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INDRANU SUHENDRO

SPIN-CURVATURE AND THE UNIFICATION OF FIELDS IN A TWISTED SPACE



Indranu Suhendro

Spin-Curvature and the Unification of Fields in a Twisted Space

Spinn-krökning
och föreningen av fält
i ett tvistat rum

2008

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Preface of the Editor

I am honoured to present this book by Indranu Suhenrdo, in which he adduces theoretical solutions to the problems of spin-curvature and the unification of fields in a twisted space. A twist of space is given herein through the appropriate formalism, and is related to the anti-symmetric metric tensor. Kaluza's theory is extended and given an appropriate integrability condition. Both matter and the isotropic electromagnetic field are geometrized through common field equations: trace-free field equations, giving the energy-momentum tensor for the electromagnetic field via only the generalized Ricci curvature tensor and scalar, are obtained. In the absence of the electromagnetic field the theory goes to Einstein's 1928 theory of distant parallelism where only the matter field is geometrized (through the twist of space-time). Therefore the above results, in common with respective wave equations, are joined into a "unified theory of semi-classical gravoelectrodynamics".

There have been few attempts to introduce spin-particles into the theory of relativity (which is the theory of fields, in the sense propounded by Landau and Lifshitz). Frankly speaking, only two of the attempts were complete. The first attempt, by A. Papapetrou (1951), was a "frontal approach" to this problem: he introduced a spin-particle as a swiftly rotating gyro (*Proc. Roy. Soc. A*, 1951, v. 209, 248–258 and 259–268). This approach however doesn't match experimental data due to that fact that, considering an electron as a solid ball, the linear velocity of its rotation at its surface should be 70 times greater than the velocity of light. The second attempt, by me and L. Borissova (2001), introduced a spin-particle through the variational principle and Lagrange's function for such a system (see Chapter 4 of the book *Fields, Vacuum, and the Mirror Universe*, Editorial URSS, Moscow, 2001). Despite some success related to its immediate application to the theory of elementary particles, this method however could not be considered as purely "geometric": spin, the fundamental property of a particle, wasn't expressed through the geometric properties of the basic space, but instead still remained a non-geometrized fundamental characteristic of matter.

Insofar as the geometrization of distributed matter is concerned, just one solution was successful before this book. It was given for an isotropic electromagnetic field. Such fields are geometrized via the

well-known Rainich condition and the Nortvedt-Pagels condition: the energy-momentum tensor of such a field expresses itself through the components of the fundamental metric tensor, so Einstein's equations contain only the "geometric" left-hand-side by moving all the right-hand-side terms to the left-hand-side so the right-hand-side becomes zero. Various solutions given by the other authors are particular to the Rainich/Nortvedt-Pagels condition, or express an electromagnetic field under a very particular condition.

Both problems are successfully resolved in this book with the use of only geometric concepts on a common basis (a twisted space), which is the great advantage of this work. This fact places this book in the same class of great ideas in the theory of fields, produced during the last century, commencing with the time Albert Einstein first formulated his field equations.

I am therefore very pleased to bring this book to the reader. I recommend it to anyone who is seriously interested in the Theory of Relativity and the geometric approach to physics in particular.

February, 2008

Dmitri Rabounski

Foreword

In the present research, I consider a unification of gravity and electromagnetism in which electromagnetic interaction is seen to produce a gravitational field. The field equations of gravity and electromagnetism are therefore completely determined by the fundamental electromagnetic laws. Insight into this unification is that although gravity and electromagnetism have different physical characteristics (e.g., they differ in strength), it can be shown through the algebraic properties of the curvature and the electromagnetic field tensors that they are just different aspects of the geometry of space-time. Another hint comes from the known speed of interaction of gravity and electromagnetism: electromagnetic and gravitational waves both travel at the speed of light. This means that they must somehow obey the same wave equation. This indeed is unity. Consequently, many different gravitational and electromagnetic phenomena may be described by a single wave equation reminiscent of the scalar Klein-Gordon equation in quantum mechanics. Light is understood to be a gravoelectromagnetic wave generated by a current-producing oscillating charge. The charge itself is generated by the torsion of space-time. This electric (or more generally, electric-magnetic) charge in turn is responsible for the creation of matter, hence also the transformation of matter into energy and vice versa. Externally, the gravitational field manifests itself as the final outcome of the entire process. Hence gravity and electromagnetism obey the same set of field equations, i.e., they derive from a common origin. As a result, the charge produces the so-called gravitational mass. Albeit the geometric non-linearity of gravity, the linearity of electromagnetism is undisturbed: an idea which is central also in quantum mechanics. Therefore I preserve the most basic properties of matter such as energy, momentum, mass, charge and spin through this linearity. It is a modest attempt to once again achieve a comprehensive unification which explains that gravity, electromagnetism, matter and light are only different aspects of a single theory.

Karlstad, Sweden, 2004

Indranu Suhendro

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This work was originally intended to be my Ph.D. thesis in 2003. However, life has led me through its own existential twists and turns. In the process of publishing this book, many individuals have made their splendid contributions. It would be rather difficult to give credit without overlooking some of these sincere, wonderful persons. I would like especially to single out my dear colleagues, Dmitri Rabounski, Stephen Crothers, and Augustina Budai, who most passionately and sincerely helped prepare the entire manuscript in the present form. I would also like to express my most sincere gratitude to Bengt and Colette Johnsson, who gave me shelter and showered me with tenderness during my most creative years in Sweden. I would also like to thank my dear friends, Barbara Śmilgin, Dima Shaheen, Quinton Westrich, Liw Råskog, Vitaliy Kazymyrovych, and Sana Rafiq, with whom I share the subtle passions that normally lie hidden in the solitary, silent depths of life. Amidst life's high tides and despite all human frailties, in our togetherness, I believe, we have done enough to motivate each other to reach the height of existence. I must also recognize the most beautiful love, patience, and understanding of my father, mother, and the rest of my family throughout my life of seasons. Finally, this little scientific creation is dedicated with infinite love, yearning, gratitude, and humility to A. S. and M. H. for always being there for me somewhere between my slumber and wakefulness in this sojourn called life.

February, 2008

Indranu Suhendro

Chapter 1

A FOUR-MANIFOLD POSSESSING AN INTERNAL SPIN

§1.1 Introduction

Attempts at a consistent unified field theory of the classical fields of gravitation and electromagnetism and perhaps also chromodynamics have been made by many great past authors since the field concept itself was introduced by the highly original physicist, M. Faraday in the 19th century. These attempts temporarily ended in the 1950's: in fact Einstein's definitive version of his unified field theory as well as other parallel constructions never comprehensively and compellingly shed new light on the relation between gravity and electromagnetism. Indeed, they were biased by various possible ways of constructing a unified field theory via different geometric approaches and interpretations of the basic geometric quantities to represent the field tensors, e.g., the electromagnetic field tensor in addition to the gravitational field tensor (for further modern reference of such attempts, especially the last version of Einstein's gravoelectrodynamics see, e.g., various works of S. Antoci). This is put nicely in the words of Infeld: "... the problem of generalizing the theory of relativity cannot be solved along a purely formal way. At first, one does not see how a choice can be made among the various non-Riemannian geometries providing us with the gravitational and Maxwell's equations. The proper world-geometry which should lead to a unified theory of gravitation and electricity can only be found by an investigation of its physical content". In my view, one way to justify whether a unified field theory of gravitation and electromagnetism is really "true" (comprehensive) or really refers to physical reality is to see if one can derive the equation of motion of a charged particle, i.e., the (generalized) Lorentz equation, if necessary, effortlessly or directly from the basic assumptions of the theory. It is also important to be able to show that while gravity is in general non-linear, electromagnetism is linear. At last, it is always our modest aim to prove that gravity and electromagnetism derive from a common source. In view of this one must be able to show that the electromagnetic field is the sole ingredient responsible for the creation of matter which in turn generates a gravitational field. Hence the two fields are inseparable.

Furthermore, I find that most of the past theories were based on the Lagrangian formulation which despite its versatility and flexibility may also cause some uneasiness due to the often excessive freedom of choosing the field Lagrangian. This strictly formal action-method looks like a short cut which does not lead along the direct route of true physical progress. In the present work we shall follow a more fundamental (natural) method and at the same time bring up again many useful classical ideas such as the notion of a mixed geometry and Kaluza's cylinder condition and five-dimensional formulation. Concerning higher-dimensional formulations of unified field theories, we must remember that there is always a stage in physics at which direct but narrow physical arguments can hardly impinge upon many hidden properties of Nature. In fact the use of projective geometries also has deep physical reason and displays a certain degree of freedom of creativity: in this sense science is an art, a creative art. But this should never exclude the elements of mathematical simplicity so as to provide us with the very conditions that Nature's manifest four-dimensional laws of physics seemingly take.

For instance, Kaluza's cylinder condition certainly meets such a requirement and as far as we speak of the physical evidence (i.e., there should possibly be no intrusion of a particular dependence upon the higher dimension(s)), such a notion must be regarded as important if not necessary. An arbitrary affine $(n+1)$ -space can be represented by a projective n -space. Such a pure higher-dimensional mathematical space should not strictly be regarded as representing a "real" higher-dimensional world space. Physically saying, in our case, the five-dimensional space only serves as a mathematical device to represent the events of the ordinary four-dimensional space-time by a collection of congruence curves. It in no way points to the factual, exact number of dimensions of the Universe with respect to which the physical four-dimensional world is only a sub-space. In this work we shall employ a five-dimensional background space simply for the sake of convenience and simplicity.

On the microscopic scales, as we know, matter and space-time itself appear to be discontinuous. Furthermore, matter arguably consists of molecules, atoms and smaller elements. A physical theory based on a continuous field may well describe pieces of matter which are so large in comparison with these elementary particles, but fail to describe their behavior. This means that the motion of individual atoms and molecules remains unexplained by physical theories other than quantum theory in which discrete representations and a full concept of the so-called material wave are taken into account. I am convinced, indeed, that if we had

a sufficient knowledge of the behavior of matter in the microcosmos, it would, and it should, be possible to calculate the way in which matter behaves in the macrocosmos by utilizing certain appropriate statistical techniques as in quantum mechanics. Unfortunately, such calculations prove to be extremely difficult in practice and only the simplest systems can be studied this way. What's more, we still have to make a number of approximations to obtain some real results. Our classical field theories alone can only deal with the behavior of elementary particles in some average sense. Perhaps we must humbly admit that our understanding and knowledge of the behavior of matter, as well as space-time which occupies it, is still in a way almost entirely based on observations and experimental tests of their behavior on the large scales. This is a matter for experimental determination but a theoretical framework is always worth constructing. As generally accepted, at this point one must abandon the concept of the continuous representation of physical fields which ignores the discrete nature of both space-time and matter although it doesn't always treat matter as uniformly distributed throughout the regions of space. Current research has centered on quantum gravity since the departure of the 1950's but we must also acknowledge the fact that a logically consistent unification of classical fields is still important. In fact we do not touch upon the "formal", i.e., standard construction of a quantum gravity theory here. We derive a wave equation carrying the information of the quantum geometry of the curved four-dimensional space-time in Chapter 4 by first assuming the discreteness of the space-time manifold on the microscopic scales in order to represent the possible inter-atomic spacings down to the order of Planck's characteristic length.

Einstein-Riemann space(-time) $\mathbb{R}_4$ (a mixed, four-dimensional one) endowed with an internal spin space $\mathbb{S}_p$ is first considered. We stick to the concept of metricity and do not depart considerably from affine-metric geometry. Later on, a five-dimensional general background space $\mathbb{I}\mathbb{R}_5$ is introduced along with the five-dimensional and (as a brief digression) six-dimensional sub-spaces $\mathbb{O}_n$ and $\mathbb{V}_6$ as special coordinate systems.

CONVENTIONS: Small Latin indices run from 1 to 4. Capital Latin indices run from 1 to 5. Round and square brackets on particular tensor indices indicate symmetric and skew-symmetric characters, respectively. The covariant derivative is indicated either by a semi-colon or the symbol ∇ . The ordinary partial derivative is indicated either by a comma or the symbol ∂ . Einstein summation convention is, as usual, employed

throughout this work. Finally, by the word *space* we may also mean *space-time*.

§1.2 Geometric construction of a mixed, metric-compatible four-manifold $\mathbb{R}_4$ possessing an internal spin space $\mathbb{S}_p$

Our four-dimensional manifold $\mathbb{R}_4$ is endowed with a general asymmetric connection Γ_{jk}^i and possesses a fundamental asymmetric tensor defined herein by

$$\gamma_{ij} = \gamma_{(ij)} + \gamma_{[ij]} \equiv \frac{1}{\sqrt{2}} (g_{(ij)} + g_{[ij]}) , \quad (1.1)$$

where $g_{(ij)} \equiv \sqrt{2}\gamma_{(ij)}$ will play the role of the usual geometric metric tensor with which we raise and lower indices of tensors while $g_{[ij]} \equiv \sqrt{2}\gamma_{[ij]}$ will play the role of a fundamental spin tensor (or of a “skew- or anti-symmetric metric tensor”). We shall also refer both to as the fundamental tensors. They satisfy the relations

$$g_{(ij)}g^{(kj)} = \delta_i^k , \quad (1.2a)$$

$$g_{[ij]}g^{[kj]} = \delta_i^k , \quad (1.2b)$$

$$g^{(ij)}g_{[jk]} = g^{[ij]}g_{(jk)} . \quad (1.2c)$$

We may construct the fundamental spin tensor as a generalization of the skew-symmetric symplectic metric tensor in any M -dimensional space(-time), where $M = 2, 4, 6, \dots, M = 2 + p$, embedded in $(M + n)$ -dimensional enveloping space(-time), where $n = 0, 1, 2, \dots$. In M dimensions, we can construct $p = M - 2$ null (possibly complex) normal vectors (null n -legs) $z_1, z_2, \dots, z_p$ with $z_m^\mu z_{n\mu} = 0$ ($m, n, \dots = 1, \dots, p$ and $\mu, \nu, \dots = 1, \dots, M$). If we define the quantity $\gamma_{\mu\nu}^{\alpha\beta} = \varphi_{\mu\nu}\varphi^{\alpha\beta}$ where the skew-symmetric, self-dual null bivector $\varphi_{\mu\nu}$ defines a null rotation, then these null n -legs are normal to the (hyper)plane $\sum_{M+n-2} \subset R_M$ (contained in R_M) defined in such a way that

$$\varphi_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta\dots\tau} z_1^\alpha z_2^\beta \dots z_p^\tau = (M - p)! (z_{1[\mu} z_{2|\nu]} + \dots + z_{1[\mu} z_{p|\nu]}) ,$$

$$\varphi_{\mu\nu}\varphi^{\mu\nu} = 0 ,$$

$$\epsilon_{\mu\nu\alpha\beta\dots} = \sqrt{\pm g} \epsilon_{\mu\nu\alpha\beta\dots} , \quad \epsilon^{\mu\nu\alpha\beta\dots} = \frac{1}{\sqrt{\pm g}} \epsilon^{\mu\nu\alpha\beta\dots} , \quad g = \det \mathbf{g} .$$

Here $\epsilon_{\mu\nu\alpha\beta\dots}$ is the Levi-Civita permutation symbol. Hence in four dimensions we have

$$\varphi_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta} z_1^\alpha z_2^\beta = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \varphi^{\alpha\beta} = z_{1\mu} z_{2\nu} - z_{1\nu} z_{2\mu} ,$$

$$\gamma_{\mu\nu}^{\alpha\beta} = \epsilon_{\mu\nu\rho\gamma} \epsilon^{\alpha\beta\delta\theta} z_1^\rho z_2^\gamma z_{\delta 1} z_{\theta 2},$$

$$\text{tr } \gamma \rightarrow \gamma_{\mu\beta}^{\alpha\beta} = \gamma_{\beta\mu}^{\beta\alpha} = \pm \delta_{\rho\gamma\mu}^{\delta\beta\alpha} z_1^\rho z_2^\gamma z_{\delta 1} z_{\beta 2} = 0,$$

where $\delta_{\mu\nu\rho\sigma}^{\alpha\beta\gamma\delta} = \pm \epsilon_{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\delta}$ is the generalized Kronecker delta and $\delta_{\rho\gamma\mu}^{\delta\beta\alpha} = \delta_{\rho\gamma\mu\nu}^{\delta\beta\alpha\nu}$. (The minus sign holds if the manifold is Lorentzian and vice versa.) The particular equation $\text{tr } \gamma = 0$ is of course valid in M dimensions as well. In M dimensions, the fundamental spin tensor of our theory is defined as a bivector satisfying

$$g_{[\mu\nu]} g^{[\alpha\beta]} = \gamma_{\mu\nu}^{\alpha\beta} + \frac{1}{M-1} (\delta_\mu^\alpha \delta_\nu^\beta - \delta_\nu^\alpha \delta_\mu^\beta). \quad (1.3a)$$

Hence the above relation leads to the identity

$$g_{[\nu\alpha]} g^{[\mu\alpha]} = \delta_\nu^\mu. \quad (1.3b)$$

In the particular case of $M=2$ and $n=1$, the $\gamma_{\mu\nu}^{\alpha\beta}$ vanishes and the fundamental spin tensor is none other than the two-dimensional Levi-Civita permutation tensor:

$$g_{[AB]} = \epsilon_{AB} = \sqrt{\pm g} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$g_{[AB]} g^{[CD]} = \delta_A^C \delta_B^D - \delta_B^C \delta_A^D,$$

$$g_{[AC]} g^{[BC]} = \delta_A^B,$$

where $A, B=1, 2$. Let's now return to our four-dimensional manifold $\mathbb{R}_4$. We now have

$$g_{[ij]} g^{[kl]} = \gamma_{ij}^{kl} + \frac{1}{3} (\delta_i^k \delta_j^l - \delta_j^k \delta_i^l),$$

$$g_{[ij]} g^{[kj]} = \delta_i^k.$$

Hence the eigenvalue equation is arrived at:

$$\lambda g_{[ij]} = \gamma_{ij}^{kl} g_{[kl]}. \quad (1.3c)$$

We can now construct the symmetric traceless matrix Q_i^k through

$$Q_i^k \equiv \gamma_{ij}^{kl} u_l u^j = \epsilon_{ijpq} \epsilon^{klrs} z_1^p z_2^q z_{r1} z_{s2} u_l u^j,$$

$$\gamma_{ij}^{kl} \equiv Q_i^k u_j u^l,$$

$$Q_{ik} = Q_{ki},$$

$$\text{tr } Q = 0,$$

$$Q_{ik}u^k = 0,$$

where u^i is the unit velocity vector, $u_i u^i = 1$. Let's introduce the unit spin vector:

$$\begin{aligned} v^i &= g^{[ik]}u_k, \\ v_i v^i &= 1, u_i v^i = 0, \\ g_{[ik]}v^i u^k &= 1. \end{aligned}$$

Multiplying both sides of (1.3c) by the unit spin tensor, we get

$$\lambda g_{[ij]}u^j = Q_i^k g_{[kr]}u^r.$$

In other words,

$$\lambda v^i = Q_i^k v^k. \quad (1.3d)$$

Now we can also verify that

$$\gamma_{ij}\gamma^{kj} = \delta_i^k. \quad (1.4)$$

Note that since the fundamental tensor is asymmetric it follows that

$$\gamma_{ij}\gamma^{jk} = g_{(ij)}g^{[jk]} \neq \gamma_{ij}\gamma^{kj} (\neq \delta_i^k). \quad (1.5)$$

The line-element of $\mathbb{R}_4$ can then be given through the asymmetric fundamental tensor:

$$ds^2 = \sqrt{2}\gamma_{ik}dx^i dx^k = g_{(ik)}dx^i dx^k.$$

There exists in general no relation such as $g^{[kj]}g_{(ji)} = \delta_i^k$. However, we have the relations

$$\left. \begin{aligned} g^{(rs)}g_{[ir]}g_{[js]} &= g_{(ij)} & (a) \\ g^{[rs]}g_{(ir)}g_{(js)} &= g_{[ij]} & (b) \end{aligned} \right\}. \quad (1.6)$$

We now introduce the basis $\{g_l\}$ which spans the metric space of $\mathbb{R}_4$ and its associate $\{\omega_l\}$ which spans the spin space $\mathbb{S}_p \subset \mathbb{R}_4$ (we identify the manifold $\mathbb{R}_4$ as having the Lorentzian signature -2 , i.e., it is a space-time). These bases satisfy the algebra

$$\left. \begin{aligned} \omega_i &= g_{[ik]}g^k & (a) \\ g_i &= g_{[ki]}\omega^k & (b) \\ (g_i \cdot g_j) &= (\omega_i \cdot \omega_j) = g_{(ij)} & (c) \\ (g^i \cdot g_j) &= (\omega^i \cdot \omega_j) = \delta_j^i & (d) \\ (\omega_i \cdot g_j) &= g_{[ij]} & (e) \end{aligned} \right\}. \quad (1.7)$$

We can derive all of (1.2) and (1.6) by means of (1.7). In a pseudo-five-dimensional space $\vartheta_n = \mathbb{R}_4 \otimes n$ (a natural extension of $\mathbb{R}_4$ which includes a microscopic fifth coordinate axis normal to all the coordinate patches of $\mathbb{R}_4$), the algebra is extended as follows:

$$\left. \begin{aligned} [g_i, g_j] &= C_{ij}^k g_k + g_{[ij]} n & (a) \\ [n, g_i] &= g_{[ij]} g^j & (b) \\ \omega_i &= [n, g_i] (= g_{[ij]} g^j) & (c) \end{aligned} \right\}, \quad (1.8)$$

where the square brackets $[]$ are the commutation operator, C_{ij}^k stands for the commutation functions and n is the unit normal vector to the manifold $\mathbb{R}_4$ satisfying $(n \cdot n) = \pm 1$. Here we shall always assume that $(n \cdot n) = +1$ anyway. In summary, the symmetric and skew-symmetric metric tensors can be written in $\vartheta_n = \mathbb{R}_4 \otimes n$ as

$$\left. \begin{aligned} g_{(ik)} &= (g_i \cdot g_k) & (d) \\ g_{[ik]} &= ([g_i, g_k] \cdot n) & (e) \end{aligned} \right\}. \quad (1.8)$$

The (intrinsic) curvature tensor of the space $\mathbb{R}_4$ is given through the relations

$$\begin{aligned} R^i_{.jkl}(\Gamma) &= \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^m_{jl} \Gamma^i_{mk} - \Gamma^m_{jk} \Gamma^i_{ml}, \\ a_{i;j;k} - a_{i;k;j} &= R^l_{.ijk} a_l - 2\Gamma^l_{[jk]} a_{i;l}, \end{aligned}$$

for an arbitrary vector a_i . The torsion tensor $\Gamma^i_{[jk]}$ is introduced through the relation

$$\phi_{;i;k} - \phi_{;k;i} = -2\Gamma^r_{[ik]} \phi_{;r},$$

which holds for an arbitrary scalar field ϕ . The connection of course can be written as $\Gamma^i_{jk} = \Gamma^i_{(jk)} + \Gamma^i_{[jk]}$. The torsion tensor $\Gamma^i_{[jk]}$ together with the spin tensor $g_{[ij]}$ shall play the role associated with the internal spin of an object moving in space-time. On the manifold $\mathbb{R}_4$, let's now turn our attention to the spin space $\mathbb{S}_p$ and evaluate the tangent component of the derivative of the spin basis $\{\omega_l\}$ with the help of (1.7):

$$(\partial_j \omega_i)_T = \left(g_{[ik],j} g^{[lk]} - g_{[ik]} g^{[lm]} \Gamma^k_{mj} \right) \omega_l \quad (1.9)$$

since $(\partial_j g_i)_T = \Gamma^k_{ij} g_k$. Now with the help of (1.3), we have

$$(\partial_j \omega_i)_T = \left(g_{[ik],j} g^{[lk]} - \frac{1}{3} (\Gamma_j \delta_i^l - \Gamma_{ij}^l) \right) \omega_l,$$

where we have put $\Gamma_j = \Gamma_{ij}^i$. On the other hand one can easily show that by imposing metricity upon the two fundamental tensors (use (1.7) to prove this), the following holds:

$$\Gamma_{jk}^i = (\partial_j g_i \cdot g^k) = (\partial_j \omega_i \cdot \omega^k). \quad (1.10)$$

