coincides with system (59) at $b_1 = b_2 = b$. Hence, it can be integrated as described in Section 2.7, where some exact solutions can also be found. With the substitution

$$\theta = \exp\left[bt + \int(k_1\bar{\varphi}_1 + k_2\bar{\psi}_1) dt\right] \zeta(x, t),$$

Equation (89) can be reduced to the standard linear heat equation

$$\zeta_t = a\zeta_{xx},$$

whose exact solutions can be found, for example, in [61].

4. Related Systems of PDEs with Several Space Variables and Delays. Other Generalizations

4.1. Reductions and Exact Solutions of a Multidimensional System with Different Diffusion Coefficients and Delays

Now, let us look at the following Lotka–Volterra-type nonlinear diffusion system of PDEs with n space variables:

$$\begin{aligned} u_t &= a_1\Delta u + u[b_1 + c_1(k_1\bar{u}_1 + k_2\bar{v}_2)], \\ v_t &= a_2\Delta v + v[b_2 + c_2(k_1\bar{u}_3 + k_2\bar{v}_4)], \end{aligned} \tag{90}$$

where $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator and x_1, \dots, x_n are Cartesian coordinates, while the other notations are the same as in system (3). As above, we assume that relation (7) holds.

System (90) admits reductions to simpler equations, which are similar to the reductions of system (3). We will illustrate this with specific examples and, in addition, present some exact solutions to (90) as well as solutions expressible in terms of solutions to linear PDEs.

1°. There is an exact solution to system (90) with $a_2 \neq a_1$ that generalizes solutions (15) and (16):

$$u = -k_2 e^{a_1\mu(t+\tau_1)}\theta(\mathbf{x}) - \frac{b_1}{c_1k_1}, \quad v = k_1 e^{a_1\mu(t+\tau_2)}\theta(\mathbf{x}), \quad \mu = \frac{b_1c_2 - b_2c_1}{c_1(a_2 - a_1)}, \tag{91}$$

where $\mathbf{x} = (x_1, \dots, x_n)$, and $\theta = \theta(\mathbf{x})$ is a function satisfying the Helmholtz equation

$$\Delta\theta = \mu\theta. \tag{92}$$

In the three-dimensional case, with $n = 3$, the linear PDE (92) admits, for example, the following exact solutions [61]:

$$\begin{aligned} \theta &= \frac{1}{r} [A_1 \exp(-\sqrt{\mu} r) + A_2 \exp(\sqrt{\mu} r)], \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad \mu > 0; \\ \theta &= \frac{1}{r} [A_1 \cos(\sqrt{-\mu} r) + A_2 \sin(\sqrt{-\mu} r)], \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad \mu < 0; \\ \theta &= \sin(\nu_1 x_1 + A_1) \sin(\nu_2 x_2 + A_2) \sin(\nu_3 x_3 + A_3), \quad \mu = -\nu_1^2 - \nu_2^2 - \nu_3^2 < 0; \\ \theta &= \sinh(\nu_1 x_1 + A_1) \sinh(\nu_2 x_2 + A_2) \sinh(\nu_3 x_3 + A_3), \quad \mu = \nu_1^2 + \nu_2^2 + \nu_3^2 > 0; \\ \theta &= \sin(\nu_1 x_1 + A_1) \sin(\nu_2 x_2 + A_2) \sinh(\nu_3 x_3 + A_3), \quad \mu = -\nu_1^2 - \nu_2^2 + \nu_3^2; \\ \theta &= \sin(\nu_1 x_1 + A_1) \sinh(\nu_2 x_2 + A_2) \sinh(\nu_3 x_3 + A_3), \quad \mu = -\nu_1^2 + \nu_2^2 + \nu_3^2, \end{aligned} \tag{93}$$

where $A_1, A_2, A_3, \nu_1, \nu_2$ and ν_3 are arbitrary constants. Any instance of the function \sinh in the last three solutions can be replaced with \cosh .