Thus, solving for a “tetrad-independent” connection, we have

$$\Gamma_{jk}^i = \frac{3}{2} g^{[ir]} g_{[jr],k} - \frac{1}{2} \delta_j^i \Gamma_k. \quad (1.11)$$

The torsion tensor is therefore

$$\begin{aligned} \Gamma_{[jk]}^i &= \frac{1}{4} (\Gamma_j \delta_k^i - \Gamma_k \delta_j^i) + \frac{3}{4} g^{[il]} (g_{[jl],k} - g_{[kl],j}) = \\ &= \frac{1}{4} (\Gamma_j \delta_k^i - \Gamma_k \delta_j^i) + \frac{3}{4} g^{[il]} g_{[jk],l}, \end{aligned} \quad (1.12)$$

where we have assumed that the fundamental spin tensor is a pure curl:

$$g_{[ij]} = \phi_{i,j} - \phi_{j,i}, \quad g_{[ij],k} + g_{[jk],i} + g_{[ki],j} = 0.$$

This expression and the symmetric part of the connection:

$$\Gamma_{(jk)}^i = \frac{1}{2} g^{(li)} (g_{(lj),k} - g_{(jk),l} + g_{(kl),j}) - g^{(li)} g_{(jm)} \Gamma_{[lk]}^m - g^{(li)} g_{(km)} \Gamma_{[lj]}^m,$$

therefore determine the connection uniquely in terms of the fundamental tensors alone:

$$\begin{aligned} \Gamma_{jk}^i &= \frac{1}{2} g^{(li)} (g_{(lj),k} - g_{(jk),l} + g_{(kl),j}) + \frac{1}{2} (\delta_k^i \Gamma_j - g^{(ir)} g_{(jk)} \Gamma_r) + \\ &+ \frac{3}{4} g^{[ir]} (g_{[jr],k} - g_{[kr],j}) - \frac{3}{4} g^{(li)} g_{(jr)} g^{[rs]} (g_{[ls],k} - g_{[ks],l}) - \\ &- \frac{3}{4} g^{(li)} g_{(kr)} g^{[rs]} (g_{[ls],j} - g_{[js],l}). \end{aligned} \quad (1.13)$$

There is, however, an alternative way of expressing the torsion tensor. The metric and the fundamental spin tensors are treated as equally fundamental and satisfy the ansatz

$$g_{(ij);k} = g_{[ij];k} = 0 \quad \text{and} \quad \gamma_{ij;k} = 0.$$

Therefore, from $g_{[ij];k} = 0$, we have the following:

$$g_{[ij],k} = g_{[mj]} \Gamma_{ik}^m + g_{[im]} \Gamma_{jk}^m.$$

Letting $W_{jik} = g_{[jm]}\Gamma_{ik}^m$, we have

$$g_{[ij],k} = W_{ijk} - W_{jik}. \quad (1.14)$$

Solving for $W_{i[jk]}$ by making cyclic permutations of i, j and k , we get

$$W_{i[jk]} = \frac{1}{2} (g_{[ij],k} - g_{[jk],i} + g_{[ki],j}) + W_{j(ik)} - W_{k(ij)}.$$

Therefore

$$\begin{aligned} W_{ijk} &= W_{i(jk)} + W_{i[jk]} = \\ &= \frac{1}{2} (g_{[ij],k} - g_{[jk],i} + g_{[ki],j}) + W_{i(jk)} + W_{j(ik)} - W_{k(ij)}. \end{aligned}$$

Now recall that $W_{ijk} = g_{[im]}\Gamma_{jk}^m$. Multiplying through by $g^{[il]}$, we get

$$\begin{aligned} \Gamma_{jk}^i &= \frac{1}{2} g^{[li]} (g_{[lj],k} - g_{[jk],l} + g_{[kl],j}) + \\ &\quad + g^{[li]} g_{[jm]}\Gamma_{(lk)}^m - g^{[li]} g_{[km]}\Gamma_{(lj)}^m + \Gamma_{(jk)}^i. \end{aligned} \quad (1.15)$$

On the other hand,

$$\begin{aligned} \Gamma_{jk}^i &= \frac{1}{2} g^{(li)} (g_{(lj),k} - g_{(jk),l} + g_{(kl),j}) - \\ &\quad - g^{(li)} g_{(jm)}\Gamma_{[lk]}^m - g^{(li)} g_{(km)}\Gamma_{[lj]}^m + \Gamma_{[jk]}^i. \end{aligned} \quad (1.16)$$

From (1.15) the torsion tensor is readily read off as

$$\begin{aligned} \Gamma_{[jk]}^i &= \frac{1}{2} g^{[li]} (g_{[lj],k} - g_{[jk],l} + g_{[kl],j}) + \\ &\quad + g^{[li]} g_{[jm]}\Gamma_{(lk)}^m - g^{[li]} g_{[km]}\Gamma_{(lj)}^m. \end{aligned} \quad (1.17)$$

We denote the familiar symmetric Levi-Civita connection by

$$\left\{ \begin{matrix} i \\ jk \end{matrix} \right\} = \frac{1}{2} g^{(li)} (g_{(lj),k} - g_{(jk),l} + g_{(kl),j}).$$

If we combine (1.13) and (1.17) with the help of (1.2), (1.3) and (1.4), after a rather lengthy but straightforward calculation we may obtain a solution:

$$\begin{aligned} \Gamma_{[jk]}^i &= \frac{1}{2} g^{[li]} (g_{[lj],k} - g_{[jk],l} + g_{[kl],j}) + \\ &\quad + g^{[li]} g_{[jm]} \left\{ \begin{matrix} m \\ lk \end{matrix} \right\} - g^{[li]} g_{[km]} \left\{ \begin{matrix} m \\ lj \end{matrix} \right\} + \frac{1}{3} (\Gamma_{[mk]}^m \delta_j^i - \Gamma_{[mj]}^m \delta_k^i). \end{aligned} \quad (1.18)$$

Now the spin vector $\Gamma_{[ik]}^i$ is to be determined from (1.12). If we contract (1.12) on the indices i and j , we have

$$\Gamma_{[ik]}^i = \frac{3}{4} g^{[ij]} (g_{[ij],k} - g_{[kj],i}) - \frac{3}{4} \Gamma_k. \quad (1.19)$$

But from (1.14):

$$\Gamma_k = \Gamma_{ik}^i = \frac{1}{2} g^{(ij)} g_{(ij),k} = \frac{1}{2} g^{[ij]} g_{[ij],k}. \quad (1.20)$$

Hence (1.19) becomes

$$\Gamma_{[ik]}^i \equiv S_k = \frac{3}{8} g^{[ij]} g_{[ij],k} - \frac{3}{4} g^{[ij]} g_{[kj],i}. \quad (1.21)$$

Then with the help of (1.21), (1.18) reads

$$\begin{aligned} \Gamma_{[jk]}^i &= \frac{1}{2} g^{[li]} (g_{[lj],k} - g_{[jk],l} + g_{[kl],j}) + \\ &+ g^{[li]} g_{[jm]} \left\{ \begin{matrix} m \\ lk \end{matrix} \right\} - g^{[li]} g_{[km]} \left\{ \begin{matrix} m \\ lj \end{matrix} \right\} + \\ &+ \frac{1}{8} g^{[mn]} (g_{[mn],k} \delta_j^i - g_{[mn],j} \delta_k^i) + \frac{1}{4} g^{[mn]} (g_{[km],n} \delta_j^i - g_{[jm],n} \delta_k^i). \end{aligned} \quad (1.22)$$

So far, we have been able to express the torsion tensor, which shall generate physical fields in our theory, in terms of the components of the fundamental tensor alone.

In a holonomic frame, $\Gamma_{[jk]}^i = 0$ and, of course, we have from (1.16) the usual Levi-Civita (or Christoffel) connection:

$$\Gamma_{jk}^i = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} = \frac{1}{2} g^{(li)} (g_{(lj),k} - g_{(jk),l} + g_{(kl),j}).$$

If therefore a theory of gravity adopts this connection, one may argue that in a strict sense, it does not admit an integral concept of internal spin in its description. Such is the classical theory of General Relativity.

In a rigid frame (constant metric) and in a pure electromagnetic gauge condition, one may have $\Gamma_{(jk)}^i = 0$ and in this special case we have from (1.15)

$$\Gamma_{jk}^i = \frac{1}{2} g^{[li]} (g_{[lj],k} - g_{[jk],l} + g_{[kl],j}),$$

which is exactly the same in structure as $\left\{ \begin{matrix} i \\ jk \end{matrix} \right\}$ with the fundamental spin tensor replacing the metric tensor. We shall call this connection

the pure spin connection, denoted by

$$L_{jk}^i = \frac{1}{2} g^{[li]} (g_{[lj],k} - g_{[jk],l} + g_{[kl],j}) . \quad (1.23)$$

Let's give an additional note to (1.20). Let's find the expression for Γ_{ki}^i , provided we know that

$$\begin{aligned} \Gamma_k &= \Gamma_{ik}^i = \frac{1}{2} g^{(ij)} g_{(ij),k} = \\ &= \frac{1}{2} g^{[ij]} g_{[ij],k} = \\ &= (\ln \sqrt{-g})_{,k} = \left\{ \begin{matrix} i \\ ik \end{matrix} \right\} . \end{aligned}$$

Meanwhile, we express the following relations:

$$\begin{aligned} S_k &= \Gamma_{[ik]}^i = -\Gamma_{[ki]}^i = \\ &= \frac{3}{8} g^{[ij]} g_{[ij],k} - \frac{3}{4} g^{[ij]} g_{[kj],i} = \\ &= \frac{3}{4} \left(\Gamma_k - g^{[ij]} g_{[kj],i} \right) , \end{aligned}$$

$$\begin{aligned} \Gamma_{(ik)}^i &= \left\{ \begin{matrix} i \\ ik \end{matrix} \right\} - g^{(li)} g_{(im)} \Gamma_{[lk]}^m - g^{(li)} g_{(km)} \Gamma_{[li]}^m = \Gamma_k - \Gamma_{[ik]}^i = \\ &= \frac{1}{4} \Gamma_k + \frac{3}{4} g^{[ij]} g_{[kj],i} . \end{aligned}$$

Therefore

$$\begin{aligned} \Gamma_{ki}^i &= \Gamma_{(ki)}^i + \Gamma_{[ki]}^i = \Gamma_{(ki)}^i - \Gamma_{[ik]}^i = \\ &= \frac{3}{2} g^{[ij]} g_{[kj],i} - \frac{1}{2} \Gamma_k , \end{aligned}$$

which can also be derived directly from (1.11). From (1.11) we also see that

$$g_{[ij],k} = \frac{2}{3} g_{[rj]} \Gamma_{ik}^r + \frac{1}{3} g_{[ij]} \Gamma_k , \quad (1.24a)$$

$$g_{[rj]} \Gamma_{ik}^r = -g_{[ri]} \Gamma_{jk}^r . \quad (1.24b)$$

Also, for later purposes, we derive the condition for the conservation of charges:

$$g_{,k}^{[ik]} = -\frac{1}{3} g^{[ik]} \Gamma_k . \quad (1.24c)$$

Having developed the basic structural equations here, we shall see in the following chapters that the gravitational and electromagnetic tensors are formed by means of the fundamental tensors $g_{(ij)} \equiv \sqrt{2} \gamma_{(ij)}$ and $g_{[ij]} \equiv \sqrt{2} \gamma_{[ij]}$ alone (see Section 4.2). In other words, gravity and electromagnetism together arise from this single tensor. We shall also investigate their fundamental relations and ultimately unveil their union.

Chapter 2

THE UNIFIED FIELD THEORY

§ 2.1 Generalization of Kaluza's projective theory

We now assume that the space-time $\mathbb{R}_4$ is embedded in a general five-dimensional Riemann space $\mathbb{M}_5$. This is referred to as embedding of class 1. We shall later define the space $\vartheta_n = \mathbb{R}_4 \otimes n$ to be a special coordinate system in $\mathbb{M}_5$. The five-dimensional metric tensor $g_{(AB)}$ of $\mathbb{M}_5$ of course satisfies the usual projective relations

$$g_{(AB)} = e_A^i e_B^j g_{(ij)} + n_A n_B,$$

$$e_i^A n_A = 0,$$

where $e_i^A = \partial_i x^A$ is the tetrad. If now $\{g_l\}$ denotes the basis of $\mathbb{R}_4$ and $\{e_A\}$ of $\mathbb{M}_5$:

$$e_A = e_A^i g_i + n_A n,$$

$$g_i = e_i^A e_A, \quad g_{(ij)} = e_i^A e_j^B g_{(AB)},$$

$$e_i^A e_B^i = \delta_B^A - n^A n_B, \quad e_i^A e_A^j = \delta_i^j.$$

The derivative of $g_i \in \mathbb{R}_4 \subset \mathbb{M}_5$ is then

$$\partial_j g_i = \Gamma_{ij}^k g_k + \phi_{ij} n.$$

We also have the following relations:

$$g_{i;j} = \phi_{ij} n, \quad \nabla_i n = -\phi_{;i}^j g_j, \quad e_{A;B} = 0, \quad (\partial_B e_A = \Gamma_{AB}^C e_C),$$

$$e_{i;j}^A = \phi_{ij} n^A, \quad n_{;i}^A = -\phi_{;i}^j e_j^A, \quad e_{A;B}^i = n_A \phi_{;B}^i e_B^j, \quad g_{(ij);k} = g_{(AB);C} = 0.$$

In our work we shall, however, emphasize that the exterior curvature tensor ϕ_{ij} is in general asymmetric: $\phi_{ij} \neq \phi_{ji}$ just as the connection Γ_{jk}^i is. This is so since in general $\partial_j e_i^A \neq \partial_i e_j^A$. Within a boundary Δ , the metric tensor $g_{(ij)}$ may possess discontinuities in its second derivatives. Now the connection and exterior curvature tensor satisfy

$$\left. \begin{aligned} \Gamma_{ij}^k &= e_A^k \partial_j e_i^A + e_A^k \Gamma_{BC}^A e_i^B e_j^C & (a) \\ \phi_{ij} &= n_A \partial_j e_i^A + n_A \Gamma_{BC}^A e_i^B e_j^C & (b) \end{aligned} \right\}, \quad (2.1)$$

where

$$\partial_j e_i^A = e_k^A \Gamma_{ij}^k - \Gamma_{BC}^A e_i^B e_j^C + \phi_{ij} n^A. \quad (2.2)$$

We can also solve for Γ_{BC}^A in (2.1) with the help of the projective relation $e_A = e_A^i g_i + n_A n$. The result is, after a quite lengthy calculation,

$$\begin{aligned} \Gamma_{BC}^A &= e_i^A \partial_C e_B^i + e_k^A \Gamma_{ij}^k e_B^i e_C^j + \phi_{ij} e_B^i e_C^j n^A + \\ &+ (\partial_C n_B) n^A - \phi_{.j}^i e_i^A e_C^j n_B. \end{aligned} \quad (2.3)$$

If we now perform the calculation $(\partial_k \partial_j - \partial_j \partial_k) g_i$ with the help of some of the above relations, we have in general

$$\begin{aligned} (\partial_k \partial_j - \partial_j \partial_k) g_i &= -R_{BCD}^A e_i^B e_j^C e_k^D e_A + \\ &+ (\partial_k e_j^C - \partial_j e_k^C) \Gamma_{BC}^A e_i^B e_A + \\ &+ (\partial_k \partial_j - \partial_j \partial_k) e_i^A e_A. \end{aligned} \quad (2.4)$$

Here we have also used the fact that

$$(\partial_C \partial_B - \partial_B \partial_C) e_A = -R_{ABC}^D e_D.$$

On the other hand, $\partial_i n = -\phi_{.i}^j g_j$, and

$$\begin{aligned} \partial_k \partial_j g_i &= (\Gamma_{ij}^l g_l + \phi_{ij} n)_{,k} = \\ &= (\Gamma_{ij,k}^l + \Gamma_{ij}^m \Gamma_{mk}^l - \phi_{ij} \phi_{.k}^l) g_l + (\phi_{ij,k} + \Gamma_{ij}^l \phi_{lk}) n. \end{aligned}$$

Therefore we obtain another expression for $(\partial_k \partial_j - \partial_j \partial_k) g_i$:

$$\begin{aligned} (\partial_k \partial_j - \partial_j \partial_k) g_i &= (-R_{ijk}^l + \phi_{ik} \phi_{.j}^l - \phi_{ij} \phi_{.k}^l) g_l + \\ &+ (\phi_{ij;k} - \phi_{ik;j} + 2\Gamma_{[jk]}^l \phi_{il}) n, \end{aligned} \quad (2.5a)$$

$$(\partial_k \partial_j - \partial_j \partial_k) e_i^A + (\partial_k e_j^C - \partial_j e_k^C) \Gamma_{BC}^A e_i^B \equiv S_{ijk}^A. \quad (2.5b)$$

Combining (2.4) and (2.5), we get, after some algebraic manipulations,

$$\left. \begin{aligned} R_{ijkl} &= \phi_{ik} \phi_{jl} - \phi_{il} \phi_{jk} + R_{ABCD} e_i^A e_j^B e_k^C e_l^D - S_{Aijk} e_i^A \quad (a) \\ \phi_{ij;k} - \phi_{ik;j} &= -R_{ABCD} n^A e_i^B e_j^C e_k^D - 2\Gamma_{[jk]}^l \phi_{il} + S_{Aijk} n^A \quad (b) \end{aligned} \right\}. \quad (2.6)$$

We have thus established the straightforward generalizations of the equations of Gauss and Codazzi.

Now the electromagnetic content of (2.6) can be seen as follows: first we split the exterior curvature tensor ϕ_{ij} into its symmetric and skew-symmetric parts:

$$\left. \begin{aligned} \phi_{ij} &= \phi_{(ij)} + \phi_{[ij]} \\ \phi_{(ij)} &\equiv k_{ij}, \phi_{[ij]} \equiv f_{ij} \end{aligned} \right\}. \quad (2.7)$$

Here the symmetric exterior curvature tensor k_{ij} has the explicit expression

$$\begin{aligned} k_{ij} &= \frac{1}{2} n_A (\partial_j e_i^A + \partial_i e_j^A) + n_A \Gamma_{(BC)}^A e_i^B e_j^C = \\ &= -\frac{1}{2} e_i^A e_j^B (n_{A;B} + n_{B;A}). \end{aligned} \quad (2.8)$$

Furthermore, in our formalism, the skew-symmetric exterior curvature tensor f_{ij} is naturally equivalent to the electromagnetic field tensor F_{ij} . It is convenient to set $f_{ij} = \frac{1}{2} F_{ij}$. Hence the electromagnetic field tensor can be written as

$$\begin{aligned} F_{ij} &= n_A (\partial_j e_i^A - \partial_i e_j^A) + 2n_A \Gamma_{[BC]}^A e_i^B e_j^C = \\ &= -e_i^A e_j^B (n_{A;B} - n_{B;A}) \end{aligned} \quad (2.9)$$

The five-dimensional electromagnetic field tensor is therefore

$$F_{AB} = \nabla_A n_B - \nabla_B n_A.$$

§2.2 Fundamental field equations of our unified field theory. Geometrization of matter

We are now in a position to simplify (2.6) by invoking two conditions. The first of these, following Kaluza, is the cylinder condition: the laws of physics in their four-dimensional form shall not depend on the fifth coordinate $x^5 \in \vartheta_n$. We also assume that $x^5 \equiv y$ is a microscopic coordinate in ϑ_n . In short, the cylinder condition is written as (by first putting $n^A = e_5^A$)

$$g_{(ij),5} = g_{(ij),A} n^A = e_i^B e_j^C (n_{B;C} + n_{C;B}) = 0,$$

where we have now assumed that in $\mathbb{M}_5$ the differential expression $e_{A,B}^i - e_{B,A}^i$ vanishes. However, from (2.2), we have the relation

$$e_{i,j}^A - e_{j,i}^A = 2e_k^A \Gamma_{[ij]}^k + F_{ij} n^A.$$

Furthermore, the cylinder condition implies that $n_{A;B} + n_{B;A} = 0$ and therefore we can nullify (2.8). This is often called “the assumption of weakness”. The second condition is the condition of integrability imposed on arbitrary vector fields, e.g., on θ_i (say) in $\mathbb{R}_4$. The necessary and sufficient condition for a vector field $\theta_{,i} \equiv \gamma_i$ (a one-form) to be integrable is $\gamma_{i,j} = \gamma_{j,i}$. If this is applied to (2.4), we will then have $R^A_{BCD} e^B_i e^C_j e^D_k = S^A_{ijk}$. Therefore (2.6) will now go into

$$R_{ijkl} = \frac{1}{4} (F_{ik} F_{jl} - F_{il} F_{jk}) , \quad (2.10a)$$

$$F_{ij;k} - F_{ik;j} = -2\Gamma^l_{[jk]} F_{il} . \quad (2.10b)$$

These are the sought unified field equations of gravity and electromagnetism. They form the basic field equations of our unified field theory.

Altogether they imply that

$$R_{ik} = -\frac{1}{4} F_{ij} F^j_{\cdot k} = R_{(ik)} , \quad (2.11a)$$

$$R_{[ik]} = 0 , \quad (2.11b)$$

$$R = \frac{1}{4} F_{ik} F^{ik} , \quad (2.11c)$$

$$F^{ij}_{\cdot j} = 2\Gamma^l_{[ik]} F^k_l \equiv J_i . \quad (2.11d)$$

These relations are necessary and sufficient following the two conditions we have dealt with. These field equations seem to satisfy a definite need. They tell us a beautiful and simple relation between gravity and electromagnetism: (2.10a) tells us that both inside and outside charges, a gravitational field originates in a non-null electromagnetic field (as in Rainich’s geometry), since according to (2.10b), the electromagnetic current is produced by the torsion of space-time: the torsion produces an electromagnetic source. The electromagnetic current is generated by dynamic “electric-magnetic” charges. In a strict sense, the gravitational field cannot exist without the electromagnetic field. Hence all matter in the Universe may have an electromagnetic origin. Denoting by $d\Omega$ a three-dimensional infinitesimal boundary enclosing several charges, we have, from (2.11d)

$$\delta e = 2 \int_{\Omega} \Gamma^r_{[ik]} u^i F^k_{\cdot r} d\Omega .$$

We may represent a negative charge by a negative spin produced by a left-handed twist (torsion) and a positive one by a positive spin produced by a right-handed twist. (For the conservation of charges (currents) see Sections 4.3 and 4.4.) Now (2.11c) tells us that when the spatial curvature, represented by the Ricci scalar, vanishes, we have a null electromagnetic field, also it is seen that the strength of the electromagnetic field is equivalent to the spatial curvature. Therefore gravity and electromagnetism are inseparable. The electromagnetic source, the charge, looks like a microscopic spinning hole in the structure of the space-time $\mathbb{R}_4$, however, the Schwarzschild singularity is non-existent in general. Consequently, outside charges our field equations read

$$R_{ijkl} = \frac{1}{4} (F_{ik}F_{jl} - F_{il}F_{jk}) , \quad (2.12a)$$

$$F_{;j}^{ij} = 0 , \quad (2.12b)$$

which, again, give us a picture of how a gravitational field emerges (outside charges).

In this way, the standard action integral of our theory may take the form

$$\begin{aligned} I &= \int \left({}^*R - (R_{ik}R^{ik})^{\frac{1}{2}} \right) \sqrt{-g} d^4x = \\ &= \int \left({}^*R - \left(\frac{1}{16} (F_{ik}F^{ik})^2 - R_{ijkl}R^{ijkl} - \frac{1}{4} R_{ijkl}F^{il}F^{jk} \right)^{\frac{1}{2}} \right) \sqrt{-g} d^4x . \end{aligned} \quad (2.13)$$

Here *R denotes the Ricci scalar built from the symmetric Christoffel connection alone.

From the variation of which, we would arrive at the standard Einstein-Maxwell equations. However, we do not wish to stress heavy emphasis upon such an action-method (which seems like a forced short cut) in order to arrive at the field equations of our unified field theory. We must emphasize that the equations (2.10)-(2.13) tell us how the electromagnetic field is incorporated into the gravitational field in a very natural manner, in other words there's no need here to construct any Lagrangian density of such. We have been led into thinking of how to couple both fields using different procedures without realizing that these fields already encapsulate each other in Nature. But here our space-time is already a polarized continuum in the sense that there exists an electromagnetic field at every point of it which in turn generates a gravitational field.

Remark 1

Without the integrability condition we have, in fairly general conditions, the relation

$$\begin{aligned} (\partial_k \partial_j - \partial_j \partial_k) e_i^A &= -R_{ijk}^l e_l^A + R_{BCD}^A e_i^B e_j^C e_k^D + \\ &+ \left(\phi_{ij;k} - \phi_{ik;j} + 2\Gamma_{[jk]}^l \phi_{il} \right) n^A + \left(\phi_{ik} \phi_{.j}^l - \phi_{ij} \phi_{.k}^l \right) e_l^A - \\ &- 2\Gamma_{BC}^A \left(\Gamma_{[jk]}^l e_l^C + \phi_{[jk]} n^C \right) e_i^B, \end{aligned} \quad (2.a)$$

$$\begin{aligned} S_{ijk}^A &\equiv (\partial_k \partial_j - \partial_j \partial_k) e_i^A + 2\Gamma_{BC}^A \left(\Gamma_{[jk]}^l e_l^C + \phi_{[jk]} n^C \right) e_i^B = \\ &= (\partial_k \partial_j - \partial_j \partial_k) e_i^A + \Gamma_{BC}^A \left(\partial_k e_j^C - \partial_j e_k^C \right) e_i^B. \end{aligned}$$

Hence we have

$$R_{ijkl} = \phi_{ik} \phi_{jl} - \phi_{il} \phi_{jk} + R_{ABCD} e_i^A e_j^B e_k^C e_l^D - S_{Aijk} e_i^A, \quad (2.b_1)$$

$$\phi_{ij;k} - \phi_{ik;j} = -R_{ABCD} n^A e_i^B e_j^C e_k^D + S_{Aijk} n^A - 2\Gamma_{[jk]}^l \phi_{il}. \quad (2.b_2)$$

These are just the equations in (2.6). Upon employing a suitable cylinder condition and putting $\phi_{[ij]} = \frac{1}{2} F_{ij}$ (within suitable units), we have the complete set of field equations of gravoelectrodynamics:

$$R_{ijkl} = \frac{1}{4} (F_{ik} F_{jl} - F_{il} F_{jk}) + R_{ABCD} e_i^A e_j^B e_k^C e_l^D - S_{Aijk} e_i^A, \quad (2.c_1)$$

$$\frac{1}{2} (F_{ij;k} - F_{ik;j}) = -R_{ABCD} n^A e_i^B e_j^C e_k^D + S_{Aijk} n^A - \Gamma_{[jk]}^l F_{il}. \quad (2.c_2)$$

End of Remark 1

Sub-remark

Let's consider the space $\mathbb{S}_5 = \mathbb{R}_4 \otimes Y$ which describes a five-dimensional "thin shell" where Y is the microscopic coordinate representation spanned by the unit normal vector to the four-manifold $\mathbb{R}_4$. The coordinates of this space are characterized by $y^\mu = (x^i, y)$ where the Greek indices run from the 1 to 5 and where the extra coordinate y is taken to be the Planck length:

$$y = \sqrt{\frac{G\hbar}{c^3}},$$

which gives the "thickness" of "thin shell". Here G is the gravitational constant of Newton, $\hbar$ is the Planck constant divided by 2π and c is the speed of light in vacuum. (From now on, since the Planck length is extremely tiny, we may drop any higher-order terms in y .) Then the

basis $\{\gamma_\mu\}$ of the space $\mathbb{S}_5$ can in general be split into

$$\gamma_i = (\delta_i^k - y\phi_{.i}^k) g_k,$$

$$\gamma_5 = n.$$

It is seen that the metric tensor of the space $\mathbb{S}_5$, i.e., $\gamma_{\mu\nu}$, has the following non-zero components:

$$\gamma_{ik} = g_{(ik)} - 2y\phi_{(ik)},$$

$$\gamma_{55} = 1.$$

The simplest sub-space of the space $\mathbb{S}_5$ is given by the basis

$$g_i = \gamma_i(x^i, 0),$$

$$g_5 = n,$$

where $\{g_l\}$ is of course the tangent basis of the manifold $\mathbb{R}_4$. We shall denote this pseudo-five-dimensional space as the special coordinate system $\vartheta_n = \mathbb{R}_4 \otimes n$ whose metric tensor $g_{\mu\nu}$ can be arrayed as

$$g_{\mu\nu} = \begin{pmatrix} [g_{(ik)}]_{4 \times 4} & 0 \\ 0 & 1 \end{pmatrix}.$$

Now the tetrad of the space $\mathbb{S}_5$ is then given by $\gamma_\mu^A = (e^A \cdot \gamma_\mu)$, which can be split into

$$\gamma_i^A = e_i^A - y\phi_{.i}^k e_k^A,$$

$$\gamma_5^A = n^A.$$

Then we may find the inverse to the tetrad γ_i^A as follows:

$$\gamma_A^i = e_A^i + y\phi_{.k}^i e_A^k,$$

$$\gamma_A^5 = n_A = y_{,A}.$$

From the above relations, we have the following:

$$\gamma_A^i \gamma_k^A = \delta_k^i,$$

$$\gamma_k^A e_A^i = \delta_k^i - y\phi_{.k}^i,$$

$$\gamma_A^i e_k^A = \delta_k^i + y\phi_{.k}^i.$$

The five-dimensional index of the tetrad γ_i^A is raised and lowered using the metric tensor $g_{(AB)}$. The inverse of the tetrad is achieved with the help of the metric tensor

$$G_{AB} = e_A^i e_B^k g_{(ik)} + n_A n_B - 2y \phi_{(ik)} e_A^i e_B^k,$$

which reduces to $g_{(AB)}$ due to the cylinder condition. The four-dimensional metric tensor $g_{(ik)}$ is used to raise and lower the four-dimensional index. Again, we bring in the electromagnetic field tensor F_{ij} via the cylinder condition, which yields $\phi_{ik} = \frac{1}{2} F_{ik}$. The connection of the space $\mathbb{S}_5$ is then

$$\theta_{\mu\nu}^\lambda(x^i, y) = \Gamma_{\mu\nu}^\lambda - \frac{1}{2} F_{\cdot\mu}^\lambda y_{,\nu} - \frac{1}{2} F_{\cdot\nu}^\lambda y_{,\mu} - \frac{1}{2} y F_{\cdot\mu,\nu}^\lambda - \frac{1}{2} y F_{\cdot\nu,\mu}^\lambda \Gamma_{\sigma\nu}^\lambda,$$

where $\Gamma_{\mu\nu}^\lambda$ is the connection of the space $\vartheta_n = \mathbb{R}_4 \otimes n$ with the electromagnetic field tensor derived from it: $F_{\mu\nu} = 2\Gamma_{\mu\nu}^5$, $F_{\cdot\nu}^\mu = -2\Gamma_{5\nu}^\mu$, $F_{\mu 5} = 0$. Therefore its only non-zero components are F_{ik} . At the base of the space $\mathbb{S}_5$, the connection is

$$\omega_{\mu\nu}^\lambda = \theta_{\mu\nu}^\lambda(x^i, 0) = \Gamma_{\mu\nu}^\lambda - \frac{1}{2} F_{\cdot\mu}^\lambda y_{,\nu} - \frac{1}{2} F_{\cdot\nu}^\lambda y_{,\mu}.$$

From the above expression, we see that

$$\begin{aligned} \omega_{\mu k}^\lambda &= \Gamma_{\mu k}^\lambda - \frac{1}{2} F_{\cdot k}^\lambda y_{,\mu}, \\ \omega_{\mu 5}^\lambda &= -\frac{1}{2} F_{\cdot\mu}^\lambda, \omega_{5\nu}^\lambda = -F_{\cdot\nu}^\lambda. \end{aligned}$$

As can be worked out, the five-dimensional connection of the background space $\mathbb{I}\mathbb{R}_5$ is related to that of $\mathbb{S}_5$ through

$$\begin{aligned} \Gamma_{BC}^A(x^i, y) &= \gamma_\mu^A \gamma_{B,C}^\mu + \gamma_\mu^A \theta_{\rho\nu}^\mu \gamma_B^\rho \gamma_C^\nu - \frac{1}{2} F_{\cdot\nu}^\mu \gamma_\mu^A \gamma_B^\nu n_C - \\ &\quad - \frac{1}{2} y \gamma_\mu^A \theta_{\nu k}^\mu \gamma_B^\nu F_{\cdot i}^k e_C^i. \end{aligned}$$

Now the five-dimensional curvature tensor $R_{ABCD} = R_{ABCD}(x^i, 0)$ is to be related once again to the four-dimensional curvature tensor of $\mathbb{R}_4$, which can be directly derived from the curvature tensor of the space $\mathbb{S}_5$ as $R_{ijkl} = {}^{\mathbb{S}_5}R_{ijkl}(x^i, 0)$. With the help of the above geometric objects, and after some laborious work-out, we arrive at the relation

$$R_{ABCD} = e_A^\mu e_B^\nu e_C^\rho e_D^\sigma R_{\mu\nu\rho\sigma} - \frac{1}{2} F_{AB} F_{CD} + \varphi_{ABCD},$$

where we have a new geometric object constructed from the electromagnetic field tensor:

$$\begin{aligned}\varphi_{ABC}^\mu &= \frac{1}{2} F_{\cdot\sigma}^\rho (e_B^\sigma n_C - e_C^\sigma n_B) e_A^\nu \omega_{\nu\rho}^\mu + \frac{1}{2} F_{\nu,\rho}^\mu (n_B e_C^\rho - n_C e_B^\rho) e_A^\nu + \\ &+ \frac{1}{2} F_{\cdot\nu}^\mu (\gamma_{A,C}^\nu(x^\alpha, 0) n_B - \gamma_{A,B}^\nu(x^\alpha, 0) n_C) + \\ &+ \frac{1}{2} F_{\cdot\nu}^\sigma (n_B e_C^\alpha - n_C e_B^\alpha) e_A^\nu \omega_{\sigma\alpha}^\mu, \\ \varphi_{BCD}^A &\equiv \varphi_{BCD}^\mu e_\mu^A.\end{aligned}$$

Define another curvature tensor:

$$\Phi_{ijkl} \equiv (R_{ABCD} - \varphi_{ABCD}) e_i^A e_j^B e_k^C e_l^D.$$

Then we have the relation

$$R_{ijkl} = \Phi_{ijkl} + \frac{1}{2} F_{ij} F_{kl}.$$

By the way, the curvature tensor of the space $\vartheta_n = \mathbb{R}_4 \otimes n$ is here given by

$$R_{\nu\rho\sigma}^\mu = \theta_{\nu\sigma,\rho}^\mu(x^\alpha, 0) - \theta_{\nu\rho,\sigma}^\mu(x^\alpha, 0) + \omega_{\nu\sigma}^\alpha \omega_{\alpha\rho}^\mu - \omega_{\nu\rho}^\alpha \omega_{\alpha\sigma}^\mu.$$

Expanding the connections in the above relation, we obtain

$$\begin{aligned}R_{\nu\rho\sigma}^\mu &= \Gamma_{\nu\sigma,\rho}^\mu - \Gamma_{\nu\rho,\sigma}^\mu + \Gamma_{\nu\sigma}^\lambda \Gamma_{\lambda\rho}^\mu - \Gamma_{\nu\rho}^\lambda \Gamma_{\lambda\sigma}^\mu + \\ &+ \frac{1}{2} (F_{\rho,\sigma}^\mu - F_{\sigma,\rho}^\mu) y_{,\nu} + \frac{1}{2} F_{\cdot\alpha}^\mu (\Gamma_{\nu\rho}^\alpha y_{,\sigma} - \Gamma_{\nu\sigma}^\alpha y_{,\rho}) + \\ &+ \frac{1}{2} (F_{\cdot\sigma}^\mu \Gamma_{\nu\rho}^\alpha - F_{\cdot\rho}^\mu \Gamma_{\nu\sigma}^\alpha) y_{,\alpha} + \frac{1}{2} (F_{\cdot\rho}^\alpha \Gamma_{\alpha\sigma}^\mu - F_{\cdot\sigma}^\alpha \Gamma_{\alpha\rho}^\mu) y_{,\nu} + \\ &+ \frac{1}{4} F_{\cdot\nu}^\alpha (F_{\rho\sigma}^\mu y_{,\sigma} - F_{\sigma\rho}^\mu y_{,\sigma}) y_{,\alpha} + \frac{1}{4} F_{\cdot\alpha}^\mu (F_{\sigma\rho}^\alpha y_{,\rho} - F_{\rho\sigma}^\alpha y_{,\sigma}) y_{,\nu} + \\ &+ \frac{1}{4} (F_{\cdot\rho}^\mu F_{\cdot\sigma}^\alpha - F_{\cdot\sigma}^\mu F_{\cdot\rho}^\alpha) y_{,\nu} y_{,\alpha}.\end{aligned}$$

We therefore see that the electromagnetic field tensor is also present in the curvature tensor of the space $\vartheta_n = \mathbb{R}_4 \otimes n$. In other words, electromagnetic and gravitational interactions are described together on an equal footing by this single curvature tensor.

Direct calculation shows that some of its four-dimensional and mixed components are

$$R_{jkl}^i = \Gamma_{jl,k}^i - \Gamma_{jk,l}^i + \Gamma_{jl}^r \Gamma_{rk}^i - \Gamma_{jk}^r \Gamma_{rl}^i,$$

$$R_{.ijk}^5 = R_{5ijk} = -\frac{1}{2} \left(F_{ij;k} - F_{ik;j} + 2\Gamma_{[jk]}^r F_{ir} \right),$$

$$R_{ij5k} = 0.$$

Furthermore, we obtain the following equivalent expressions:

$$\begin{aligned} R_{ABCD} &= R_{ijkl} e_A^i e_B^j e_C^k e_D^l + \frac{1}{2} \left(F_{ik;l} - F_{il;k} + 2\Gamma_{[kl]}^m F_{im} \right) e_A^i n_B e_C^k e_D^l + \\ &+ \frac{1}{2} \left(F_{jl;k} - F_{jk;l} + 2\Gamma_{[lk]}^m F_{jm} \right) n_A e_B^j e_C^k e_D^l - \frac{1}{2} F_{AB} F_{CD} + \varphi_{ABCD}, \\ R_{ABCD} &= \frac{1}{2} (F_{AC;D} - F_{AD;C}) n_B + \frac{1}{2} (F_{BD;C} - F_{BC;D}) n_A - \\ &- \frac{1}{2} F_{AB} F_{CD} - \psi_{ABCD} + R_{ijkl} e_A^i e_B^j e_C^k e_D^l + \\ &+ 2\Gamma_{[kl]}^m F_{im} e_A^i n_B e_C^k e_D^l + 2\Gamma_{[lk]}^m F_{jm} n_A e_B^j e_C^k e_D^l, \end{aligned}$$

where

$$\begin{aligned} \psi_{ABCD} &= \frac{1}{2} \left(F_{MA} (F_{.C}^M n_D - F_{.D}^M n_C) n_B - \right. \\ &\left. - F_{MB} (F_{.C}^M n_D - F_{.D}^M n_C) n_A \right) - \varphi_{ABCD}. \end{aligned}$$

When the torsion tensor of the space $\mathbb{R}_4$ vanishes, we have the relation

$$\begin{aligned} R_{ABCD} &= \frac{1}{2} (F_{AC;D} - F_{AD;C}) n_B + \frac{1}{2} (F_{BD;C} - F_{BC;D}) n_A - \\ &- \frac{1}{2} F_{AB} F_{CD} - \psi_{ABCD} + R_{ijkl} e_A^i e_B^j e_C^k e_D^l \end{aligned}$$

which, again, relates the curvature tensors to the electromagnetic field tensor.

Finally, if we define yet another five-dimensional curvature tensor:

$$\mathbb{R}_{ABCD} \equiv \tilde{R}_{ABCD} + \frac{1}{2} \tilde{F}_{AB} \tilde{F}_{CD} - \tilde{\varphi}_{ABCD},$$

where $\tilde{R}_{ABCD}$, $\tilde{F}_{AB}$ and $\tilde{\varphi}_{ABCD}$ are the extensions of R_{ABCD} , F_{AB} and φ_{ABCD} which are dependent on y , we may obtain the relation

$$\begin{aligned} \mathbb{R}_{ABCD} e_i^A e_j^B e_k^C e_l^D &= \\ &= R_{ijkl} + \frac{1}{2} y F_{.i}^r R_{rjkl} + \frac{1}{2} y F_{.j}^r R_{irkl} + \frac{1}{2} y F_{.k}^r R_{ijrl} + \frac{1}{2} y F_{.l}^r R_{ijk r}. \end{aligned}$$

End of Sub-remark

Let's now write the field equations of our unified field theory as

$$R_{ijkl} = \frac{1}{4} (F_{ik}F_{jl} - F_{il}F_{jk}) + \phi_{ijkl}, \quad (2.d_1)$$

$$F_{ij;k} - F_{ik;j} = -2\Gamma_{[jk]}^l F_{il} + \lambda_{ijk}, \quad (2.d_2)$$

$$\phi_{ijkl} = R_{ABCD} e_i^A e_j^B e_k^C e_l^D - S_{Aijk} e_i^A, \quad (2.e_1)$$

$$\lambda_{ijk} = -2 (R_{ABCD} e_i^B e_j^C e_k^D - S_{Aijk}) n^A, \quad (2.e_2)$$

Consider the invariance of the curvature tensor under the gauge transformation

$$\Gamma_{jk}^i = \Gamma_{jk}^i + \delta_j^i \psi_{,k} \quad (2.f)$$

for some function $\psi = \psi(x)$. This is analogous to the gauge transformation of the electromagnetic potential, i.e., $\varphi_i = \varphi_i + \sigma \psi_{,i}$, with a scaling constant σ , which leaves the electromagnetic field tensor invariant.. We define the electromagnetic potential vector φ_i and pseudo-vector ζ_i via $\Gamma_{ki}^k = \alpha \varphi_i + \zeta_i$ where α is a constant.

Then we see that the electromagnetic field tensor can be expressed as

$$F_{ik} = \varphi_{i,k} - \varphi_{k,i} = \frac{1}{\alpha} (\zeta_{i,k} - \zeta_{k,i}). \quad (2.g)$$

More specifically, the two possible electromagnetic potentials φ_i and ζ_i transform homogeneously and inhomogeneously, respectively, according to

$$\varphi_i = e_i^A \varphi_A,$$

$$\zeta_i = e_i^A \zeta_A + e_A^k e_{k,i}^A - n_A \Gamma_{BC}^A n^B e_i^C.$$

The two potentials become equivalent in a coordinate system where $\sqrt{-g}$ equals a constant. Following (2.g), we can express the curvature tensor as

$$R_{ijkl} = \lambda_1 (F_{ik}F_{jl} - F_{il}F_{jk}) + \lambda_2 (g_{(ik)}g_{(jl)} - g_{(il)}g_{(jk)}), \quad (2.h)$$

where λ_1 and λ_2 are invariants. (The term $\lambda_0 F_{ij}F_{kl}$ would contribute nothing.) Hence

$$R_{ik} = \lambda_1 F_{il}F_{k.}^l + 3\lambda_2 g_{(ik)}. \quad (2.i)$$

Putting $\lambda_1 = \frac{1}{4}$ in accordance with (2.d₁) and contracting (2.i) on the indices i and k we see that $\lambda_2 = \frac{1}{12}R - \frac{1}{48}F_{ik}F^{ik}$. Consequently, we

have the important relations

$$R_{ijkl} = \frac{1}{4} (F_{ik}F_{jl} - F_{il}F_{jk}) + \frac{1}{12} (g_{(ik)}g_{(jl)} - g_{(il)}g_{(jk)}) R - \frac{1}{48} (g_{(ik)}g_{(jl)} - g_{(il)}g_{(jk)}) F_{rs}F^{rs}, \quad (2.j)$$

$$R_{ik} = \frac{1}{4} F_{il}F_k^l + \frac{1}{4} g_{(ik)}R - \frac{1}{16} g_{(ik)}F_{rs}F^{rs}. \quad (2.k)$$

Comparing (2.j) and (2.e₁) we find

$$\begin{aligned} \phi_{ijkl} &= R_{ABCD} e_i^A e_j^B e_k^C e_l^D - S_{Aijkl} e_i^A = \\ &= \frac{1}{12} (g_{(ik)}g_{(jl)} - g_{(il)}g_{(jk)}) R - \frac{1}{48} (g_{(ik)}g_{(jl)} - g_{(il)}g_{(jk)}) F_{rs}F^{rs}. \end{aligned} \quad (2.l)$$

Hence also

$$\phi_{ik} = \frac{1}{4} g_{(ik)}R - \frac{1}{16} g_{(ik)}F_{rs}F^{rs}, \quad (2.m_1)$$

$$\phi = R - \frac{1}{4} F_{rs}F^{rs}. \quad (2.m_2)$$

Note that our above consideration produces the following traceless field equation:

$$R_{ik} - \frac{1}{4} g_{(ik)}R = \frac{1}{4} \left(F_{il}F_k^l - \frac{1}{4} g_{(ik)}F_{rs}F^{rs} \right). \quad (2.n)$$

In a somewhat particular case, we may set $-\frac{1}{4} g_{(ik)}R = \kappa \times (\rho c^2 u_i u_k + t_{ik})$ where κ is a coupling constant and t_{ik} is the generalized stress-metric tensor, such that $R = -\kappa \rho_e$, where now $\rho_e = \rho c^2 + t$ is the effective material density. We also have

$$R_{ik} - \frac{1}{4} g_{(ik)}R = \kappa (\rho c^2 u_i u_k + t_{ik}) + \frac{1}{4} \left(F_{il}F_k^l - \frac{1}{4} g_{(ik)}F_{rs}F^{rs} \right). \quad (2.o)$$

The above looks slightly different from the standard field equation of General Relativity:

$${}^*R_{ik} - \frac{1}{2} g_{(ik)} {}^*R = k \left(\rho c^2 u_i u_k + t_{ik} - \left(F_{il}F_k^l - \frac{1}{4} g_{(ik)}F_{rs}F^{rs} \right) \right), \quad (2.p)$$

which is usually obtained by summing altogether the matter and electromagnetic terms. Here ${}^*R_{ik}$ and *R are the Ricci tensor and scalar built out of the Christoffel connection and $k \neq \kappa$ is the usual coupling constant of General Relativity. Let's denote by m_0 , ρ and c , the point-mass, material density and speed of light in vacuum. Then the vanishing

of the divergence of (2.p) leads to the equation of motion for a charged particle:

$$\frac{Du^i}{Ds} \equiv u^i_{;k} u^k = \frac{e}{m_0 c^2} F^i_k u^k.$$

However, this does not provide a real hint to the supposedly missing link between matter and electromagnetism. We hope that there's no need to add an external matter term to the stress-energy tensor. We may interpret (2.n), (2.o) and (2.p) as telling us that matter and electromagnetism are already incorporated, in other words, the electromagnetic field produces material density out of the electromagnetic current J . In fact these are all acceptable field equations. Now, for instance, we have $R = -\kappa \rho_e = -\kappa (c^2 J^i u_i + t)$. From (2.m₂), the classical variation follows:

$$\begin{aligned} \delta I &= \delta \int \phi \sqrt{-g} d^4 x = \\ &= \delta \int \left(R - \frac{1}{4} F_{ik} F^{ik} \right) \sqrt{-g} d^4 x = 0, \end{aligned} \quad (2.q)$$

which yields the gravitational and electromagnetic equations of Einstein and Maxwell endowed with source since the curvature scalar here contains torsion as well.