2°. Suppose that

$$c_1 = a_1, \quad c_2 = a_2 \quad (a_1 \neq a_2). \tag{94}$$

Then, system (90) admits a radially symmetric exact solution of the form

$$u = -k_2 e^{\lambda(t+\tau_1)} \theta(r) + \varphi(r), \quad v = k_1 e^{\lambda(t+\tau_2)} \theta(r), \quad \lambda = \frac{a_2 b_1 - a_1 b_2}{a_2 - a_1}, \tag{95}$$

where $r = \sqrt{x_1^2 + \dots + x_n^2}$. The functions $\varphi = \varphi(r)$ and $\theta = \theta(r)$ are described by the nonautonomous system of ODEs

$$\varphi''_{rr} + \frac{n-1}{r} \varphi'_r + (b + k\varphi)\varphi = 0, \tag{96}$$

$$\theta''_{rr} + \frac{n-1}{r} \theta'_r + (\beta + k\varphi)\theta = 0, \tag{97}$$

where $b = b_1/a_1, k = k_1$, and $\beta = (b_2 - b_1)/(a_2 - a_1)$. Systems (96) and (97) are a generalization of systems (24) and (25).

Equation (96) with $b = 0$ admits the exact solution

$$\varphi = \frac{2n - 8}{kr^2}. \tag{98}$$

Substituting (98) into (97) and multiplying by r^2 , we arrive at the equation

$$r^2 \theta''_{rr} + (n - 1)r\theta'_r + (\beta r^2 + 2n - 8)\theta = 0, \tag{99}$$

which coincides, up to notation, with a special case of an ODE discussed in the book [58] (see Equation (132) with $n = 2$ on page 535). It follows that the general solution of ODE (99) is given by

$$\begin{aligned} \theta &= \begin{cases} r^{(2-n)/2} [C_1 J_\nu(\sqrt{\beta} r) + C_2 Y_\nu(\sqrt{\beta} r)] & \text{if } \beta > 0, \\ r^{(2-n)/2} [C_1 I_\nu(\sqrt{-\beta} r) + C_2 K_\nu(\sqrt{-\beta} r)] & \text{if } \beta < 0, \end{cases} \\ \nu &= \frac{1}{2}|6 - n|, \end{aligned} \tag{100}$$

where C_1 and C_2 are arbitrary constants, $J_\nu(\xi)$ and $Y_\nu(\xi)$ are the Bessel functions of the first and second kind, and $I_\nu(\xi)$ and $K_\nu(\xi)$ are the modified Bessel functions of the first and second kind. In the three-dimensional case, which corresponds to $n = 3$ and $\nu = 3/2$, Formula (100) can be rewritten in terms of elementary functions.

4.2. Reductions and Exact solutions of a Multidimensional System with Identical Diffusion Coefficients and Delays

We assume that

$$a_1 = a_2 = a, \quad b_1 = b_2 = b, \quad c_1 = c_2 = 1. \tag{101}$$

System (90) with coefficients (101) admits exact solutions of the form

$$u = -k_2 e^{\lambda(t+\tau_1)} \theta(\mathbf{x}, t) + \varphi(\mathbf{x}), \quad v = k_1 e^{\lambda(t+\tau_2)} \theta(\mathbf{x}, t) + \psi(\mathbf{x}), \tag{102}$$

where λ is a free parameter, while $\theta = \theta(\mathbf{x}, t)$, $\varphi = \varphi(\mathbf{x})$, and $\psi = \psi(\mathbf{x})$ are described by the mixed algebraic-differential system of equations

$$\begin{aligned} a\Delta\rho + (b + \rho)\rho &= 0, \\ a\Delta\varphi + (b + \rho)\varphi &= 0, \\ \rho &= k_1\varphi + k_2\psi, \\ \theta_t &= a\Delta\theta + (b - \lambda + \rho)\theta, \end{aligned} \tag{103}$$

with θ additionally adopted to satisfy the periodicity condition $\theta(\mathbf{x}, t + \tau_2 - \tau_1) = \theta(\mathbf{x}, t)$. System (103) generalizes systems (74)–(77).

Presented below are two simple stationary solutions to system (103) with $\rho = \text{const}$.

1°. Solution with $\rho = 0$ expressible in terms of solutions to two independent Helmholtz equations:

$$\begin{aligned} \Delta\varphi + (b/a)\varphi &= 0, \\ \Delta\theta + [(b - \lambda)/a]\theta &= 0, \end{aligned} \tag{104}$$

where ψ is expressed via φ by the formula $\psi = -(k_1/k_2)\varphi$.

In the three-dimensional case, a few exact solutions to the linear PDEs (104) can be obtained using Formula (93) in which one must set $\mu = -b/a$ and $\mu = -(b - \lambda)/a$, respectively. For other solutions to the linear PDEs (104), see [61].

2°. Solution with $\rho = -b$ expressible in terms of solutions to the independent Laplace and Helmholtz equations

$$\begin{aligned} \Delta\varphi &= 0, \\ \Delta\theta - (\lambda/a)\theta &= 0, \end{aligned} \tag{105}$$

where ψ is expressed via φ by the formula $\psi = -(k_1\varphi + b)/k_2$. For solutions to the linear PDEs (105), see [61].