Finally, let's investigate the explicit relation between the Weyl tensor and the electromagnetic field tensor in this theory. In four dimensions the Weyl tensor is

$$\begin{aligned} C_{ijkl} &= R_{ijkl} - \frac{1}{2} (g_{(ik)} R_{jl} + g_{(jl)} R_{ik} - g_{(il)} R_{jk} - g_{(jk)} R_{il}) + \\ &+ \frac{1}{6} (g_{(ik)} g_{(jl)} - g_{(il)} g_{(jk)}) R. \end{aligned}$$

Comparing the above equation(s) with (2.j) and (2.k), we have

$$\begin{aligned} C_{ijkl} &= \frac{1}{4} (F_{ik} F_{jl} - F_{il} F_{jk}) + \frac{1}{24} (g_{(ik)} g_{(jl)} - g_{(il)} g_{(jk)}) F_{rs} F^{rs} - \\ &- \frac{1}{8} (g_{(ik)} F_{jr} F^r_{l.} + g_{(jl)} F_{ir} F^r_{k.} - g_{(il)} F_{jr} F^r_{k.} - g_{(jk)} F_{ir} F^r_{l.}). \end{aligned} \quad (2.r)$$

We see that the Weyl tensor is composed solely of the electromagnetic field tensor in addition to the metric tensor. Hence we come to the conclusion that the space-time $\mathbb{R}_4$ is conformally flat if and only if the electromagnetic field tensor vanishes. This agrees with the fact that, when treating gravitation and electromagnetism separately, it is

the Weyl tensor, rather than the Riemann tensor, which is compatible with the electromagnetic field tensor. From the structure of the Weyl tensor as revealed by (2.r), it is understood that the Weyl tensor actually plays the role of an electromagnetic polarization tensor in the space-time $\mathbb{R}_4$. In an empty region of the space-time $\mathbb{R}_4$ with a vanishing torsion tensor, when the Weyl tensor vanishes, that region possesses a constant sectional curvature which conventionally corresponds to a constant energy density.

Let's for a moment turn back to (2.6). We shall show how to get the source-torsion relation, i.e., (2.11d) in a different way. For this purpose we also set a constraint $S^A_{ijk} = 0$ and assume that the background five-dimensional space is an Einstein space:

$$R_{AB} = \Lambda g_{(AB)},$$

where Λ is a cosmological constant. Whenever $\Lambda = 0$ we say that the space is Ricci-flat or energy-free, devoid of matter. Taking into account the cosmological constant, this consideration therefore takes on a slightly different path than our previous one. We only wish to see what sort of field equations it will produce.

We first write

$$\begin{aligned} R_{ijkl} &= \frac{1}{4} (F_{ik}F_{jl} - F_{il}F_{jk}) + R_{ABCD} e_i^A e_j^B e_k^C e_l^D, \\ F_{ij;k} - F_{ik;j} &= -2R_{ABCD} n^A e_i^B e_j^C e_k^D - 2\Gamma_{[jk]}^l F_{il}. \end{aligned}$$

Define a symmetric tensor:

$$B_{ik} \equiv -R_{ABCD} e_i^A n^B e_k^C n^D = B_{ki}. \quad (2.14)$$

It is immediately seen that

$$R_{AB} e_i^A e_k^B = \Lambda g_{(ik)}, \quad (2.15a)$$

$$R_{AB} n^A n^B = \Lambda, \quad (2.15b)$$

$$R_{AB} e_i^A n^B = 0. \quad (2.15c)$$

Therefore

$$\begin{aligned} R_{ik} &= -\frac{1}{4} F_{il} F_{.k}^l + R_{AB} e_i^A e_k^B - R_{ABCD} e_i^A n^B e_k^C n^D = \\ &= -\frac{1}{4} F_{il} F_{.k}^l + \Lambda g_{(ik)} + B_{ik}. \end{aligned} \quad (2.16)$$

From (2.14) we also have, with the help of (2.15b), the following:

$$\begin{aligned} B &= g^{(ik)} B_{ik} = -R_{ABCD} n^B n^D \left(g^{(AC)} - n^A n^C \right) = \\ &= -R_{AB} n^A n^B + R_{ABCD} n^A n^B n^C n^D = \\ &= -\Lambda. \end{aligned}$$

Hence we have

$$R = \frac{1}{4} F_{ik} F^{ik} + 3\Lambda. \quad (2.17)$$

The Einstein tensor (or rather, the generalized Einstein tensor endowed with torsion)

$$G_{ik} \equiv R_{ik} - \frac{1}{2} g_{(ik)} R$$

up to this point is therefore

$$G_{ik} = B_{ik} - \frac{1}{8} g_{(ik)} F_{rs} F^{rs} - \frac{1}{2} \Lambda g_{(ik)} - \frac{1}{4} F_{il} F_{\cdot k}^l. \quad (2.18)$$

From the relation

$$F_{ij;k} - F_{ik;j} = -2R_{ABCD} n^A e_i^B e_j^C e_k^D - 2\Gamma_{[jk]}^l F_{il}$$

we see that

$$\begin{aligned} F_{\cdot k}^{kj} &= -2R_{A\cdot D}^{BC} n^A e_B^k e_C^j e_k^D - 2\Gamma_{l[k]}^j F^{kl} = \\ &= -2R_{A\cdot}^C e_C^j n^A - 2R_{A\cdot D}^{BC} n^A n_B e_C^j n^D - 2\Gamma_{l[k]}^j F^{kl} = \\ &= -2\Gamma_{l[k]}^j F^{kl} = -J^j. \end{aligned}$$

In other words,

$$J_i = 2\Gamma_{[ik]}^l F_{\cdot l}^k,$$

which is just (2.11d). We will leave this consideration here and commit ourselves to the field equations given by (2.10) and (2.11) for the rest of our work.

Let's obtain the (generalized) Bianchi identity with the help of (2.10) and (2.11). Recall once again that

$$R_{ijkl} = \frac{1}{4} (F_{ik} F_{jl} - F_{il} F_{jk}),$$

$$F_{ij;k} - F_{ik;j} = -2\Gamma_{[jk]}^l F_{il}.$$

Performing the covariant derivative on R_{ijkl} , the result of the cyclic summation over the indices k, l and m is

$$\begin{aligned}
 R_{ijkl;m} + R_{ijlm;k} + R_{ijmk;l} &= \\
 &= \frac{1}{4} (F_{ik;m} - F_{im;k}) F_{jl} + \frac{1}{4} (F_{jl;m} - F_{jm;l}) F_{ik} + \\
 &+ \frac{1}{4} (F_{il;k} - F_{ik;l}) F_{jm} + \frac{1}{4} (F_{jm;k} - F_{jk;m}) F_{il} + \\
 &+ \frac{1}{4} (F_{im;l} - F_{il;m}) F_{jk} + \frac{1}{4} (F_{jk;l} - F_{jl;k}) F_{im} .
 \end{aligned} \tag{2.19}$$

Or equivalently,

$$\begin{aligned}
 R_{ijkl;m} + R_{ijlm;k} + R_{ijmk;l} &= \\
 &= \frac{1}{2} \Gamma_{[mk]}^n F_{in} F_{jl} + \frac{1}{2} \Gamma_{[ml]}^n F_{jn} F_{ik} + \frac{1}{2} \Gamma_{[kl]}^n F_{in} F_{jm} + \\
 &+ \frac{1}{2} \Gamma_{[km]}^n F_{jn} F_{il} + \frac{1}{2} \Gamma_{[lm]}^n F_{in} F_{jk} + \frac{1}{2} \Gamma_{[lk]}^n F_{jn} F_{im} .
 \end{aligned} \tag{2.20}$$

From (2.10) if we raise the index i and then perform a contraction with respect to the indices i and k , we have

$$\begin{aligned}
 R_{jl;m} - R_{jml;i} - R_{jm;l} &= \\
 &= \frac{1}{4} J_m F_{jl} - \frac{1}{4} J_l F_{jm} + \frac{1}{4} F_{\cdot l}^i F_{jm;i} - \frac{1}{4} F_{\cdot l}^i F_{ji;m} + \\
 &+ \frac{1}{4} F_{ji} F_{\cdot m;l} - \frac{1}{4} F_{ji} F_{\cdot l;m} + \frac{1}{4} F_{\cdot m}^i F_{ji;l} - \frac{1}{4} F_{\cdot m}^i F_{jl;i} .
 \end{aligned}$$

If we raise the index j and then contract on the indices j and l , we have the expression

$$\begin{aligned}
 R_{;m} - 2R_{\cdot m;i}^i &= (\delta_m^i R - 2R_{\cdot m}^i)_{;i} = \\
 &= -\frac{1}{2} J^i F_{im} + \frac{1}{2} F_{\cdot l}^i (F_{\cdot m;i}^l - F_{\cdot i;m}^l) .
 \end{aligned}$$

Therefore we have

$$\begin{aligned}
 \left(R_{\cdot m}^i - \frac{1}{2} \delta_m^i R \right)_{;i} &= \frac{1}{4} J^i F_{im} + \frac{1}{4} F_{\cdot l}^i (F_{\cdot i;m}^l - F_{\cdot m;i}^l) = \\
 &= \frac{1}{4} J^i F_{im} + \lambda_m ,
 \end{aligned} \tag{2.21}$$

where

$$\lambda_m = \frac{1}{4} F_{\cdot l}^i (F_{\cdot i;m}^l - F_{\cdot m;i}^l) = -\frac{1}{2} F_{\cdot l}^i \Gamma_{[im]}^k F_{\cdot k}^l ,$$

We can also write

$$\left(R^{ik} - \frac{1}{2} g^{(ik)} R \right)_{;k} = -\frac{1}{4} J_k F^{ik} + \lambda^i. \quad (2.22)$$

On the other hand, repeating the same contraction steps on (2.20) gives

$$\begin{aligned} R_{;m} - 2R^i_{m;i} &= (\delta_m^i R - 2R^i_m)_{;i} \equiv g_m = \\ &= \frac{1}{2} \Gamma_{[kl]}^n F_{\cdot n}^k F_{\cdot m}^l + \frac{1}{2} \Gamma_{[km]}^n F_{\cdot n}^l F_{\cdot l}^k + \frac{1}{2} \Gamma_{[lm]}^n F_{\cdot n}^k F_{\cdot k}^l + \\ &+ \frac{1}{2} \Gamma_{[lk]}^n F_{\cdot n}^l F_{\cdot m}^k = \\ &= \Gamma_{[kl]}^n F_{\cdot n}^k F_{\cdot m}^l + \Gamma_{[km]}^n F_{\cdot n}^l F_{\cdot l}^k = \\ &= \Gamma_{[kl]}^n F_{\cdot n}^k F_{\cdot m}^l - 2\lambda_m. \end{aligned}$$

Therefore we have

$$\left(R^{ik} - \frac{1}{2} g^{(ik)} R \right)_{;k} = -\frac{1}{2} g^i, \quad (2.23a)$$

where g^i , a non-linear quantity, can be seen as a complementary torsional current:

$$g^i = -\frac{1}{2} J_k F^{ik} + \lambda^i = \Gamma_{[kl]}^n F_{\cdot n}^k F^{li} - 2\lambda^i. \quad (2.23b)$$

In the most general case, by the way, the Ricci tensor is asymmetric. If we proceed further, the generalized Bianchi identity and its contracted form will be given by

$$\begin{aligned} R_{ijkl;m} + R_{ijlm;k} + R_{ijmk;l} &= \\ &= 2 \left(\Gamma_{[kl]}^r R_{ijrm} + \Gamma_{[lm]}^r R_{ijrk} + \Gamma_{[mk]}^r R_{ijrl} \right), \end{aligned} \quad (2.24a)$$

$$\left(R^{ik} - \frac{1}{2} g^{(ik)} R \right)_{;i} = 2g^{(ik)} \Gamma_{[ji]}^r R_{\cdot r}^j + \Gamma_{[ij]}^r R_{\cdot \dots r}^{ijk}. \quad (2.24b)$$

Remark 2

Consider a uniform charge density. Again, our resulting field equation (2.21) reads

$$\left(R^{ik} - \frac{1}{2} g^{(ik)} R \right)_{;k} = -\frac{1}{4} F_{\cdot k}^i J^k + \frac{1}{4} g^{(ik)} (F_{\cdot s}^r (F_{\cdot r;k}^s - F_{\cdot k;r}^s)),$$

where we have set $\lambda^i = \frac{1}{4} g^{(ik)} (F_{\cdot s}^r (F_{\cdot r; k}^s - F_{\cdot k; r}^s))$. Recall the Lorentz equation of motion: $m_0 c^2 \frac{Du^i}{Ds} = e F_{\cdot k}^i u^k$. Setting $\gamma = -\frac{1}{4} \left(\frac{\rho}{e} \right)$, we obtain

$$m_0 c^2 \frac{Du^i}{Ds} = \frac{1}{\gamma} \left(\left(R^{ik} - \frac{1}{2} g^{(ik)} R \right)_{;k} - \lambda^i \right)$$

or

$$m_0 c^2 \frac{Du^i}{Ds} = \frac{1}{\gamma} \left(\left(R^{ik} - \frac{1}{2} g^{(ik)} R \right)_{;k} - 2g^{(ik)} \Gamma_{[rk]}^s R_s^r \right).$$

In the absence of charge density (when the torsion tensor is zero), i.e., in the limit $\gamma \rightarrow \infty$, we get the usual geodesic equation of motion of General Relativity:

$$\frac{d^2 x^i}{ds^2} + \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0.$$

The field equation can always be brought into the form

$$R_{ik} = k \left(T_{ik} - \frac{1}{2} g_{(ik)} T \right).$$

Here the energy-momentum tensor need not always vanish outside the world-tube and in general we can write $T_{;k}^{ik} = -\frac{1}{2} g^i$. Now the Einstein tensor is

$$G_{ik} = R_{ik} - \frac{1}{2} g_{(ik)} R = -\frac{1}{4} F_{il} F_{\cdot k}^l - \frac{1}{8} g_{(ik)} F_{rs} F^{rs} = \varepsilon T_{ik}.$$

The right-hand side stands more appropriately as the field strength rather than the classical conservative source term as $T = -\frac{1}{2\varepsilon} F_{ik} F^{ik}$ (again, ε is a coupling constant). The equation of gravoelectrodynamics can immediately be written in the form

$$R_{ik} - \frac{1}{4} g_{(ik)} R = \frac{1}{4} \left(F_{il} F_{\cdot k}^l - \frac{1}{4} g_{(ik)} F_{rs} F^{rs} \right) = \kappa T_{ik},$$

$$T_{ik} = F_{il} F_{\cdot k}^l - \frac{1}{4} g_{(ik)} F_{rs} F^{rs}$$

as expected. In this field equation, as can be seen, the material density arises directly from electromagnetic interaction.

End of Remark 2

The (sub-)spaces $\vartheta_n = \mathbb{R}_4 \otimes n$ and $\mathbb{V}_6 = \vartheta_n \otimes m$. The ϑ_n -covariant derivative

We now consider the space $\vartheta_n = \mathbb{R}_4 \otimes n \subset \mathbb{M}_5$, a sub-space of $\mathbb{M}_5$ with basis $\{\gamma_A\}$ satisfying $\gamma_A = (g_i, n)$. Therefore this basis spans a special coordinate system in $\mathbb{M}_5$. We define the ϑ_n -covariant derivative to be a projective derivative which acts upon an arbitrary vector field of the form $\psi = (\psi^i, \sigma)$ or, more generally, upon an arbitrary tensor field of the form $T = \left(T_{kl\dots}^{ij\dots}, t_{kl\dots}^{i\dots}, t_{kl\dots}^{j\dots}, \dots, t_{kl\dots}^{ij\dots}, t_{kl\dots}^{ij\dots} \right)$, where $t_{kl\dots}^{i\dots} = T_{kl\dots}^{i5\dots}$, $t_{kl\dots}^{j\dots} = T_{kl\dots}^{5j\dots}$ etc., and which is projected onto the four-dimensional physical space-time $\mathbb{R}_4$. For instance, for $\psi = \psi^i g_i + \sigma n$ then

$$\begin{aligned} \nabla_j \psi &= \psi_{;j}^i g_i + \psi^i g_{i;j} + \sigma_{;j} n + \sigma n_{;j} = \\ &= \psi_{;j}^i g_i + \frac{1}{2} \psi^i F_{ij} n + \sigma_{;j} n - \frac{1}{2} \sigma F_{\cdot j}^i g_i = \\ &= \left(\psi_{;j}^i - \frac{1}{2} \sigma F_{\cdot j}^i \right) g_i + \left(\sigma_{;j} + \frac{1}{2} F_{ij} \psi^i \right) n. \end{aligned}$$

The projection of this onto $\mathbb{R}_4$ manifests in

$$\psi_{|j}^i = \psi_{;j}^i - \frac{1}{2} \sigma F_{\cdot j}^i.$$

The stroke $|$ represents the ϑ_n -covariant derivative which takes up the notion of cylindricity. When this is applied, for instance, to a second rank tensor field T , we have $T_{|k}^{ij} = T_{;k}^{ij} - \frac{1}{2} t_{(1)}^i F_{\cdot k}^j - \frac{1}{2} t_{(2)}^j F_{\cdot k}^i$. For a tensor of arbitrary rank, we therefore have

$$\begin{aligned} T_{kl\dots|m}^{ij\dots} &= T_{kl\dots;m}^{ij\dots} - \frac{1}{2} t_{(1)}^{i\dots} F_{\cdot m}^j - \frac{1}{2} t_{(2)}^{j\dots} F_{\cdot m}^i - \\ &\quad - \dots - \frac{1}{2} t_{(N-1)}^{ij\dots} F_{lm} - \frac{1}{2} t_{(N)}^{ij\dots} F_{km} - \dots \end{aligned} \quad (2.25)$$

So the electromagnetic field extends the covariant derivative. To pass on from General Relativity to our unified field theory we merely need to replace ordinary (“horizontal”) covariant derivatives with ϑ_n -covariant derivatives. Recall that the equation of geodesic motion in General Relativity is $u_{;j}^i u^j = 0$ where $u^i = dx^i/ds$ is of course the unit velocity vector. Generalizing by letting $u = (u^i, \epsilon)$ and setting $u_{;j}^i u^j \rightarrow u_{|j}^i u^j = 0$, we have $u_{|j}^i u^j = \left(u_{;j}^i - \frac{1}{2} \epsilon F_{\cdot j}^i \right) u^j = 0$.

Setting $\in = 2\left(\frac{e}{m_0 c^2}\right)$ where, again, e is the electric charge, m_0 is the point-mass and c is the speed of light in vacuum, we have the Lorentz equation of motion:

$$\frac{Du^i}{Ds} = \frac{e}{m_0 c^2} F_{;j}^i u^j, \quad \frac{Du^i}{Ds} \equiv u_{;j}^i u^j.$$

The fifth component of the momentum is consequently given by

$$p_5 = 2 \left(\frac{e}{c^2} \right). \quad (2.26)$$

We will now show that the quantity $\in$ is indeed constant along the world-line. First we write $u = u^i g_i + \in n$, then, as before, we have

$$\nabla_j u = \left(u_{;j}^i - \frac{1}{2} \in F_{;j}^i \right) g_i + \left(\in_{;j} + \frac{1}{2} F_{ij} u^i \right) n.$$

Applying the law of parallel transport in $\vartheta_n = \mathbb{R}_4 \otimes n$, i.e., $\nabla_{4u} u = 0$ (in the direction of 4u) where 4u represents the ordinary tangent four-velocity field, we get two equations of motion:

$$\left(u_{;j}^i - \frac{1}{2} \in F_{;j}^i \right) u^j = 0, \quad (2.27a)$$

$$\left(\in_{;j} + \frac{1}{2} F_{ij} u^i \right) u^j = 0. \quad (2.27b)$$

The first of these is just the usual Lorentz equation in a general coordinate system, the one we've just obtained before using the straightforward notion of "vertical-horizontal" ϑ_n -covariant derivative. Meanwhile, the second reads, due to the vanishing of its second term: $\frac{d\in}{ds} = 0$, which establishes the constancy of $\in$ with respect to the world-line. Therefore it is justified that $\in$ forms a fundamental constant of Nature in the sense of a correct parameterization. We also have $\frac{de}{ds} = \frac{dm_0}{ds} = 0$. Again, there's a certain possibility for the electric charge and the mass, to vary with time, perhaps slowly in reality. We shall now consider the unit spin vector field in the spin space $\mathbb{S}_p$:

$${}^4v \equiv u^i \omega_i = g_{[ik]} u^i g^k, \quad (2.28)$$

which has been defined in Section 1.2. This spin (rotation) vector is analogous to the ordinary velocity vector in the spin space representation. For the moment, let $v = (v^i, \alpha)$; $v^i = g^{[ik]} u_k$ where $v = v^i \omega_i + \alpha n$.

Therefore we see that

$$\begin{aligned}
\nabla_j v &= u_{;j}^i g_i + u^i g_{i;j} + \alpha_{;j} n + \alpha n_{;j} = \\
&= v_{;j}^i \omega_i + v^i \omega_{i;j} + \alpha_{;j} n + \alpha n_{;j} = \\
&= \left(v_{;j}^i - \frac{1}{2} \alpha g^{[ik]} F_{kj} \right) \omega_i + \left(\alpha_{;j} + \frac{1}{2} g_{[kl]} F_{;j}^l v^k \right) n,
\end{aligned} \tag{2.29}$$

with the help of (1.7). If the law of parallel transport $\nabla_u u = 0$ applies for the velocity field u , it is intuitive that in the same manner it must also apply to the spin field v :

$$\nabla_u v = 0. \tag{2.30}$$

This states that spin is geometrically conserved. We then get

$$\left(v_{;j}^i - \frac{1}{2} \alpha g^{[ik]} F_{kj} \right) u^j = 0, \tag{2.31a}$$

$$\left(\alpha_{;j} + \frac{1}{2} g_{[kl]} F_{;j}^l v^k \right) u^j = 0, \tag{2.31b}$$

which are completely equivalent to the equations of motion in (2.27a) and (2.27b).

Let's also observe that

$$G_{|k}^{ik} = G_{;k}^{ik}, \tag{2.32a}$$

$$F_{ij|k} + F_{jk|i} + F_{ki|j} = F_{ij;k} + F_{jk;i} + F_{ki;j}, \tag{2.32b}$$

i.e., the “vertical-horizontal” ϑ_n -covariant derivative operator when applied to the Einstein tensor and the electromagnetic field tensor equals the ordinary covariant derivative operator. We shall be able to prove this statement. First $G_{;j}^{ik} \rightarrow G_{|j}^{ik} = G_{;j}^{ik} - \frac{1}{2} X^i F_{;j}^k - \frac{1}{2} Y^k F_{;j}^i$ where $X^i = G^{i5} = Y^i$ (due to the symmetry of the tensor G^{ik} , $G^{5i} = R^{i5}$), so that $G_{|k}^{ik} = G_{;k}^{ik} - \frac{1}{2} Y^k F_{;k}^i$. Now the five-dimensional curvature tensors (the Riemann and Ricci tensors) in ϑ_n are

$$R_{BCD}^A = \Gamma_{BD,C}^A - \Gamma_{BC,D}^A + \Gamma_{BD}^E \Gamma_{EC}^A - \Gamma_{BC}^E \Gamma_{ED}^A,$$

$$R_{AB} = \Gamma_{AB,C}^C - \Gamma_{AC,B}^C + \Gamma_{AB}^E \Gamma_{EC}^C - \Gamma_{AC}^E \Gamma_{EB}^C.$$

In this special coordinate system we have

$$\Gamma_{5i}^k = g^k \cdot (\nabla_i n) = -\frac{1}{2} g^k \cdot F_{;i}^j g_j = -\frac{1}{2} F_{;i}^k,$$

$$\begin{aligned}\Gamma_{5i}^5 &= n \cdot (\nabla_i n) = -\frac{1}{2} n \cdot F_{\cdot i}^k g_k = 0 = \Gamma_{i5}^5 = \Gamma_{55}^k, \\ \Gamma_{5k}^k &= g^k \cdot (\nabla_k n) = -\frac{1}{2} g^k \cdot F_{\cdot k}^j g_j = -\frac{1}{2} F_{\cdot j}^j = 0 = \Gamma_{k5}^k,\end{aligned}$$

$$\begin{aligned}R_{5i} &= \Gamma_{5i,k}^k - \Gamma_{5k,i}^k + \Gamma_{5i}^k \Gamma_{kl}^l - \Gamma_{5k}^l \Gamma_{li}^k = \\ &= -\frac{1}{2} (F_{\cdot i,k}^k + \Gamma_{lk}^k F_{\cdot i}^l - \Gamma_{li}^k F_{\cdot k}^l) = \\ &= -\frac{1}{2} (F_{\cdot i;k}^k + 2\Gamma_{[ik]}^l F_{\cdot l}^k) = \\ &= -\frac{1}{2} (-J_i + J_i) = 0\end{aligned}$$

with the help of (2.11d). Hence $G_{|k}^{ik} = G_{;k}^{ik}$. (2.32b) can also be easily proven this way.

As a brief digression, consider a six-dimensional manifold $\mathbb{V}_6 = \mathcal{V}_n \otimes m$ where m is the second normal coordinate with respect to $\mathbb{R}_4$. Let $\bar{\lambda}^i \equiv g^{[5i]}$, $a \equiv \bar{\lambda}^5$, $b \equiv \bar{\lambda}^6$, $\Delta_{\cdot k}^i \equiv \left\{ \begin{smallmatrix} i \\ 5k \end{smallmatrix} \right\}$, $\Theta_{\cdot k}^i \equiv \left\{ \begin{smallmatrix} i \\ 6k \end{smallmatrix} \right\}$, $\theta = g_{[56]}$, and $\omega_i \equiv g_{[6i]}$. Casting (1.12) and (1.22) into six dimensions, the electromagnetic field tensor can be written in terms of the fundamental spin tensor as the following equivalent expressions:

$$\begin{aligned}F_{ik} &= \frac{5}{4} (\bar{\lambda}^r (g_{[ir],k} - g_{[kr],i}) - \theta (\omega_{i,k} - \omega_{k,i})) = \\ &= \frac{5}{4} (\bar{\lambda}^r g_{[ik],r} - \theta (\omega_{i,k} - \omega_{k,i})),\end{aligned}\tag{2.33a}$$

$$\begin{aligned}F_{ik} &= -\bar{\lambda}^l (g_{[li],k} - g_{[ik],l} + g_{[kl],i}) - a (\bar{\lambda}_{i,k} - \bar{\lambda}_{k,i}) - b (\omega_{i,k} - \omega_{k,i}) - \\ &- 2\bar{\lambda}^l \left(g_{[im]} \left\{ \begin{smallmatrix} m \\ lk \end{smallmatrix} \right\} - g_{[km]} \left\{ \begin{smallmatrix} m \\ li \end{smallmatrix} \right\} \right) - \\ &- 2a (g_{[im]} \Delta_{\cdot k}^m - g_{[km]} \Delta_{\cdot i}^m) - 2b (g_{[im]} \Theta_{\cdot k}^m - g_{[km]} \Theta_{\cdot i}^m).\end{aligned}\tag{2.33b}$$

Since the basis in this space is given by $g^\mu = (g^i, n, m)$, the fundamental tensors are

$$g_{(\mu\nu)} = \begin{pmatrix} [g_{(ik)}]_{4 \times 4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\tag{2.34a}$$

$$g_{[\mu\nu]} = \begin{pmatrix} [g_{[ik]}]_{4 \times 4} & -\bar{\lambda}_i & -\theta \\ \bar{\lambda}_i & 0 & -\omega_i \\ \theta & \omega_i & 0 \end{pmatrix}. \quad (2.34b)$$

Remark 3 (on the modified Maxwell's equations)

Using (2.25) we can now generalize Maxwell's field equations through the new extended electromagnetic tensor $\bar{F}_{ik} = \varphi_{i|k} - \varphi_{k|i}$ where

$$\bar{F}_{ik} = \varphi_{i;k} - \varphi_{k;i} = \varphi_{i,k} - \varphi_{k,i} + 2\Gamma_{[ik]}^l \varphi_l = F_{ik} + 2\Gamma_{[ik]}^l \varphi_l, \quad (2.35)$$

where $\varphi_{i|k} = \varphi_{i;k} - \frac{1}{2}\phi F_{ik}$. Now $\varphi_5 = \phi$ is taken to be an extra scalar potential. Therefore $F_{ik} = \varphi_{i,k} - \varphi_{k,i} - F_{ik}\phi$ or

$$F_{ik} = \frac{1}{1+\phi} (\varphi_{i,k} - \varphi_{k,i}) \equiv \gamma (\varphi_{i,k} - \varphi_{k,i}). \quad (2.a)$$

For instance, the first pair of Maxwell's equations can therefore be generalized into

$$\vec{E} = \frac{1}{1+\phi} \left(-\frac{1}{c} \frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \varphi \right), \quad (2.b_1)$$

$$\vec{B} = \frac{1}{1+\phi} \vec{\nabla} x \vec{A}, \quad (2.b_2)$$

$$\text{div } \vec{B} = -\frac{1}{(1+\phi)^2} (\vec{\nabla} \phi \cdot \vec{\nabla} x \vec{A}) + \frac{1}{1+\phi} \vec{\nabla} \cdot \vec{\nabla} x \vec{A}, \quad (2.b_3)$$

$$\text{curl} \left(\frac{1}{1+\phi} \vec{E} \right) = -\frac{1}{c} \frac{\partial}{\partial t} \frac{1}{1+\phi} \vec{B} - \frac{1}{c} \frac{\partial \vec{B}}{\partial t} \frac{1}{1+\phi}, \quad (2.b_4)$$

where $\vec{A}$ is the three-dimensional electromagnetic vector potential: $\vec{A} = (A_a)$, φ is the electromagnetic scalar potential, $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field and $\vec{\nabla}$ is the three-dimensional (curvilinear) gradient operator and $\nabla^2 \varphi = -4\pi\rho$. Here we define the electric and magnetic fields in such a way that

$$F^{4a} = (1+\phi)^{-1} E^a, \quad F^{12} = (1+\phi)^{-1} B_3,$$

$$F^{31} = (1+\phi)^{-1} B_2, \quad F^{23} = (1+\phi)^{-1} B_1.$$

We also note that in (2.b₃) the divergence $\vec{\nabla} \cdot \vec{\nabla} x \vec{A}$ is in general non-vanishing when torsion is present in the three-dimensional curved

sub-space. Direct calculation gives

$$\begin{aligned}\operatorname{div} \operatorname{curl} \vec{A} &= -\frac{1}{2} \epsilon^{abc} \left(R^d_{\cdot cab} A_d - 2\Gamma^d_{[ab]} A_{c;d} \right), \\ (\operatorname{curl} \operatorname{grad} \varphi)_c &= \epsilon^{ab}_{\cdot c} \Gamma^d_{[ab]} \varphi_{,d} .\end{aligned}$$

So the magnetic charge μ with density ρ_m in the infinitesimal volume $d\Omega$ is given by

$$\delta\mu = -\frac{1}{2} \int_{\Omega} \epsilon^{abc} \left(R^d_{\cdot cab} A_d - 2\Gamma^d_{[ab]} A_{c;d} \right) d\Omega .$$

End of Remark 3

Chapter 3

SPIN-CURVATURE

§3.1 Dynamics in the microscopic limit

We now investigate the microscopic dynamics of our theory. Let's introduce an infinitesimal coordinate transformation into ϑ_n through the diffeomorphism

$$x'^i = x^i + \xi^i$$

with an external Killing-like vector $\xi = (\xi^i, \psi) : \xi^i = \xi^i(x), \psi = \psi(x)$ (not to be confused with the internal Killing vector which describes the internal symmetry of a particular configuration of space-time or which maps a particular space-time onto itself). The function ψ here shall play the role of the amplitude of the quantum mechanical state vector $|\psi\rangle$. Recall that $\mathbb{R}_4$ represents the four-dimensional physical world and n is a microscopic dimension. In its most standard form $\psi(x) \equiv C e^{-(2\pi i/\hbar)(Et - \mathbf{p} \cdot \mathbf{r}(x, y, z))}$ is the quantum mechanical scalar wave function; $\hbar$ is the Planck constant, E is energy and $\mathbf{p}$ is the three-momentum. Define the “extension” of the space-time $\vartheta_n = \mathbb{R}_4 \otimes n$ by

$$\tau_{ij} = \frac{1}{2} D_\xi g_{(ij)} . \quad (3.1)$$

We would like to express the most general symmetry, first, of the structure of ϑ_n and then find out what sort of symmetry (expressed in terms of the Killing-like vector) is required to describe the non-local “statics” or “non-deformability” of the structure of the metric tensor g (the lattice arrangement). Our exterior derivative is defined as the variation of an arbitrary quantity with respect to the external field ξ . Unlike the ordinary Killing vector which maps a space-time onto itself, the external field and hence also the derivative $D_\xi g_{(ij)}$ map $\mathbb{R}_4$ onto, say, $\mathbb{R}'_4$ which possesses a deformed metrical structure of $g \in \mathbb{R}_4, g'$.

We calculate the change in the metric tensor with respect to the external field, according to the scheme $g_{(ij)} = (g_i \cdot g_j) \rightarrow g'_{(ij)} = (g'_i \cdot g'_j)$, as follows:

$$D_\xi g_{(ij)} = (g_i \cdot D_\xi g_j) + (D_\xi g_i \cdot g_j); D_\xi g_i = g'_i - g_i = \nabla_i \xi .$$

The Lie derivative D_ξ denotes the exterior change with respect to the infinitesimal exterior field, i.e., it dynamically measures the deformation

of the geometry of the space-time $\mathbb{R}_4$.

Thus

$$D_\xi g_{(ij)} = (g_i \cdot \nabla_j \xi) + (\nabla_i \xi \cdot g_j) ,$$

where

$$\nabla_i \xi = \left(\xi_{;i}^k - \frac{1}{2} F_{\cdot i}^k \psi \right) g_k + \left(\psi_{,i} + \frac{1}{2} F_{ki} \xi^k \right) n .$$

By direct calculation, we thus obtain

$$D_\xi g_{(ij)} = \xi_{i;j} + \xi_{j;i} . \quad (3.2)$$

§3.2 Spin-curvature tensor of $\mathbb{S}_p$

As an interesting feature, we point out that the change $D_\xi g_{(ij)}$ in the structure of the four-dimensional metric tensor does not involve the wave function ψ . The space-time $\mathbb{R}_4$ will be called “static” if g does not change with respect to ξ . The four-dimensional (but not the five-dimensional) metric is therefore “static” whenever $\xi_{i;j} + \xi_{j;i} = 0$.

Now let $\overline{\mathbb{R}}_4$ be an infinitesimal copy of $\mathbb{R}_4$. To arrive at the lattice picture, let also $\mathbb{R}_4'', \mathbb{R}_4''', \mathbb{R}_4''', \dots$ be n such copies of $\overline{\mathbb{R}}_4$. Imagine the space $\mathbb{S}_n$ consisting of these copies. This space is therefore populated by $\overline{\mathbb{R}}_4$ and its copies. If we assume that each of the copies of $\overline{\mathbb{R}}_4$ has the same metric tensor as $\overline{\mathbb{R}}_4$, then we may have

$$\xi = (0, \psi) .$$

We shall call this particular “fundamental symmetry” *normal symmetry* or “spherical” *world-symmetry*. Then the n copies of $\overline{\mathbb{R}}_4$ exist simultaneously and each history is independent of the four-dimensional external field ${}^4\xi$ and is dependent on the wave function ψ only. In other words, the special lattice arrangement $\xi = (0, \psi)$ gives us a condition for $\overline{\mathbb{R}}_4$ and its copies to co-exist simultaneously independently of how the four-dimensional external field deforms their interior metrical structure. Thus the many sub-manifolds $n\overline{\mathbb{R}}_4$ represent many simultaneous realities which we call world-pictures. More specifically, at one point in $\mathbb{R}_4$, there may at least exist two world-pictures. In other words, a point seen by an observer confined to lie in $\mathbb{R}_4$ may actually be a line or curve whose two end points represent the two solutions to the wave function ψ . The splitting of $\overline{\mathbb{R}}_4$ into its copies occurs and can only be perceived on the microscopic scales with the wave function ψ describing the entire process. Conversely, on the macroscopic scales the inhabitants of $\vartheta_n = \mathbb{R}_4 \otimes n$ may perceive the collection of the $n\overline{\mathbb{R}}_4$ (sub-spaces of $\mathbb{R}_4$)

representing a continuous four manifold $\mathbb{R}_4 = \overline{\mathbb{R}}_4 \oplus \overline{\mathbb{R}}_4' \oplus \overline{\mathbb{R}}_4'' \oplus \dots$. There may be an infinite number of $n\mathbb{R}_4$ composing the space-time $\mathbb{R}_4$, possessing the same fluctuating metric tensor g , i.e., that of $\mathbb{R}_4$ (the total space). On the microscopic scales, fluctuations in the metric do occur. In special cases, the cylinder condition may prevent the topology of the spaces $n\mathbb{R}_4$ to change which cannot be perceived directly by an external observer in $\mathbb{R}_4$ for it takes place along the microscopic coordinate y . The fluctuations induce the many different world-pictures. Our task now is to find the wave equation describing the entire process. Now, with the help of (1.7) and (1.8) we can write

$$\begin{aligned}
\nabla_i \xi &= (\nabla_i \xi \cdot g_j) g^j + (\nabla_i \xi \cdot n) n = \\
&= \frac{1}{2} ((\nabla_i \xi \cdot g_j) + (\nabla_j \xi \cdot g_i)) g^j + \\
&+ \frac{1}{2} ((\nabla_i \xi \cdot g_j) - (\nabla_j \xi \cdot g_i)) g^j + (\nabla_i \xi \cdot n) n = \\
&= \frac{1}{2} D_\xi g_{(ij)} g^j - [(\nabla_j \xi \cdot n) \omega^j, g_i] + \\
&+ g^{[jk]} (\nabla_j \xi \cdot n) C_{ki}^l g_l + \frac{3}{2} g^{[kl]} (\nabla_k \xi \cdot g_l) g_{[ij]} g^j - \\
&- \frac{3}{2} \gamma_{ij}^{kl} (\nabla_k \xi \cdot g_l) g^j = \\
&= \frac{1}{2} D_\xi g_{(ij)} g^j - \left[\frac{3}{2} g^{[kl]} (\nabla_k \xi \cdot g_l) n - (\nabla_j \xi \cdot n) \omega^j \right], g_i + \\
&+ g^{[jk]} (\nabla_j \xi \cdot n) C_{ki}^l g_l - \frac{3}{2} \gamma_{ij}^{kl} (\nabla_k \xi \cdot g_l) g^j.
\end{aligned} \tag{3.3}$$

Setting

$$\begin{aligned}
S &= \frac{3}{2} g^{[kl]} (\nabla_k \xi \cdot g_l) n - (\nabla_j \xi \cdot n) \omega^j, \\
X_i &= g^{[jk]} (\nabla_j \xi \cdot n) C_{ki}^l g_l - \frac{3}{2} \gamma_{ij}^{kl} (\nabla_k \xi \cdot g_l) g^j,
\end{aligned}$$

we have

$$\nabla_i \xi = \frac{1}{2} D_\xi g_{(ij)} g^j + [S, g_i] + X_i. \tag{3.4}$$

To see that the vector S represents a spin (rotation) vector is not difficult as we know that $(g_i \cdot n) = 0$ and hence $D(g_i \cdot n) = 0$ where the change in n can be represented by an internal rotation: $Dn = [S, n]$. With the help of (1.8), we now have

$$(g_i \cdot Dn) = (g_i \cdot [S, n]) = (S \cdot [n, g_i]) = (S \cdot \omega_i) = S_i.$$

Therefore

$$S_i = (S \cdot \omega_i) = -(\nabla_i \xi \cdot n) = -(n \cdot \nabla_i \xi) .$$

Now

$$\begin{aligned} S &= S_i \omega^i + \frac{3}{2} g^{[ij]} (\nabla_i \xi \cdot g_j) n = \\ &= S_i \omega^i + \bar{\phi} n , \end{aligned} \quad (3.5)$$

$$S_i = -(n \cdot \nabla_i \xi) = -\psi_{,i} - \frac{1}{2} F_{\cdot i}^k \xi_k , \quad (3.6a)$$

$$\begin{aligned} \bar{\phi} &= \frac{3}{2} g^{[ij]} (\nabla_i \xi \cdot g_j) = \\ &= \frac{3}{2} g^{[ij]} \left(\xi_{j;i} - \frac{1}{2} \psi F_{ji} \right) . \end{aligned} \quad (3.6b)$$

We can also calculate the exterior variation of the electromagnetic field tensor F_{ik} :

$$\begin{aligned} D_\xi F_{ik} &= 2D_\xi (\nabla_i n \cdot g_k) = \\ &= 2(\nabla_i D_\xi n \cdot g_k) + 2(\nabla_i n \cdot D_\xi g_k) = \\ &= 2(\nabla_i [S, n] \cdot g_k) + 2(\nabla_i n \cdot D_\xi g_k) = \\ &= 2(\nabla_i S \cdot \omega_k) - 2\left([S, \frac{1}{2} F_{\cdot i}^j g_j] \cdot g_k\right) - \left(F_{\cdot i}^j g_j \cdot \nabla_k \xi\right) . \end{aligned} \quad (3.7a)$$

Hence we can write

$$\begin{aligned} D_\xi F_{ik} &= 2(\nabla_i S \cdot \omega_k) - \left(F_{\cdot i}^j g_j \cdot (\nabla_k \xi - [S, g_k])\right) = \\ &= 2(\nabla_i S \cdot \omega_k) - F_{\cdot i}^j \left(g_j \cdot \frac{1}{2} D_\xi g_{(kl)} g^l\right) - F_{\cdot i}^j (g_j \cdot X_k) = \\ &= -\left(2H_{ik} + \frac{1}{2} F_{\cdot i}^j D_\xi g_{(jk)} + F_{\cdot i}^j X_{jk}\right) , \end{aligned} \quad (3.7b)$$

where we have just defined the spin-curvature tensor H_{ik} :

$$H_{ik} = -(\nabla_i S \cdot \omega_k) , \quad (3.8)$$

which measures the internal change of the spin in the direction of the spin basis. We further posit that the spin-curvature tensor satisfies the supplementary identities (which are deduced from the conditions $\nabla_{4_u} u = 0$ and $\nabla_{4_u} S = 0$)

$$H_{ik} u^i = 0 , \quad (3.9a)$$

$$\text{tr } H = 0 . \quad (3.9b)$$

The transverse condition (3.9a) reproduces the Lorentz equation of motion while (3.9b) describes the internal properties of the structure of physical fields which corresponds to the quantum limit on our manifold (for details see Section 4.2).

The explicit expression of H_{ik} can be found to be

$$\begin{aligned} H_{ik} &= -(\nabla_i S \cdot \omega_k) = \\ &= \frac{1}{2} g_{[kl]} F_{\cdot i}^l \bar{\phi} - S_{k;i} \end{aligned} \quad (3.10)$$

from (3.6a) and (3.6b). Furthermore, still with the help of (3.6a) and (3.6b), we obtain, after some simplifications,

$$H_{ik} = \psi_{;k;i} + \frac{1}{4} F_{\cdot i}^l (\xi_{l;k} - \xi_{k;l}) - \frac{1}{4} F_{\cdot i}^l F_{lk} \psi + \frac{1}{2} (F_{\cdot k}^l \xi_l)_{;i} . \quad (3.11)$$

However, we recall that

$$R_{ijkl} = \frac{1}{4} (F_{ik} F_{jl} - F_{il} F_{jk}), \quad R_{ik} = -\frac{1}{4} F_{il} F_{\cdot k}^l$$

and therefore we obtain the spin-curvature relation in the form

$$H_{ik} = \psi_{;k;i} - R_{ik} \psi + \frac{1}{4} F_{\cdot i}^l (\xi_{l;k} - \xi_{k;l}) + \frac{1}{2} (F_{\cdot k}^l \xi_l)_{;i} . \quad (3.12)$$

§3.3 Wave equation describing the geometry of $\mathbb{R}_4$

If we contract (3.12) with respect to the indices i and k , we have

$$H = (\square - R)\psi + F^{ik} \xi_{;k} + \frac{1}{2} J^i \xi_i , \quad (3.13)$$

where $\square$ is the covariant four-dimensional Laplacian, again, R is the curvature scalar and J^i is the current density vector. However, using (3.9a) and (3.9b) and associating with the space $\vartheta_n = \mathbb{R}_4 \otimes n$ the “fundamental world-symmetry” $\xi = (0, \psi)$, then we obtain, from (3.12), the equation of motion:

$$\psi_{;k;i} = H_{ik} + R_{ik} \psi . \quad (3.14)$$

From (3.9a), (3.9b) and (3.14), we obtain the wave equation:

$$(\square - R)\psi = 0 . \quad (3.15a)$$

This resembles the scalar Klein-Gordon wave equation except that we have the curvature scalar R in place of $M^2 = (m_0 c/\hbar)^2$ (we normally expect this in generalizing the scalar Klein-Gordon equation). Note also that the ordinary Klein-Gordon and Dirac equations do not explicitly

contain any electromagnetic terms. This means that the electromagnetic field must somehow already be incorporated into gravity in terms of M . Since ψ is just the amplitude of the state vector $|\psi\rangle$, we can also write

$$(\square - R)|\psi\rangle = 0. \quad (3.15b)$$

If the curvature scalar vanishes, there is no “source” (or actually, no electromagnetic field strength) and we have $\square|\psi\rangle = 0$ which is the wave equation of massless particles.

Remark 4

Recall (3.7):

$$D_\xi F_{ik} = - \left(2H_{ik} + \frac{1}{2} F_{.i}^j D_\xi g_{(jk)} + F_{.i}^j X_{jk} \right), \quad (3.a)$$

where

$$D_\xi g_{(ik)} = \xi_{i;k} + \xi_{k;i} = 2\tau_{ik},$$

$$H_{ik} = -(\nabla_i S \cdot \omega_k).$$

Meanwhile, for an arbitrary tensor field T , we have in general

$$\begin{aligned} D_\xi T_{kl...}^{ij...} &= \xi^m T_{kl...;m}^{ij...} + T_{ml...}^{ij...} \xi_{.k}^m + T_{km...}^{ij...} \xi_{.l}^m + \dots - \\ &- T_{kl...}^{mj...} \xi_{.m}^i - T_{kl...}^{im...} \xi_{.m}^j - \dots \end{aligned}$$

Therefore

$$D_\xi F_{ik} = \xi^l F_{ik;l} + F_{lk} \xi_{.i}^l + F_{il} \xi_{.k}^l. \quad (3.b)$$

Comparing this with (3.a), we have, for the spin-curvature tensor H_{ik} ,

$$H_{ik} = -\frac{1}{2} (\xi^l F_{ik;l} + F_{lk} \xi_{.i}^l + F_{il} \xi_{.k}^l) - \frac{1}{4} F_{.i}^l (\xi_{l;k} + \xi_{k;l}) - \frac{1}{2} F_{.i}^l X_{lk} \quad (3.c)$$

in terms of the electromagnetic field tensor.

End of Remark 4

Finally, let's define the following tensor:

$$A_{ij} \equiv S_{i;j} - \frac{1}{2} F_{.j}^k \xi_{k|i}, \quad (3.16)$$

where, as before,

$$\xi_{k|i} = \xi_{k;i} - \frac{1}{2} \psi F_{ki}$$

is the ϑ_n -covariant derivative of ξ_k , the notion of which we have developed in Section 2.1 of this work, and

$$S_i = -\psi_{,i} - \frac{1}{2} F_{.i}^k \xi_k$$

is the spin vector (3.6a). The meaning of the tensor (3.16) will become clear soon. It has no classical analogue. We are now in a position to decompose (3.16) into its symmetric and alternating parts. The symmetric part of (3.16):

$$A_{(ij)} = \frac{1}{2} \left(S_{i;j} + S_{j;i} - \frac{1}{2} F_{\cdot i}^k \xi_{k|j} - \frac{1}{2} F_{\cdot j}^k \xi_{k|i} \right) \quad (3.17a)$$

may be interpreted as the tension of the spin field.