4.3. Further Generalizations and Modifications

1°. The nonlinear system of delay PDEs (3) can be generalized to the anisotropic case (when the diffusion coefficients depend on the spatial coordinate):

$$\begin{aligned} u_t &= a_1[\sigma(x)u_x]_x + u[b_1 + c_1(k_1\bar{u}_1 + k_2\bar{v}_2)], \\ v_t &= a_2[\sigma(x)v_x]_x + v[b_2 + c_2(k_1\bar{u}_3 + k_2\bar{v}_4)], \end{aligned} \tag{106}$$

where $\sigma(x)$ is a given function.

As previously, the constants (5) and (6) are the simplest exact solutions of system (106). If condition (7) is satisfied, then system (106) admits generalized separable solutions of the form (8), where the function $\theta(x)$ is a solution to the second-order linear ODE

$$[\sigma(x)\theta'_x]' = \mu\theta. \tag{107}$$

The constants λ and μ appearing in (8) and (107) are defined by Formulas (12) and (13).

Exact solutions of the linear ODE (107) for various $\sigma(x)$ can be found in the hand-book [58]. In particular, for $\sigma(x) = x^k$ and $\sigma(x) = e^{kx}$, the general solutions of ODE (107) can be expressed in terms of Bessel functions or modified Bessel functions.

Under conditions (7) and (21), system (106) admits more complicated generalized separable solutions of the form (17).

2°. A further generalization of the system of delay PDEs (3) is

$$\begin{aligned} u_t &= a_1 L[u] + u[b_1 + c_1 F(x, k_1 \bar{u}_1 + k_2 \bar{v}_2)], \\ v_t &= a_2 L[v] + v[b_2 + c_2 F(x, k_1 \bar{u}_3 + k_2 \bar{v}_4)]. \end{aligned} \tag{108}$$

Here, $F(x, \rho)$ is an arbitrary function of two arguments, while $L[u]$ is an arbitrary linear differential operator with respect to the spatial variable of the form

$$L = \sum_{j=1}^m \sigma_j(x) \frac{\partial^j}{\partial x^j}, \tag{109}$$

where $\sigma_j(x)$ are some given functions and m is a positive integer.

Under conditions (7) and (21), systems (108) and (109) admit a generalized separable solution of the form (17), where the functions $\varphi = \varphi(x)$, $\psi = \psi(x)$, and $\theta = \theta(x)$ satisfy the system of ODEs

$$\begin{aligned} a_1 L[\varphi] + [b_1 + a_1 F(x, \rho)]\varphi &= 0, \\ a_2 L[\psi] + [b_2 + a_2 F(x, \rho)]\psi &= 0, \\ L[\theta] + \left[\frac{b_2 - b_1}{a_2 - a_1} + F(x, \rho) \right] \theta &= 0, \end{aligned} \tag{110}$$

where $\rho = k_1 \varphi + k_2 \psi$.

If the function $F(x, \rho)$ is explicitly independent of x , so that $F(x, \rho) = f(\rho)$, then system (110) admits particular solutions of the form

$$\varphi = \varphi^\circ = \text{const}, \quad \psi = \psi^\circ = \text{const}, \quad \theta = \theta(x).$$

In this case, if all the coefficients σ_j of the differential operator (109) are independent of x , then the last equation in (110) for θ will be a linear ODE with constant coefficients.

3°. Nonlinear Klein–Gordon-type systems with delay, which are obtained from the reaction–diffusion delay systems (3), (106) and (108) by formally replacing the first time derivative u_t with the second derivative u_{tt} , also admit exact generalized separable solutions of the form (4).

5. Brief Conclusions

We have described several reductions of a Lotka–Volterra-type nonlinear multi-parameter reaction–diffusion system with several delay times to simpler systems of ODEs with or without delay, as well as reductions to mixed systems consisting of ODEs and a single linear PDE without delay.

We have found many new exact solutions to the reaction–diffusion delay system in question, including over 50 solutions expressible in terms of elementary functions. Moreover, setting the delay times in these solutions to zero can result in many new exact solutions in terms of elementary functions to a simpler nonlinear Lotka–Volterra-type system without delays.

The exact solutions presented in this paper all involve several free constants of integration. Such solutions may be suitable for testing approximate analytical and numerical methods for solving reaction–diffusion equations as well as other nonlinear PDEs with delays.

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