Now its alternating part:

$$A_{[ij]} = \frac{1}{2} \left(S_{i;j} - S_{j;i} + \frac{1}{2} F_{\cdot i}^k \xi_{k|j} - \frac{1}{2} F_{\cdot j}^k \xi_{k|i} \right) \quad (3.17b)$$

represents a non-linear spin field. (However, this becomes linear when we invoke the “fundamental world-symmetry” $\xi = (0, \psi)$.) If we employ this “fundamental world- symmetry”, (3.17a) and (3.17b) become

$$A_{(ij)} = \frac{1}{2} (S_{i;j} + S_{j;i}) + R_{ij} , \quad (3.18a)$$

$$A_{[ij]} = \frac{1}{2} (S_{i;j} - S_{j;i}) . \quad (3.18b)$$

With the help of (3.6a) and the relation

$$\psi_{;i;k} - \psi_{;k;i} = -2\Gamma_{[ik]}^r \psi_{;r} ,$$

(3.18b) can also be written

$$A_{[ij]} = \Gamma_{[ij]}^k \psi_{;k} . \quad (3.19)$$

Now

$$\begin{aligned} A_{ij} &= -\psi_{;i;j} - \frac{1}{2} (F_{\cdot i}^k \xi_k)_{;j} - \frac{1}{2} F_{\cdot j}^k \xi_{k;i} + \frac{1}{4} F_{\cdot j}^k F_{ki} \psi = \\ &= - \left(\psi_{;i;j} - R_{ij} \psi + \frac{1}{2} (F_{\cdot i}^k \xi_k)_{;j} + \frac{1}{2} F_{\cdot j}^k \xi_{k;i} \right) . \end{aligned}$$

Therefore the Ricci tensor can be expressed as

$$\begin{aligned} R_{ik} &= -\frac{1}{2} \psi^{-1} F_{\cdot k}^r \xi_{r|i} - \\ &\quad - \psi^{-1} \left(\psi_{;i;k} + S_{i;k} + \frac{1}{2} (F_{\cdot i}^r \xi_r)_{;k} - \frac{1}{2} F_{\cdot k}^r \xi_{r;i} \right) . \end{aligned} \quad (3.20)$$

We can still obtain another form of the wave equation of our quantum gravity theory. Taking the world-symmetry $\xi = (0, \psi)$, we have, from $r = r' - \psi n$,

$$g_i = h_i - \psi_{,i} n + \frac{1}{2} \psi F_{.i}^r g_r, \quad (3.21)$$

where $h_i \equiv r'_{,i}$ is the basis of the space-time $\mathbb{R}'_4$. Now the metric tensor of the space-time $\mathbb{R}_4$ is

$$\begin{aligned} g_{(ik)} &= (g_i \cdot g_k) = \\ &= h_{ik} - \psi_{,k} \eta_i + \frac{1}{2} \psi F_{.k}^r \eta_{ir} - \psi_{,i} \eta_k + \psi_{,i} \psi_{,k} + \\ &+ \frac{1}{2} \psi F_{.i}^r \eta_{kr} + \frac{1}{4} \psi^2 g_{(rs)} F_{.i}^r F_{.k}^s, \end{aligned}$$

where $h_{ik} \equiv (h_i \cdot h_k)$ is the metric tensor of $\mathbb{R}'_4$, $\eta_i \equiv (h_i \cdot n)$ and $\eta_{ik} \equiv (h_i \cdot g_k)$. Direct calculation shows that

$$\begin{aligned} \eta_i &= \psi_{,i}, \\ \eta_{ik} &= g_{(ik)} + \frac{1}{2} \psi F_{ik}. \end{aligned}$$

Then we arrive at the relation

$$g_{(ik)} = h_{ik} - \psi_{,i} \psi_{,k} - \frac{1}{4} \psi^2 g_{(rs)} F_{.i}^r F_{.k}^s. \quad (3.22)$$

Now from (3.2) we find that this is subject to the condition

$$D_\xi g_{(ij)} = 0. \quad (3.23)$$

Hence we obtain the wave equation

$$\psi_{,i} \psi_{,k} = -\frac{1}{4} \psi^2 g_{(rs)} F_{.i}^r F_{.k}^s. \quad (3.24a)$$

Expressed in terms of the Ricci tensor, the equivalent form of (3.24a) is

$$\psi_{,i} \psi_{,k} = -\psi^2 R_{ik}. \quad (3.24b)$$

Expressed in terms of the Einstein tensor $G_{ik} = R_{ik} - \frac{1}{2} g_{(ik)} R$, (3.24b) becomes

$$\left(\delta_i^r \delta_k^s - \frac{1}{2} g_{(ik)} g^{(rs)} \right) \psi_{,r} \psi_{,s} = -\psi^2 G_{ik}. \quad (3.25)$$

If in particular the space-time $\mathbb{R}_4$ has a constant sectional curvature, then $R_{ijkl} = \frac{1}{12} (g_{(ik)}g_{(jl)} - g_{(il)}g_{(jk)})$ and $R_{ik} = \lambda g_{(ik)}$, where $\lambda \equiv \frac{1}{4} R$ is constant, so (3.24b) reduces to

$$\psi_{,i}\psi_{,k} = -\lambda\psi^2 g_{(ik)}. \quad (3.26)$$

Any axisymmetric solution of (3.26) would then yield equations that could readily be integrated, giving the wave function in a relatively simple form. Multiplying now (3.24b) by the contravariant metric tensor $g^{(ik)}$, we have the wave equation in terms of the curvature scalar as follows:

$$g^{(ik)}\psi_{,i}\psi_{,k} = -\psi^2 R. \quad (3.27)$$

Finally, let's consider a special case. In the absence of the scalar source, i.e., in "void", the wave equation becomes

$$g^{(ik)}\psi_{,i}\psi_{,k} = 0. \quad (3.28)$$

This wave equation therefore describes a massless, null electromagnetic field where

$$F_{ik}F^{ik} = 0.$$

In this case the electromagnetic field tensor is a null bivector. Therefore, according to our theory, there are indeed "seemingly void" regions in the Universe that are governed by null electromagnetic fields only.

Chapter 4

ADDITIONAL CONSIDERATIONS

§4.1 Embedding of generalized Riemannian manifolds (with twist) in $N = n + p$ dimensions

In connection to Chapter 2 of this work where we considered an embedding of “class 1”, we outline the most general formulation of embedding theory of “class p” in $N = n + p$ dimensions where n now is the number of dimensions of the embedded Riemannian manifold. First, let the embedding space $\mathbb{R}_N$ be an N -dimensional Riemannian manifold spanned by the basis $\{e_A\}$. For the sake of generality we take $\mathbb{R}_N$ to be an N -dimensional space-time. Let also $\mathbb{R}_n$ be an n -dimensional Riemannian sub-manifold (possessing torsion) in $\mathbb{R}_N$ spanned by the basis $\{g_i\}$ where now the capital Latin indices $A, B, \dots$ run from 1 to N and the ordinary ones $i, j, \dots$ from 1 to n . If now $g_{AB} = (e_A \cdot e_B)$ and $g_{ij} = (g_i \cdot g_j)$ denote the metric tensors of $\mathbb{R}_N$ and $\mathbb{R}_n$, respectively, and if we introduce the p -unit normal vectors (also called n -legs) $n^{(\alpha)}$ (where the Greek indices run from 1 to p and summation over any repeated Greek indices is explicitly indicated otherwise there is no summation), then

$$\begin{aligned}
 g_{ij} &= e_i^A e_j^B g_{AB}, \\
 g_{AB} &= e_A^i e_B^j g_{ij} + \sum_{\mu} n_A^{(\mu)} n_B^{(\mu)}, \\
 e_i^A n_A^{(\mu)} &= 0, \\
 (n^{(\mu)} \cdot n^{(\nu)}) &= \gamma^{(\mu)} \delta^{\mu\nu}, \quad \gamma^{(\mu)} = \pm 1, \\
 g_{i,j} &= \Gamma_{ij}^k g_k + \sum_{\mu} \gamma^{(\mu)} \phi_{ij}^{(\mu)} n^{(\mu)}, \\
 g_{i;j} &= \sum_{\mu} \gamma^{(\mu)} \phi_{ij}^{(\mu)} n^{(\mu)}, \\
 e_{A,B} &= \Gamma_{AB}^C e_C, \\
 \Gamma_{ij}^k &= e_A^k e_{i,j}^A + e_A^k \Gamma_{BC}^A e_i^B e_j^C,
 \end{aligned}$$

$$\begin{aligned}
\Gamma_{BC}^A &= e_i^A e_{B,C}^i + e_k^A \Gamma_{ij}^k e_B^i e_C^j + \sum_{\mu} \gamma^{(\mu)} \phi_{ij}^{(\mu)} e_B^i e_C^j n^{(\mu)A} + \\
&+ \sum_{\mu} n^{(\mu)A} n_{B,C}^{(\mu)} - \sum_{\mu} \gamma^{(\mu)} \phi_{kj}^{(\mu)} g^{ki} e_i^A e_C^j n_B^{(\mu)}, \\
\phi_{ij}^{(\mu)} &= n_A^{(\mu)} e_{i,j}^A + n_A^{(\mu)} \Gamma_{BC}^A e_i^B e_j^C, \\
e_{i,j}^A &= \sum_{\mu} \gamma^{(\mu)} \phi_{ij}^{(\mu)} n^{(\mu)A}, \\
e_{i,j}^A &= e_k^A \Gamma_{ij}^k - \Gamma_{BC}^A e_i^B e_j^C + \sum_{\mu} \gamma^{(\mu)} \phi_{ij}^{(\mu)} n^{(\mu)A}.
\end{aligned}$$

Now since $e_A = e_A^i g_i + \sum_{\mu} \gamma^{(\mu)} n_A^{(\mu)} n^{(\mu)}$ and $\phi_{ij}^{(\mu)} = -e_i^A n_{A;j}^{(\mu)}$ for the asymmetric p -extrinsic curvatures, we see that

$$\begin{aligned}
e_B^i \phi_{ij}^{(\mu)} &= -\left(\delta_B^A - \sum_{\nu} \gamma^{(\nu)} n^{(\nu)A} n_B^{(\nu)} \right) n_{A;j}^{(\mu)} = \\
&= -n_{B;j}^{(\mu)} + \sum_{\nu} \gamma^{(\nu)} n_{A;j}^{(\mu)} n^{(\nu)A} n_B^{(\nu)}.
\end{aligned}$$

Let's define the p -torsion vectors by

$$\begin{aligned}
\theta_i^{\mu\nu} &= n_{A;i}^{(\mu)} n^{(\nu)A}, \\
\theta_i^{\mu\nu} &= -\theta_i^{\nu\mu}.
\end{aligned}$$

Hence

$$n_{A;i}^{(\mu)} = -\phi_{ki}^{(\mu)} e_A^k + \sum_{\nu} \gamma^{(\nu)} \theta_i^{\mu\nu} n_A^{(\nu)}$$

or

$$n_{;i}^{(\mu)} = -\phi_{ki}^{(\mu)} g^k + \sum_{\nu} \gamma^{(\nu)} \theta_i^{\mu\nu} n^{(\nu)}.$$

Now

$$\begin{aligned}
n_{A;i;k}^{(\mu)} &= -\phi_{ik}^{(\mu)} e_A^i - \sum_{\nu} g^{rs} \phi_{ri}^{(\mu)} \phi_{sk}^{(\nu)} n_A^{(\nu)} + \sum_{\nu} \gamma^{(\nu)} \theta_{i;k}^{\mu\nu} n_A^{(\nu)} - \\
&- \sum_{\nu} \gamma^{\nu} \theta_i^{\mu\nu} \phi_{rk}^{(\nu)} e_A^r + \sum_{\nu, \tau} \gamma^{(\tau)} \theta_i^{\mu\nu} \theta_k^{\nu\tau} n_A^{(\tau)}.
\end{aligned}$$

Hence we obtain the expression

$$n_{A;i;k}^{(\mu)} n^{(\nu)A} = -g^{rs} \phi_{ri}^{(\mu)} \phi_{sk}^{(\nu)} + \theta_{i;k}^{\mu\nu} + \sum_{\tau} \gamma^{(\tau)} \theta_i^{\mu\tau} \theta_k^{\nu\tau}.$$

Meanwhile,

$$n_{A;i}^{(\mu)} = n_{A;B}^{(\mu)} e_i^B,$$

$$\begin{aligned} n_{A;i;k}^{(\mu)} &= \left(n_{A;B}^{(\mu)} e_i^B \right)_{;C} e_k^C = \\ &= n_{;B;C}^{(\mu)} e_i^B e_k^C + n_{A;B}^{(\mu)} e_{i;k}^A = \\ &= n_{A;B;C}^{(\mu)} e_i^B e_k^C + \sum_{\nu} n_{A;B}^{(\mu)} \gamma^{(\nu)} \phi_{ik}^{(\nu)} n^{(\nu)B}, \end{aligned}$$

but

$$n_{A;B}^{(\mu)} = n_{A;i}^{(\mu)} e_B^i = -\phi_{ki}^{(\mu)} e_A^k e_B^i.$$

Hence

$$n_{A;B}^{(\mu)} n^{(\nu)B} = 0,$$

$$n_{A;B}^{(\mu)} n^{(\nu)A} = 0.$$

We also see that

$$n_{A;i;k}^{(\mu)} n^{(\nu)A} = n_{A;B;C}^{(\mu)} n^{(\nu)A} e_i^B e_k^C.$$

Consequently, we have

$$\begin{aligned} \left(n_{A;i;k}^{(\mu)} - n_{A;k;i}^{(\mu)} \right) n^{(\nu)A} &= \left(n_{A;B;C}^{(\mu)} - n_{A;C;B}^{(\mu)} \right) n^{(\nu)A} e_i^B e_k^C = \\ &= R_{ABCD} n^{(\mu)A} n^{(\nu)B} e_i^C e_k^D. \end{aligned}$$

On the other hand, we see that

$$\begin{aligned} \left(n_{A;i;k}^{(\mu)} - n_{A;k;i}^{(\mu)} \right) n^{(\nu)A} &= \theta_{i;k}^{\mu\nu} - \theta_{k;i}^{\mu\nu} + g^{rs} \left(\phi_{rk}^{(\mu)} \phi_{si}^{(\nu)} - \phi_{ri}^{(\mu)} \phi_{sk}^{(\nu)} \right) + \\ &+ \sum_{\tau} \gamma^{(\tau)} \left(\theta_i^{\mu\tau} \theta_k^{\nu\tau} - \theta_k^{\mu\tau} \theta_i^{\nu\tau} \right) + \\ &+ R_{ABCD} n^{(\mu)A} n^{(\nu)B} e_i^C e_k^D. \end{aligned}$$

Combining the last two equations we get the Ricci equations:

$$\theta_{i;k}^{\mu\nu} - \theta_{k;i}^{\mu\nu} = g^{rs} \left(\phi_{ri}^{(\mu)} \phi_{sk}^{(\nu)} - \phi_{rk}^{(\mu)} \phi_{si}^{(\nu)} \right) - \sum_{\tau} \gamma^{(\tau)} \left(\theta_i^{\mu\tau} \theta_k^{\nu\tau} - \theta_k^{\mu\tau} \theta_i^{\nu\tau} \right).$$

Now from the relation

$$g_{i,j} = \Gamma_{ij}^k g_k + \sum_{\mu} \gamma^{(\mu)} \phi_{ij}^{(\mu)} n^{(\mu)},$$

we obtain the expression

$$g_{i,jk} = \left(\Gamma_{ij,k}^r + \Gamma_{ij}^s \Gamma_{sk}^r - \sum_{\mu} g^{rs} \gamma^{(\mu)} \phi_{ij}^{(\mu)} \phi_{sk}^{(\mu)} \right) g_r + \\ + \sum_{\mu} \left(\gamma^{(\mu)} \phi_{rk}^{(\mu)} \Gamma_{ij}^r + \gamma^{(\mu)} \phi_{ij,k}^{(\mu)} + \sum_{\nu} \gamma^{(\nu)} \phi_{ij}^{(\nu)} \theta_k^{\nu\mu} \right) n^{(\mu)}.$$

Hence consequently,

$$g_{i,jk} - g_{i,kj} = \left(-R_{ijk}^r + \sum_{\mu} g^{rs} \gamma^{(\mu)} \left(\phi_{sj}^{(\mu)} \phi_{ik}^{(\mu)} - \phi_{sk}^{(\mu)} \phi_{ij}^{(\mu)} \right) \right) g_r + \\ + \sum_{\mu} \gamma^{(\mu)} \left(\phi_{ij;k}^{(\mu)} - \phi_{ik;j}^{(\mu)} + 2\Gamma_{[jk]}^r \phi_{ir}^{(\mu)} \right) n^{(\mu)} + \\ + \sum_{\mu,\nu} \gamma^{(\nu)} \left(\phi_{ij}^{(\nu)} \theta_k^{\nu\mu} - \phi_{ik}^{(\nu)} \theta_j^{\nu\mu} \right) n^{(\mu)}.$$

On the other hand, we have

$$g_{i,jk} = \left(\Gamma_{BC,D}^A + \Gamma_{BC}^E \Gamma_{ED}^A \right) e_i^B e_j^C e_k^D e_A + \\ + \left(e_{i,jk}^A + \Gamma_{BC}^A e_{i,j}^B e_k^C + \Gamma_{BC}^A e_{i,k}^B e_j^C + \Gamma_{BC}^A e_i^B e_{j,k}^C \right) e_A, \\ g_{i,jk} - g_{i,kj} = \left(-R_{BCD}^A e_i^B e_j^C e_k^D + S_{ijk}^A \right) e_A,$$

where, just as in Chapter 2,

$$S_{ijk}^A = e_{i,jk}^A - e_{i,kj}^A + (e_{j,k}^C - e_{k,j}^C) \Gamma_{BC}^A e_i^B.$$

Combining the above, we generalize the Gauss-Codazzi equations into

$$R_{ijkl} = \sum_{\mu} \gamma^{(\mu)} \left(\phi_{ik}^{(\mu)} \phi_{jl}^{(\mu)} - \phi_{il}^{(\mu)} \phi_{jk}^{(\mu)} \right) + R_{ABCD} e_i^A e_j^B e_k^C e_l^D - S_{Aijkl} e_i^A, \\ \phi_{ij;k}^{(\mu)} - \phi_{ik;j}^{(\mu)} = -R_{ABCD} n^{(\mu)} e_i^B e_j^C e_k^D + S_{Aijk} n^{(\mu)A} - 2\Gamma_{[jk]}^r \phi_{ir}^{(\mu)} + \\ + \sum_{\nu} \gamma^{(\nu)} \left(\phi_{ij}^{(\nu)} \theta_k^{\mu\nu} - \phi_{ik}^{(\nu)} \theta_j^{\mu\nu} \right).$$

Finally, when $\mathbb{R}_n$ is embedded isometrically in $\mathbb{R}_N$, i.e., when the embedding manifold $\mathbb{R}_N$ is an Euclidean or pseudo-Euclidean N -dimensional space(-time) or if we impose a particular integrability condition on the n -vectors in $\mathbb{R}_n$ the way we derived (2.10) in Chap-

ter 2, we have the system of equations

$$\begin{aligned}
 R_{ijkl} &= \sum_{\mu} \gamma^{(\mu)} \left(\phi_{ik}^{(\mu)} \phi_{jl}^{(\mu)} - \phi_{il}^{(\mu)} \phi_{jk}^{(\mu)} \right), \\
 \phi_{ij;k}^{(\mu)} - \phi_{ik;j}^{(\mu)} &= -2\Gamma_{[jk]}^r \phi_{ir}^{(\mu)} + \sum_{\nu} \gamma^{(\nu)} \left(\phi_{ij}^{(\nu)} \theta_k^{\mu\nu} - \phi_{ik}^{(\nu)} \theta_j^{\mu\nu} \right), \\
 \theta_{i;k}^{\mu\nu} - \theta_{k;i}^{\mu\nu} &= g^{rs} \left(\phi_{ri}^{(\mu)} \phi_{sk}^{(\nu)} - \phi_{rk}^{(\mu)} \phi_{si}^{(\nu)} \right) - \sum_{\tau} \gamma^{(\tau)} \left(\theta_i^{\mu\tau} \theta_k^{\nu\tau} - \theta_k^{\mu\tau} \theta_i^{\nu\tau} \right).
 \end{aligned}$$

Turning to physics, a unified field theory of gravity and electromagnetism may be framed by the geometric quantities above either in $5 = 4+1$ background dimensions or, if we wish to extend it, in $N = 4+p$ background dimensions. As an alternative, in $N = 4+p$ dimensions we may also observe that the following assumptions can be made possible:

1. The first alternating p -exterior curvature $\phi_{[ik]}^{(1)}$ is equivalent to the ordinary electromagnetic field tensor F_{ik} ;
2. The remaining p -exterior curvatures $\phi_{[ik]}^{(2)}, \dots, \phi_{[ik]}^{(p)}, \phi_{(ik)}^{(1)}, \dots, \phi_{(ik)}^{(p)}$ represent fields beyond the known electromagnetic and gravitational fields;
3. As in Chapter 2, if desired, the four-dimensional metric tensor g_{ij} may not depend on the extra p -coordinates $\{x^{N-4}, x^{N-3}, \dots, x^N\}$. Thus the cylinder condition represented by the equations

$$n_{A;B}^{(\mu)} + n_{B;A}^{(\mu)} = 0 \text{ or } \phi_{(ij)}^{(\mu)} = 0$$

is, again, arrived at.

§4.2 Formulation of our gravoelectrodynamics by means of the theory of distributions. Massive quantum electromagnetic field tensor

In Chapter 2 we have assumed a type of parallel transport applied to the pseudo-five-dimensional velocity field $u \equiv (\bar{u}^A) = u^i g_i + \epsilon \in n$ in $\vartheta_n = \mathbb{R}_4 \otimes n$, i.e., $\nabla_4 u = 0$ where ${}^4u = u^i g_i$; $u^i = \frac{dx^i}{ds}$ and $\epsilon = \frac{2e}{m_0 c^2} = \bar{u}^A n_A$; $\frac{d\epsilon}{ds} = 0$. To describe non-diverging point-like objects which may experience no change in energy even when accelerated, like electrons, we now introduce a special kind of autoparallelism through the relation $\nabla_i u = 0$. This says that the pseudo-five-dimensional velocity field $u = (u^i, \epsilon)$ is an autoparallel vector field in the sense of the theory of distributions, whose

magnitude is independent of the four coordinates of $\mathbb{R}_4$. In other words,

$$\begin{aligned} u_{;k}^i &= \frac{1}{2} \in F_{;k}^i, \\ \in_{;i} &= -\frac{1}{2} F_{;i}^k u_k. \end{aligned}$$

From these we have

$$\begin{aligned} \frac{Du^i}{Ds} &= \frac{e}{m_0 c^2} F_{;k}^i u^k, \\ u_{;i}^i &= 0, \\ \frac{d}{ds} &= 0. \end{aligned}$$

We're now in a position to derive the field equations of gravoelectrodynamics with the help of these relations. Afterwards, we shall show that these relations, indeed, lead to acceptable equations of motion in both four and five dimensions. We may also emphasize that the five-dimensional space $\mathbb{M}_5$ is a Riemann space. First we note that

$$\begin{aligned} u_{i;k;l} &= \frac{1}{2} \in F_{ik;l} + \frac{1}{2} \in_{;l} F_{ik} = \\ &= \frac{1}{2} \in F_{ik;l} - \frac{1}{4} F_{;l}^r F_{ik} u_r. \end{aligned}$$

Now

$$\begin{aligned} u_{i;k;l} - u_{i;l;k} &= R_{;ikl}^r u_r - 2u_{i;r} \Gamma_{[kl]}^r = \\ &= \frac{1}{2} (F_{ik;l} - F_{il;k}) \in + \frac{1}{4} (F_{;k}^r F_{il} - F_{;l}^r F_{ik}) u_r. \end{aligned}$$

Therefore we have

$$R_{;jkl}^i u_i = \frac{1}{4} (F_{;k}^i F_{jl} - F_{;l}^i F_{jk}) u_i + \frac{1}{2} (F_{jk;l} - F_{jl;k} + 2\Gamma_{[kl]}^r F_{jr}) \in.$$

But $u_i = \gamma_i^A \bar{u}_A$ and $\in = \bar{u}^A n_A$, so by means of symmetry, we can lop off the $\bar{u}^A$:

$$R_{;jkl}^i \gamma_i^A = \frac{1}{4} (F_{;k}^i F_{jl} - F_{;l}^i F_{jk}) \gamma_i^A + \frac{1}{2} (F_{jk;l} - F_{jl;k} + 2\Gamma_{[kl]}^r F_{jr}) n^A.$$

From this relation, we derive the unified field equations

$$\begin{aligned} R_{ijkl} &= \frac{1}{4} (F_{ik} F_{jl} - F_{il} F_{jk}), \\ F_{ij;k} - F_{ik;j} &= -2\Gamma_{[jk]}^r F_{ir} \end{aligned}$$

as expected. We have thus no need to assume that the curvature of the background five-dimensional space $\mathbb{M}_5$ vanishes. To strengthen our proof, recall that $g_{i;j} = \phi_{ij}n$. Hence

$$\begin{aligned} g_{i;j;k} &= \phi_{ij;k}n + \phi_{ij}n_{;k} = \\ &= \phi_{ij;k}n - \phi_{ij}\phi_{.k}^r g_r, \\ g_{i;j;k} - g_{i;k;j} &= (\phi_{ij;k} - \phi_{ik;j})n + (\phi_{ik}\phi_{.j}^r - \phi_{ij}\phi_{.k}^r)g_r. \end{aligned}$$

However,

$$g_{i;j;k} - g_{i;k;j} = R_{.ijk}^r g_r - 2\Gamma_{[jk]}^r \phi_{ir}^n.$$

Combining the above, we have

$$\begin{aligned} R_{ijkl} &= \phi_{ik}\phi_{jl} - \phi_{il}\phi_{jk} \cdot \\ \phi_{ij;k} - \phi_{ik;j} &= -2\Gamma_{[jk]}^r \phi_{ir}. \end{aligned}$$

Invoking the cylinder condition, again we get

$$\begin{aligned} R_{ijkl} &= \frac{1}{4} (F_{ik}F_{jl} - F_{il}F_{jk}), \\ F_{ij;k} - F_{ik;j} &= -2\Gamma_{[jk]}^r F_{ir}. \end{aligned}$$

We now assume that the five-dimensional equation of motion in $\mathbb{M}_5$ is in general not a geodesic equation of motion. Instead, we expect an equation of motion of the form

$$\frac{d^2 x^A}{ds^2} + \Gamma_{BC}^A \frac{dx^B}{ds} \frac{dx^C}{ds} = \beta \tilde{F}_{.B}^A u^B,$$

where $u^A = e_i^A u^i$, and

$$\begin{aligned} ds^2 &= {}^5ds^2 = g_{(AB)} dx^A dx^B = (e_A^i e_B^k g_{(ik)} + n_A n_B) dx^A dx^B = \\ &= g_{(ik)} dx^i dx^k = {}^4ds^2, \end{aligned}$$

$$\Gamma_{BC}^A = \frac{1}{2} g^{(DA)} (g_{(DB),C} - g_{(BC),D} + g_{(CD),B}),$$

$$\tilde{F}_{AB} \equiv F_{AB} + \Omega_{AB},$$

$$F_{AB} = -(n_{A;B} - n_{B;A}),$$

$$F_{ik} = e_i^A e_k^B F_{AB},$$

$$F_{AB} n^B = 0,$$

$$\Omega_{AB} e_i^A e_k^B = 0.$$

Note also that the connection transforms as

$$\begin{aligned}\Gamma_{BC}^A &= e_i^A e_{B,C}^i + e_i^A \Gamma_{jk}^i e_B^j e_C^k + \\ &\quad + \frac{1}{2} F_{ik} e_B^i e_C^k n^A + n^A n_{B,C} - \frac{1}{2} F_{\cdot k}^i e_i^A e_C^k n_B.\end{aligned}$$

Now with the help of the relation

$$e_{i,k}^A = e_r^A \Gamma_{ik}^r - \Gamma_{BC}^A e_i^B e_k^C + \frac{1}{2} F_{ik} n^A,$$

we have

$$\begin{aligned}\frac{d^2 x^A}{ds^2} &= \frac{d}{ds} \left(e_i^A \frac{dx^i}{ds} \right) = e_{i,k}^A \frac{dx^i}{ds} \frac{dx^k}{ds} + e_i^A \frac{d^2 x^i}{ds^2} = \\ &= e_i^A \frac{d^2 x^i}{ds^2} + e_i^A \Gamma_{jk}^i \frac{dx^j}{ds} \frac{dx^k}{ds} - \Gamma_{BC}^A \frac{dx^B}{ds} \frac{dx^C}{ds}.\end{aligned}$$

Hence

$$\frac{d^2 x^A}{ds^2} + \Gamma_{BC}^A \frac{dx^B}{ds} \frac{dx^C}{ds} = e_i^A \left(\frac{d^2 x^i}{ds^2} + \Gamma_{jk}^i \frac{dx^j}{ds} \frac{dx^k}{ds} \right) = \beta \tilde{F}_{\cdot B}^A u^B.$$

Setting $\Omega_{AB} = 0$ and $\beta = \frac{e}{m_0 c^2}$, since $F_{\cdot B}^A e_A^i u^B = F_{\cdot B}^A e_A^i e_k^B u^k = F_{\cdot k}^i u^k$, we then obtain the equation of motion:

$$\frac{Du^i}{Ds} = \frac{e}{m_0 c^2} F_{\cdot k}^i u^k.$$

A straightforward way to obtain the five-dimensional equation of motion is as follows: from our assumptions we have

$$(e_i^A u_A)_{;k} = \frac{1}{2} \in F_{ik},$$

$$F_{ik} u_A n^A + u_{A;B} e_i^A e_k^B = \frac{1}{2} \in F_{ik}.$$

Therefore recalling that $u^A n_A = 0$ and multiplying through by u^k , we get

$$u_{;B}^A e_A^i u^B = \frac{1}{2} \in F_{\cdot k}^i u^k.$$

Again, noting that $u_{;B}^A u^B = u_{;k}^A u^k = \frac{Du^A}{Ds}$ and multiplying through by e_i^C , we get

$$\frac{Du^C}{Ds} - n_A \frac{Du^A}{Ds} n^C = \frac{e}{m_0 c^2} F_{\cdot A}^C u^A.$$

But

$$\begin{aligned} u^A \frac{Dn_A}{Ds} &= -n_A \frac{Du^A}{Ds} = -n_{A;B} u^A u^B = -n_{A;k} u^A u^k = \\ &= \frac{1}{2} u^A F_{ik} e_A^i u^k = \\ &= \frac{1}{2} F_{ik} u^i u^k = 0. \end{aligned}$$

Therefore

$$\frac{Du^A}{Ds} = \frac{e}{m_0 c^2} F_{\cdot B}^A u^B.$$

Let's now recall the spin-curvature tensor (3.8) as well as (3.5) and (3.6):

$$\begin{aligned} H_{ik} &= -(\nabla_i S \cdot \omega_k) = \\ &= \frac{1}{2} g_{[k]l} F_{\cdot i}^l \bar{\phi} - S_{k;i}, \\ S &= S_i \omega^i + \frac{3}{2} g^{[ij]} (\nabla_i \xi \cdot g_j) n = \\ &= S_i \omega^i + \bar{\phi} n, \\ S_i &= -(n \cdot \nabla_i \xi) = -\psi_{,i} - \frac{1}{2} F_{\cdot i}^k \xi_k, \\ \bar{\phi} &= \frac{3}{2} g^{[ij]} (\nabla_i \xi \cdot g_j) = \\ &= \frac{3}{2} g^{[ij]} \left(\xi_{j;i} - \frac{1}{2} \psi F_{ji} \right). \end{aligned}$$

Recall (3.9a):

$$H_{ik} u^i = 0.$$

Multiplying (3.10) by u^i , we get the equation of motion:

$$\frac{DS_k}{Ds} = \frac{1}{2} \bar{\phi} g_{[kr]} F_{\cdot i}^r u^i,$$

Equivalently,

$$\frac{DS^i}{Ds} = \frac{1}{2} \bar{\phi} g^{[ik]} F_{kr} u^r.$$

We can now compare this with (2.31a). Now this leads us to consider a case in our theory in which the spin vector S^i is normalized (in the quantum limit). Then it is given by

$$S^i = v^i = g^{[ik]} u_k.$$

Multiplying once again by $g_{[ij]}$, we get

$$g_{[ij]} \frac{DS^i}{Ds} = \frac{1}{2} \bar{\phi} F_{jr} u^r$$

$$\frac{Du^i}{Ds} = \frac{1}{2} \bar{\phi} F_{,k}^i u^k,$$

which is the Lorentz equation of motion. In this case then it automatically follows that

$$\bar{\phi} = 2 \left(\frac{e}{m_0 c^2} \right) = \epsilon,$$

$$\frac{d\bar{\phi}}{ds} = 0.$$

Corresponding to our ongoing analysis, let's also recall (3.9b):

$$\text{tr } H = 0.$$

This gives the divergence equation

$$S_{;i}^i = -\frac{1}{2} \bar{\phi} g_{[ik]} F^{ik}$$

or

$$g_{[ik]} \left(\frac{1}{2} \bar{\phi} F^{ik} - u^{i;k} \right) = 0.$$

On the other hand, from (3.6a), and by employing our fundamental symmetry, we have

$$S_{;i}^i = -\square \psi.$$

Therefore, with the help of (3.15a), we see that

$$\square \psi = \frac{1}{2} \bar{\phi} g_{[ik]} F^{ik} = R \psi.$$

$$\frac{1}{2} \bar{\phi} g_{[ik]} F^{ik} = \frac{1}{4} F_{ik} F^{ik} \psi.$$

The simplest solution for massive, electrically charged particles of this would then be

$$F_{ik} \psi = 2 \bar{\phi} g_{[ik]}.$$

which expresses the proportionality of the “already quantized” electromagnetic field tensor to the fundamental spin tensor of our unified field theory. In other words,

$$F_{ik} = 4 \left(\frac{e}{m_0 c^2} \right) \psi^{-1} g_{[ik]}.$$

Finally, with the help of (3.27), we get the fundamental quantum relations (here we mean $R \neq 0$, $\psi \neq 0$):

$$F_{ik} = -4 \left(\frac{e}{m_0 c^2} \right) R \psi g_{[ik]},$$

$$R = -\frac{1}{\psi^2}.$$

Now, following (1.1), we can write our asymmetric fundamental tensor γ_{ik} as

$$\gamma_{ik} = \frac{1}{\sqrt{2}} \left(g_{(ik)} + \frac{1}{4} \left(\frac{m_0 c^2}{e} \right) \psi F_{ik} \right),$$

which satisfies (1.4): $\gamma_{ij} \gamma^{kj} = \delta_i^k$. On the other hand, now we see that

$$\gamma_{ir} \gamma^{kr} = \frac{1}{2} \left(\delta_i^k + \frac{1}{4} \left(\frac{m_0 c^2}{e} \right)^2 \psi^2 R_{i.}^k \right),$$

$$\gamma_{ir} \gamma^{rk} = \frac{1}{2} \left(\delta_i^k + \frac{1}{2} \left(\frac{m_0 c^2}{e} \right) \psi F_{i.}^k - \frac{1}{4} \left(\frac{m_0 c^2}{e} \right)^2 \psi^2 R_{i.}^k \right).$$

Hence we get the following expression for the Ricci tensor:

$$R_{ik} = 4 \left(\frac{e}{m_0 c^2} \right)^2 \psi^{-2} g_{(ik)}.$$

We now define an inverse wave function:

$$\Psi = 2 \left(\frac{e}{m_0 c^2} \right) \psi^{-1}.$$

We can then express the Ricci and the electromagnetic field tensors as

$$R_{ik} = \Psi^2 g_{(ik)},$$

$$F_{ik} = 2 \Psi g_{[ik]}.$$

Then it follows from (1.2b) and (1.2c) that

$$F_{ik} = -2 \Psi^{-1} g_{[kr]} R_{i.}^r,$$

$$R = \frac{1}{2} \Psi g^{[ik]} F_{ik} = 4 \Psi^2.$$

We have now therefore fulfilled our promise in the beginning (for instance, at the end of Section 1.2) to express gravity and electromagnetism in terms of the components of the fundamental tensor, i.e., $g_{(ik)} \equiv \sqrt{2} \gamma_{(ik)}$ and $g_{[ik]} \equiv \sqrt{2} \gamma_{[ik]}$, alone.

§4.3 On the conservation of currents

We need to recall the basic field equations:

$$\begin{aligned}
R_{ijkl} &= \frac{1}{4} (F_{ik}F_{jl} - F_{il}F_{jk}) , \\
F_{ij;k} - F_{ik;j} &= -2\Gamma_{[jk]}^r F_{ir} , \\
R^{ik} &= -\frac{1}{4} F_{\cdot r}^i F^{rk} , \\
R_{\cdot k}^{ik} &= -\frac{1}{4} F_{\cdot r;k}^i F^{rk} - \frac{1}{4} F_{\cdot r}^i F_{\cdot k}^{rk} = \\
&= -\frac{1}{4} F_{\cdot k}^i J^k + \frac{1}{4} F_{\cdot k}^i \Gamma_{[rs]}^k F^{rs} , \\
J^i &= 2g^{(ik)} \Gamma_{[ks]}^r F_{\cdot r}^s .
\end{aligned}$$

To guarantee conservation of currents, we now introduce the one-form $\zeta^i \in \vartheta_n$:

$$\zeta^i = R^{ik} g_k + \chi^i n ,$$

where χ^i represents another current. Requiring, in the sense of the theory of distributions, that the covariant derivative of ζ^i vanishes at all points of $\mathbb{R}_4$ also means that its covariant divergence also vanishes:

$$\zeta_{;i}^i = 0 .$$

Hence

$$\begin{aligned}
R_{\cdot i}^{ik} g_k + R^{ik} g_{i;k} + \chi_{\cdot i}^i n + \chi^i n_{\cdot i} &= 0 , \\
R_{\cdot i}^{ik} g_k - \frac{1}{2} R^{ik} F_{ik} n + \chi_{\cdot i}^i n - \frac{1}{2} \chi^i F_{\cdot i}^k g_k &= 0 .
\end{aligned}$$

Then we have

$$\begin{aligned}
R_{\cdot k}^{ik} &= \frac{1}{2} F_{\cdot k}^i \chi^k , \\
\chi_{\cdot i}^i &= 0 .
\end{aligned}$$

Comparing the last two equations above with the fourth equation, one must find

$$\chi^i = -\frac{1}{2} \left(J^i - \Gamma_{[kl]}^i F^{kl} \right) .$$

Meanwhile, in the presence of torsion the Bianchi identity for the electromagnetic field tensor and the covariant divergence of the four-current J are

$$F_{ij;k} + F_{jk;i} + F_{ki;j} = -2 \left(\Gamma_{[ij]}^r F_{kr} + \Gamma_{[jk]}^r F_{ir} + \Gamma_{[ki]}^r F_{jr} \right) ,$$

$$J^i_{;i} = \left(\Gamma^i_{[kl]} F^{kl} \right)_{;i} .$$

Now since $\chi^i_{;i} = 0$, we see that

$$\left(\Gamma^i_{[kl]} F^{kl} \right)_{;i} = 0$$

and (in a general setting)

$$J^i = \alpha \Gamma^i_{[kl]} F^{kl} \neq -2\chi^i$$

for some constant α , are necessary and sufficient conditions for the current density vector J to be conserved. Otherwise both χ^i and J^i are directly equivalent to each other. Of course one may also define another conserved current through

$$j^i = F^{ik}_{||k} \rightarrow j^i_{||i} = 0 ,$$

where the double stroke represents a covariant derivative with respect to the symmetric Levi-Civita connection.

In general the current J will automatically be conserved if the orthogonality condition imposed on the twist vector, derived from the torsion tensor, and the velocity vector:

$$\tau_i u^i = 0 ,$$

where

$$\tau_i = \Gamma^k_{[ki]}$$

holds. For more details of the conservation law for charges, see Section 4.4 below.

§4.4 On the wave equations of our unified field theory

We start again with the basic field equations of our unified field theory:

$$R_{ijkl} = \frac{1}{4} (F_{ik} F_{jl} - F_{il} F_{jk}) ,$$

$$F_{ij;k} - F_{ik;j} = -2\Gamma^r_{[jk]} F_{ir} ,$$

$$J^i = 2g^{(ik)} \Gamma^s_{[kr]} F^r_{.s} .$$

We remind ourselves that these field equations give us a set of complete relations between the curvature tensor, the torsion tensor, the electromagnetic field tensor and the current density vector. From these field equations, we are then able to derive the following insightful algebraic

relations:

$$\begin{aligned}
F^{ij}R_{ijkl} &= R_{.k}^r F_{rl} - R_{.l}^r F_{rk} , \\
F_{ik} &= R^{-1} (F^{rs} R_{risk} + F_{ir} R_{.k}^r) , \\
R^2 &= R_{ik} R^{ik} + \frac{1}{4} R_{ijkl} F^{ik} F^{jl} = \\
&= R_{ik} R^{ik} + R_{ijkl} R^{ijkl} + \frac{1}{4} R_{ijkl} F^{il} F^{jk} , \\
J^i &= 2R^{-1} g^{(ik)} \Gamma_{[kp]}^q (F^{rs} R_{r.sq}^p + F_{.r}^p R_{q.}^r) , \\
R &= 2\rho^{-2} J^i \Gamma_{[ik]}^q (F^{rs} R_{r.sq}^k + F_{.r}^k R_{q.}^r) ,
\end{aligned}$$

where the density ρ corresponds to possible electric-magnetic charge distribution. As has been shown previously, we also have the traceless field equation:

$$R_{ik} - \frac{1}{4} g_{(ik)} R = \frac{1}{4} \left(F_{il} F_{k.}^l - \frac{1}{4} g_{(ik)} F_{rs} F^{rs} \right) .$$

Now recall that the tetrad and the unit normal vector satisfy

$$\begin{aligned}
e_{i;k}^A &= \frac{1}{2} F_{ik} n^A , \\
n_{;i}^A &= -\frac{1}{2} F_{.i}^k e_k^A .
\end{aligned}$$

Then we see that

$$\begin{aligned}
e_{i;j;k}^A &= \frac{1}{2} F_{ij;k} n^A - \frac{1}{4} F_{ij} F_{.k}^r e_r^A , \\
g^{(jk)} e_{i;j;k}^A &= \frac{1}{2} J_i n^A - \frac{1}{4} F_{ik} F^{rk} e_r^A .
\end{aligned}$$

With the help of the basic field equations, we obtain the tetrad wave equation of our unified field theory:

$$\square e_i^A = \frac{1}{2} J_i n^A - R_{i.}^k e_k^A .$$

This expression gives a wave-type equation of the tetrad endowed with two sources: the electromagnetic source, i.e., the electromagnetic current density vector and the Ricci curvature tensor which represents the gravitational source in standard General Relativity.

In addition, we can also obtain the following wave equation:

$$\square n^A = -\frac{1}{2} J^i e_i^A - R n^A.$$

In other words,

$$(\square + R) n^A = -\frac{1}{2} J^i e_i^A.$$

Furthermore, it follows that

$$e_A^k \square e_i^A = -R_{i.}^k,$$

$$n_A \square e_i^A = \frac{1}{2} J_i,$$

$$n_A \square n^A = -R,$$

$$e_A^i \square n^A = -\frac{1}{2} J^i.$$

We may now express the current density vector as

$$J^i = 2 n^A \square e_A^i.$$

Again, we obtain the conservation law of electromagnetic currents as follows:

$$\begin{aligned} J_{;i}^i &= 2 n_{;i}^A \square e_A^i + 2 n^A \square e_{A;i}^i = \\ &= -F_{.i}^k e_k^A \square e_A^i = \\ &= F_{.i}^k R_{.k}^i = \\ &= 0. \end{aligned}$$

(Here we have also used the relation $\square e_{A;i}^i = \frac{1}{2} \square (F_{.i}^i n_A) = 0$.)

Meanwhile, since the equation $\square g_{(ik)} = 0$ must be satisfied unconditionally by the metric tensor $g_{(ik)} = e_{Ai} e_k^A$ and also since $g_{(AB);i} = g_{(AB);C} e_i^C = 0$, we then get

$$e_{Ak} \square e_i^A = -e_i^A \square e_{Ak} - 2 R_{ik}.$$

Now the curvature tensor can be expressed in terms of the tetrad as

$$R_{ijkl} = g_{(AB)} (\nabla_k e_i^A \nabla_l e_j^B - \nabla_l e_i^A \nabla_k e_j^B).$$

Then the Ricci tensor is

$$R_{ik} = g_{(AB)} \nabla_r e_i^A \nabla^r e_k^B.$$

We can also express this as

$$R_{ik} = -g_{(AB)} e_i^A \square e_k^B,$$

which can be written equivalently as

$$R_{ik} = -\frac{1}{2} g_{(AB)} (e_i^A \nabla_r \nabla^r e_k^B + e_k^A \nabla_r \nabla^r e_i^B).$$

Consider now a source-free region in the space-time $\mathbb{R}_4$. As we've seen, the absence of source is characterized by the vanishing of the torsion tensor. In such a case, if the space-time has a constant sectional curvature K , we obtain the tetrad wave equation for the “empty” region:

$$(\square + \frac{1}{4} R) e_i^A = 0.$$

In other words,

$$(\square + K) e_i^A = 0.$$

Combining our tetrad wave equation $\square e_i^A = \frac{1}{2} J_i n^A - R_{i \cdot}^k e_k^A$ with the equation for the Ricci tensor we have derived in the quantum limit (in Section 4.2), which is

$$R_{ik} = 4 \left(\frac{e}{m_0 c^2} \right)^2 \psi^{-2} g_{(ik)},$$

we obtain the following wave equation in the presence of the electromagnetic current density:

$$(\square + \frac{1}{4} R) e_i^A = \frac{1}{2} J_i n^A,$$

$$R = C + 4 \int \left(\Gamma_{[ik]}^r R_{\cdot \cdot rs}^{ik} - 2 \Gamma_{[rs]}^i R_{\cdot i}^r \right) dx^s,$$

where $C = 4K$ is constant.

Finally, let's have a look back at the wave equation given by (3.15b):

$$(\square - R) |\psi\rangle = 0.$$

If we fully assume that the space-time $\mathbb{R}_4$ is embedded isometrically in $\mathbb{M}_5$ spanned by the time coordinate $\tau = ct$ and the four space coordinates u, v, w, y which together form the line-element $ds^2 = d\tau^2 - du^2 - dv^2 - dw^2 - dy^2 \equiv d\tau^2 - d\sigma^2 - dy^2$, and if the wave function represented by the state vector $|\psi\rangle$ does not depend on the microscopic fifth coordinate y , we can write the wave equation in the simple form:

$$\left(\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \sigma^2} \right) |\psi\rangle = R |\psi\rangle.$$

§4.5 A more compact form of the generalized Gauss-Codazzi equations in $\mathbb{R}_5$

We write the generalized Gauss-Codazzi equations (see (2.6) for instance) in $\mathbb{R}_5$ once again:

$$R_{ijkl} = \phi_{ik}\phi_{jl} - \phi_{il}\phi_{jk} + R_{ABCD} e_i^A e_j^B e_k^C e_l^D - S_{Aijk} e_i^A,$$

$$\phi_{ij;k} - \phi_{ik;j} = -R_{ABCD} n^A e_i^B e_j^C e_k^D - 2\Gamma_{[jk]}^r \phi_{ir} + S_{Aijk} n^A$$

in terms of the general asymmetric extrinsic curvature tensor. Here, as usual,

$$S_{ijk}^A = e_{i,jk}^A - e_{i,kj}^A + (e_{j,k}^C - e_{k,j}^C) \Gamma_{BC}^A e_i^B.$$

From the fundamental relations

$$e_{i;j}^A = \phi_{ij} n^A,$$

$$n_{;i}^A = -\phi_{,i}^k e_k^A,$$

we have

$$e_{i;j;k}^A = \phi_{ij;k} n^A - \phi_{ij} \phi_{,k}^r e_r^A.$$

Hence

$$e_{i;j;k}^A - e_{i;k;j}^A = (\phi_{ij;k} - \phi_{ik;j}) n^A + (\phi_{,j}^r \phi_{ik} - \phi_{,k}^r \phi_{ij}) e_r^A.$$

With the help of the generalized Gauss-Codazzi equations above and the identity

$$e_A^i e_i^B = \delta_A^B - n_A n^B,$$

we see that that the generalized Gauss-Codazzi equations in $\mathbb{R}_5$ can be written somewhat more compactly as a “single equation”:

$$e_{i;j;k}^A - e_{i;k;j}^A = R_{,ijk}^r e_r^A - R_{ABCD} e_i^B e_j^C e_k^D - 2\Gamma_{[jk]}^r \phi_{ir} n^A + S_{,ijk}^A.$$

Appendix: The Fundamental Geometric Properties of a Curved Manifold

Let us present the fundamental geometric objects of an n -dimensional curved manifold. Let $\omega_a = \frac{\partial X^i}{\partial x^a} E_i = \partial_a X^i E_i$ (the Einstein summation convention is assumed throughout this work) be the covariant (*frame*) basis spanning the n -dimensional base manifold $\mathbb{C}^\infty$ with local coordinates $x^a = x^a(X^k)$. The contravariant (*coframe*) basis θ^b is then given via the orthogonal projection $\langle \theta^b, \omega_a \rangle = \delta_a^b$, where δ_a^b are the components of the Kronecker delta (whose value is unity if the indices coincide or null otherwise). The set of linearly independent local directional derivatives $E_i = \frac{\partial}{\partial X^i} = \partial_i$ gives the coordinate basis of the locally flat tangent space $\mathbb{T}_x(\mathbb{M})$ at a point $x \in \mathbb{C}^\infty$. Here M denotes the topological space of the so-called n -tuples $h(x) = h(x^1, \dots, x^n)$ such that relative to a given chart $(U, h(x))$ on a neighborhood U of a local coordinate point x , our $\mathbb{C}^\infty$ -differentiable manifold itself is a topological space. The dual basis to E_i spanning the locally flat cotangent space $\mathbb{T}_x^*(\mathbb{M})$ will then be given by the differential elements dX^k via the relation $\langle dX^k, \partial_i \rangle = \delta_i^k$. In fact and in general, the *one-forms* dX^k indeed act as a linear map $\mathbb{T}_x(\mathbb{M}) \rightarrow \mathbb{R}$ when applied to an arbitrary vector field $F \in \mathbb{T}_x(\mathbb{M})$ of the explicit form $F = F^i \frac{\partial}{\partial X^i} = f^a \frac{\partial}{\partial x^a}$. Then it is easy to see that $F^i = F X^i$ and $f^a = F x^a$, from which we obtain the usual transformation laws for the contravariant components of a vector field, i.e., $F^i = \partial_a X^i f^a$ and $f^i = \partial_i x^a F^i$, relating the localized components of F to the general ones and vice versa. In addition, we also see that $\langle dX^k, F \rangle = F X^k = F^k$.

The components of the symmetric metric tensor $g = g_{ab} \theta^a \otimes \theta^b$ of the base manifold $\mathbb{C}^\infty$ are readily given by

$$g_{ab} = \langle \omega_a, \omega_b \rangle$$

satisfying

$$g_{ac} g^{bc} = \delta_a^b,$$

where $g^{ab} = \langle \theta^a, \theta^b \rangle$. It is to be understood that the covariant and contravariant components of the metric tensor will be used to raise and the (component) indices of vectors and tensors.

The components of the metric tensor $g(x_N) = \eta_{ik} dX^i \otimes dX^k$ describing the locally flat tangent space $\mathbb{T}_x(\mathbb{M})$ of rigid frames at a point

$x_N = x_N(x^a)$ are given by

$$\eta_{ik} = \langle E_i, E_k \rangle = \text{diag}(\pm 1, \pm 1, \dots, \pm 1).$$

In four dimensions, the above may be taken to be the components of the Minkowski metric tensor, i.e., $\eta_{ik} = \langle E_i, E_k \rangle = \text{diag}(1, -1, -1, -1)$.

Then we have the expression

$$g_{ab} = \eta_{ik} \partial_a X^i \partial_b X^k.$$

The line-element of $\mathbb{C}^\infty$ is then given by

$$ds^2 = g = g_{ab} (\partial_i x^a \partial_k x^b) dX^i \otimes dX^k,$$

where $\theta^a = \partial_i x^a dX^i$.

Given the existence of a local coordinate transformation via $x^i = x^i(\bar{x}^\alpha)$ in $\mathbb{C}^\infty$, the components of an arbitrary tensor field $T \in \mathbb{C}^\infty$ of rank (p, q) transform according to

$$T_{cd\dots h}^{ab\dots g} = T_{\mu\nu\dots\eta}^{\alpha\beta\dots\lambda} \partial_\alpha x^a \partial_\beta x^b \dots \partial_\lambda x^g \partial_c \bar{x}^\mu \partial_d \bar{x}^\nu \dots \partial_h \bar{x}^\eta,$$

Let $\delta_{j_1 j_2 \dots j_p}^{i_1 i_2 \dots i_p}$ be the components of the generalized Kronecker delta. They are given by

$$\delta_{j_1 j_2 \dots j_p}^{i_1 i_2 \dots i_p} = \epsilon_{j_1 j_2 \dots j_p} \epsilon^{i_1 i_2 \dots i_p} = \det \begin{pmatrix} \delta_{j_1}^{i_1} & \delta_{j_1}^{i_2} & \dots & \delta_{j_1}^{i_p} \\ \delta_{j_2}^{i_1} & \delta_{j_2}^{i_2} & \dots & \delta_{j_2}^{i_p} \\ \dots & \dots & \dots & \dots \\ \delta_{j_p}^{i_1} & \delta_{j_p}^{i_2} & \dots & \delta_{j_p}^{i_p} \end{pmatrix}$$

where $\epsilon_{j_1 j_2 \dots j_p} = \sqrt{\det(g)} \epsilon_{j_1 j_2 \dots j_p}$ and $\epsilon^{i_1 i_2 \dots i_p} = \frac{1}{\sqrt{\det(g)}} \epsilon^{i_1 i_2 \dots i_p}$ are the covariant and contravariant components of the completely anti-symmetric Levi-Civita permutation tensor, respectively, with the ordinary permutation symbols being given as usual by $\epsilon_{j_1 j_2 \dots j_q}$ and $\epsilon^{i_1 i_2 \dots i_p}$. Again, if ω is an arbitrary tensor, then the object represented by

$$*\omega_{j_1 j_2 \dots j_p} = \frac{1}{p!} \delta_{j_1 j_2 \dots j_p}^{i_1 i_2 \dots i_p} \omega_{i_1 i_2 \dots i_p}$$

is completely anti-symmetric.

Introducing a generally asymmetric connection Γ via the covariant derivative

$$\partial_b \omega_a = \Gamma_{ab}^c \omega_c,$$

i.e.,

$$\Gamma_{ab}^c = \langle \theta^c, \partial_b \omega_a \rangle = \Gamma_{(ab)}^c + \Gamma_{[ab]}^c,$$

where the round index brackets indicate symmetrization and the square ones indicate anti-symmetrization, we have, by means of the local coordinate transformation given by $x^a = x^a(\bar{x}^\alpha)$ in $\mathbb{C}^\infty$

$$\partial_b e_a^\alpha = \Gamma_{ab}^c e_c^\alpha - \bar{\Gamma}_{\beta\lambda}^\alpha e_a^\beta e_b^\lambda,$$

where the tetrads of the *moving frames* are given by $e_a^\alpha = \partial_a \bar{x}^\alpha$ and $e_\alpha^a = \partial_\alpha x^a$. They satisfy $e_\alpha^a e_b^\alpha = \delta_b^a$ and $e_a^\alpha e_\beta^\alpha = \delta_\beta^a$. In addition, it can also be verified that

$$\partial_\beta e_\alpha^a = \bar{\Gamma}_{\alpha\beta}^\lambda e_\lambda^a - \Gamma_{bc}^a e_\alpha^b e_\beta^c,$$

$$\partial_b e_\alpha^a = e_\lambda^a \bar{\Gamma}_{\alpha\beta}^\lambda e_b^\beta - \Gamma_{cb}^a e_\alpha^c e_b^\beta.$$

We know that Γ is a non-tensorial object, since its components transform as

$$\Gamma_{ab}^c = e_\alpha^c \partial_b e_a^\alpha + e_\alpha^c \bar{\Gamma}_{\beta\lambda}^\alpha e_a^\beta e_b^\lambda.$$

However, it can be described as a kind of displacement field since it is what makes possible a comparison of vectors from point to point in $\mathbb{C}^\infty$. In fact the relation $\partial_b \omega_a = \Gamma_{ab}^c \omega_c$ defines the so-called metricity condition, i.e., the change (during a displacement) in the basis can be measured by the basis itself. This immediately translates into

$$\nabla_c g_{ab} = 0.$$

where we have just applied the notion of a covariant derivative to an arbitrary tensor field T :

$$\begin{aligned} \nabla_m T_{cd\dots h}^{ab\dots g} &= \partial_m T_{cd\dots h}^{ab\dots g} + \Gamma_{pm}^a T_{cd\dots h}^{pb\dots g} + \Gamma_{pm}^b T_{cd\dots h}^{ap\dots g} + \dots + \Gamma_{pm}^g T_{cd\dots h}^{ab\dots p} - \\ &\quad - \Gamma_{cm}^p T_{pd\dots h}^{ab\dots g} - \Gamma_{dm}^p T_{cp\dots h}^{ab\dots g} - \dots - \Gamma_{hm}^p T_{cd\dots p}^{ab\dots g} \end{aligned}$$

such that $(\partial_m T)_{cd\dots h}^{ab\dots g} = \nabla_m T_{cd\dots h}^{ab\dots g}$.

The condition $\nabla_c g_{ab} = 0$ can be solved to give

$$\Gamma_{ab}^c = \frac{1}{2} g^{cd} (\partial_b g_{da} - \partial_d g_{ab} + \partial_a g_{bd}) + \Gamma_{[ab]}^c - g^{cd} (g_{ae} \Gamma_{[db]}^e + g_{be} \Gamma_{[da]}^e)$$

from which it is customary to define

$$\Delta_{ab}^c = \frac{1}{2} g^{cd} (\partial_b g_{da} - \partial_d g_{ab} + \partial_a g_{bd})$$

as the Christoffel symbols (symmetric in their two lower indices) and

$$K_{ab}^c = \Gamma_{[ab]}^c - g^{cd} (g_{ae} \Gamma_{[db]}^e + g_{be} \Gamma_{[da]}^e)$$

as the components of the so-called contorsion tensor (anti-symmetric in the first two mixed indices).

Note that the components of the torsion tensor are given by

$$\Gamma_{[bc]}^a = \frac{1}{2} e_\alpha^a \left(\partial_c e_b^\alpha - \partial_b e_c^\alpha + e_b^\beta \bar{\Gamma}_{\beta c}^\alpha - e_c^\beta \bar{\Gamma}_{\beta b}^\alpha \right),$$

where we have set $\bar{\Gamma}_{\beta c}^\alpha = \bar{\Gamma}_{\beta\lambda}^\alpha e_c^\lambda$, such that for an arbitrary scalar field Φ we have

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) \Phi = 2\Gamma_{[ab]}^c \nabla_c \Phi.$$

The components of the curvature tensor R of $\mathbb{C}^\infty$ are then given via the relation

$$\begin{aligned} (\nabla_q \nabla_p - \nabla_p \nabla_q) T_{cd\dots r}^{ab\dots s} &= T_{wd\dots r}^{ab\dots s} R_{cpq}^w + T_{cw\dots r}^{ab\dots s} R_{dpq}^w + \dots + T_{cd\dots w}^{ab\dots s} R_{rpq}^w - \\ &\quad - T_{cd\dots r}^{wb\dots s} R_{wpq}^a - T_{cd\dots r}^{aw\dots s} R_{wpq}^b - \dots - T_{cd\dots r}^{ab\dots w} R_{wpq}^s - \\ &\quad - 2\Gamma_{[pq]}^w \nabla_w T_{cd\dots r}^{ab\dots s}. \end{aligned}$$

where

$$\begin{aligned} R_{abc}^d &= \partial_b \Gamma_{ac}^d - \partial_c \Gamma_{ab}^d + \Gamma_{ac}^e \Gamma_{eb}^d - \Gamma_{ab}^e \Gamma_{ec}^d = \\ &= B_{abc}^d(\Delta) + \hat{\nabla}_b K_{ac}^d - \hat{\nabla}_c K_{ab}^d + K_{ac}^e K_{eb}^d - K_{ab}^e K_{ec}^d, \end{aligned}$$

where $\hat{\nabla}$ denotes covariant differentiation with respect to the Christoffel symbols alone, and where

$$B_{abc}^d(\Delta) = \partial_b \Delta_{ac}^d - \partial_c \Delta_{ab}^d + \Delta_{ac}^e \Delta_{eb}^d - \Delta_{ab}^e \Delta_{ec}^d$$

are the components of the Riemann-Christoffel curvature tensor of $\mathbb{C}^\infty$.

From the components of the curvature tensor, namely, R_{abc}^d , we have (using the metric tensor to raise and lower indices)

$$R_{ab} \equiv R_{acb}^c = B_{ab}(\Delta) + \hat{\nabla}_c K_{ab}^c - K_{ad}^c K_{cb}^d - 2\hat{\nabla}_b \Gamma_{[ac]}^c + 2K_{ab}^c \Gamma_{[cd]}^d,$$

$$R \equiv R_a^a = B(\Delta) - 4g^{ab} \hat{\nabla}_a \Gamma_{[bc]}^c - 2g^{ac} \Gamma_{[ab]}^b \Gamma_{[cd]}^d - K_{abc} K^{acb},$$

where $B_{ab}(\Delta) \equiv B_{acb}^c(\Delta)$ are the components of the symmetric Ricci tensor and $B(\Delta) \equiv B_a^a(\Delta)$ is the Ricci scalar. Note that $K_{abc} \equiv g_{ad} K_{bc}^d$ and $K^{acb} \equiv g^{cd} g^{be} K_{de}^a$.

Now since

$$\Gamma_{ba}^b = \Delta_{ba}^b = \Delta_{ab}^b = \partial_a \left(\ln \sqrt{\det(g)} \right),$$

$$\Gamma_{ab}^b = \partial_a \left(\ln \sqrt{\det(g)} \right) + 2\Gamma_{[ab]}^b,$$

we see that for a continuous metric determinant, the so-called homothetic curvature vanishes:

$$H_{ab} \equiv R_{cab}^c = \partial_a \Gamma_{cb}^c - \partial_b \Gamma_{ca}^c = 0.$$

Introducing the traceless Weyl tensor W , we have the following decomposition theorem:

$$\begin{aligned} R_{abc}^d &= W_{abc}^d + \frac{1}{n-2} (\delta_b^d R_{ac} + g_{ac} R_{ab}^d - \delta_c^d R_{ab} - g_{ab} R_{ac}^d) + \\ &+ \frac{1}{(n-1)(n-2)} (\delta_c^d g_{ab} - \delta_b^d g_{ac}) R, \end{aligned}$$

which is valid for $n > 2$. For $n=2$, we have

$$R_{abc}^d = K_G (\delta_b^d g_{ac} - \delta_c^d g_{ab}),$$

where

$$K_G = \frac{1}{2} R$$

is the Gaussian curvature of the surface. Note that (in this case) the Weyl tensor vanishes.

Any n -dimensional manifold (for which $n > 1$) with constant sectional curvature R and vanishing torsion is called an Einstein space. It is described by the following simple relations:

$$R_{abc}^d = \frac{1}{n(n-1)} (\delta_b^d g_{ac} - \delta_c^d g_{ab}) R,$$

$$R_{ab} = \frac{1}{n} g_{ab} R.$$

In the above, we note especially that

$$R_{abc}^d = B_{abc}^d(\Delta),$$

$$R_{ab} = B_{ab}(\Delta),$$

$$R = B(\Delta).$$

Furthermore, after some lengthy algebra, we obtain, in general, the following *generalized* Bianchi identities:

$$\begin{aligned} R_{bcd}^a + R_{cdb}^a + R_{dbc}^a &= \\ &= -2 \left(\partial_d \Gamma_{[bc]}^a + \partial_b \Gamma_{[cd]}^a + \partial_c \Gamma_{[db]}^a + \Gamma_{eb}^a \Gamma_{[cd]}^e + \Gamma_{ec}^a \Gamma_{[db]}^e + \Gamma_{ed}^a \Gamma_{[bc]}^e \right), \end{aligned}$$

$$\begin{aligned}\nabla_e R^a_{.bcd} + \nabla_c R^a_{.bde} + \nabla_d R^a_{.bec} &= 2 \left(\Gamma^f_{[cd]} R^a_{.bfe} + \Gamma^f_{[de]} R^a_{.bfc} + \Gamma^f_{[ec]} R^a_{.bfd} \right), \\ \nabla_a \left(R^{ab} - \frac{1}{2} g^{ab} R \right) &= 2 g^{ab} \Gamma^c_{[da]} R^d_{.c} + \Gamma^a_{[cd]} R^{cdb},\end{aligned}$$

for any metric-compatible manifold endowed with both curvature and torsion.

In the last of the above set of equations, we have introduced the generalized Einstein tensor, i.e.,

$$G_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R.$$

In particular, we also have the following specialized identities, i.e., the *regular* Bianchi identities:

$$\begin{aligned}B^a_{.bcd} + B^a_{.cdb} + B^a_{.dbc} &= 0, \quad \hat{\nabla}_e B^a_{.bcd} + \hat{\nabla}_c B^a_{.bde} + \hat{\nabla}_d B^a_{.bec} = 0, \\ \hat{\nabla}_a \left(B^{ab} - \frac{1}{2} g^{ab} B \right) &= 0.\end{aligned}$$

In general, these hold in the case of a symmetric, metric-compatible connection. Non-metric differential geometry is beyond the scope of our present consideration.

We now define the so-called Lie derivative which can be used to define a diffeomorphism invariant in $\mathbb{C}^\infty$. For a vector field U and a tensor field T , both arbitrary, the invariant derivative represented (in component notation) by

$$\begin{aligned}L_U T^{ab\dots g}_{cd\dots h} &= \partial_m T^{ab\dots g}_{cd\dots h} U^m + T^{ab\dots g}_{md\dots h} \partial_c U^m + T^{ab\dots g}_{cm\dots h} \partial_d U^m + \dots + \\ &+ \dots + T^{ab\dots g}_{cd\dots m} \partial_h U^m - T^{mb\dots g}_{cd\dots h} \partial_m U^a - T^{am\dots g}_{cd\dots h} \partial_m U^b - \dots - T^{ab\dots m}_{cd\dots h} \partial_m U^g\end{aligned}$$

defines the Lie derivative of T with respect to U . With the help of the torsion tensor and the relation

$$\partial_b U^a = \nabla_b U^a - \Gamma^a_{cb} U^c = \nabla_b U^a - \left(\Gamma^a_{bc} - 2\Gamma^a_{[bc]} \right) U^c,$$

we can write

$$\begin{aligned}L_U T^{ab\dots g}_{cd\dots h} &= \nabla_m T^{ab\dots g}_{cd\dots h} U^m + T^{ab\dots g}_{md\dots h} \nabla_c U^m + T^{ab\dots g}_{cm\dots h} \nabla_d U^m + \dots + \\ &+ T^{ab\dots g}_{cd\dots m} \nabla_h U^m - T^{mb\dots g}_{cd\dots h} \nabla_m U^a - T^{am\dots g}_{cd\dots h} \nabla_m U^b - \dots - \\ &- T^{ab\dots m}_{cd\dots h} \nabla_m U^g + 2\Gamma^a_{[mp]} T^{mb\dots g}_{cd\dots h} U^p + 2\Gamma^b_{[mp]} T^{am\dots g}_{cd\dots h} U^p + \dots + \\ &+ 2\Gamma^g_{[mp]} T^{ab\dots m}_{cd\dots h} U^p - 2\Gamma^m_{[cp]} T^{ab\dots g}_{md\dots h} U^p + 2\Gamma^m_{[dp]} T^{ab\dots g}_{cm\dots h} U^p - \dots - \\ &- 2\Gamma^m_{[hp]} T^{ab\dots g}_{cd\dots m} U^p.\end{aligned}$$

Hence, noting that the components of the torsion tensor, namely, $\Gamma_{[kl]}^i$, indeed transform as components of a tensor field, it is seen that the $L_U T_{kl\dots r}^{ij\dots s}$ do transform as components of a tensor field. Apparently, the beautiful property of the Lie derivative (applied to an arbitrary tensor field) is that it is connection-independent even in a curved manifold.

Conclusion

We have shown that gravity and electromagnetism are intertwined in a very natural manner, both ensuing from the melting of the same underlying space-time geometry. They obey the same set of field equations. However, there are actually no objectively existing elementary particles in this theory. Based on the wave equation (73), we may suggest that what we perceive as particles are only singularities which may be interpreted as wave centers. In the microcosmos everything is essentially a wave function that also contains particle properties. Individual wave function is a fragment of the universal wave function represented by the wave function of the Universe in (73). Therefore all objects are essentially interconnected. We have seen that the electric(-magnetic) charge is none other than the torsion of space-time. This charge can also be described by the wave function alone. This doesn't seem to be contradictory evidence if we realize that nothing exists in the quantum realm save the quantum mechanical wave function (unfortunately, we have not made it possible here to carry a detailed elaboration on this statement). Although we have not approached and constructed a quantum theory of gravity in the strictly formal way (through the canonical quantization procedure), internal consistency of our theory awaits further justification. For a few more details of the underlying unifying features of our theory, see Chapter "Additional Considerations".

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(Other Attempts at Unification by the Author)

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Spin-Curvature and the Unification of Fields in a Twisted Space
by Indranu Suhendro

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The book draws theoretical findings for spin-curvature and the unification of fields in a twisted space. A space twist, represented through the appropriate formalism, is related to the anti-symmetric metric tensor. Kaluza's theory is extended and given an appropriate integrability condition. Both matter and the isotropic electromagnetic field are geometrized through common field equations: trace-free field equations giving the energy-momentum tensor for such an electromagnetic field solely via the (generalized) Ricci curvature tensor and scalar are obtained. In the absence of electromagnetic fields the theory goes to Einstein's 1928 theory of distant parallelism where only matter field is geometrized (through the twist of space-time). The above results in common with respective wave equations are joined into a "unified field theory of semi-classical gravoelectrodynamics".

Spinn-krökning och föreningen av fält i ett tvistat rum
av Indranu Suhendro

ISBN 978-91-85917-01-3. Svenska fysikarkivet, 2008, 78 sidor

Boken skildrar teoretiska forskningsresultat för spinn-krökning och förening av fält i ett tvistat rum. En rumsartad tvist, representerat genom lämplig formalism, relateras till den antisymmetriska metriska tensorn. Kaluzas teori utökas och ges ett lämpligt villkor för integrabilitet. Både materiefält och isotropiska elektromagnetiska fält geometrifieras genom gemensamma fältekvationer som är spårfria fältekvationer som ger energi-impulstensorn för ett elektromagnetiskt fält enbart via den (generaliserade) Ricci krökningstensorn och skalären. I avsaknad av elektromagnetiska fält leder teorin till Einsteins teori från 1928 om avlägsen parallellism där bara materiefält geometrifieras (genom en tvistning av rumstiden). Ovanstående resultat tillsammans med respektive vågekvationer förenas till en "förenad fältteori av semiklassisk gravitationselektrodynamik".

