Solution of Nonlinear Functional and Functional-Differential Equations by Reduction to the Bilinear Functional Equation

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Here we describe a method for solving some classes of nonlinear functional and functional-differential equations that relies upon the reduction to a standard bilinear functional equation. The classes of nonlinear functional and functional-differential equations in question arise in constructing exact solutions to nonlinear partial differential equations with the methods of generalized and functional separation of variables [1–4].

1. Bilinear Functional Equations

1.1. Simple bilinear functional equations and their solutions

1°. A binomial bilinear functional equation has the form

$$f_1(x)g_1(y) + f_2(x)g_2(y) = 0, \tag{1}$$

where $f_n = f_n(x)$ and $g_n = g_n(y)$ $(n = 1, 2)$ are unknown functions of the different arguments; $f_n \not\equiv 0$, $g_n \not\equiv 0$.

Separating variables in (1) yields the solution

$$f_1 = Af_2, \quad g_2 = -Ag_1, \tag{2}$$

where A is an arbitrary constant. The functions on the right-hand sides of the formulae in (2) are assumed to be arbitrary.

2°. The trinomial bilinear functional equation

$$f_1(x)g_1(y) + f_2(x)g_2(y) + f_3(x)g_3(y) = 0, \tag{3}$$

where $f_n = f_n(x)$ and $g_n = g_n(y)$ $(n = 1, 2, 3)$ are unknown functions, has two solutions:

$$\begin{aligned} &f_1 = A_1 f_3, \quad f_2 = A_2 f_3, \quad g_3 = -A_1 g_1 - A_2 g_2;\\ &g_1 = A_1 g_3, \quad g_2 = A_2 g_3, \quad f_3 = -A_1 f_1 - A_2 f_2, \end{aligned} \tag{4}$$

where A_1 and A_2 are arbitrary constants. The functions on the right-hand sides of the equations in (4) are assumed to be arbitrary.

3°. The four-term functional equation

$$f_1(x)g_1(y) + f_2(x)g_2(y) + f_3(x)g_3(y) + f_4(x)g_4(y) = 0, \tag{5}$$

where the f_i are all functions of the same argument and the g_i are all functions of another argument, has a solution

$$\begin{aligned} &f_1 = A_1 f_3 + A_2 f_4, \quad f_2 = A_3 f_3 + A_4 f_4,\\ &g_3 = -A_1 g_1 - A_3 g_2, \quad g_4 = -A_2 g_1 - A_4 g_2 \end{aligned} \tag{6}$$

dependent on four arbitrary constants $A_1, \ldots, A_4$. The functions on the right-hand sides of the equations in (4) are assumed to be arbitrary.

Equation (5) has also two other solutions

$$\begin{aligned} &f_1 = A_1 f_4, \quad f_2 = A_2 f_4, \quad f_3 = A_3 f_4, \quad g_4 = -A_1 g_1 - A_2 g_2 - A_3 g_3;\\ &g_1 = A_1 g_4, \quad g_2 = A_2 g_4, \quad g_3 = A_3 g_4, \quad f_4 = -A_1 f_1 - A_2 f_2 - A_3 f_3 \end{aligned} \tag{7}$$

involving three arbitrary constants.

1.2. Bilinear functional equation of the general form

Consider a bilinear functional equation of the general form

$$f_1(x)g_1(y) + f_2(x)g_2(y) + \cdots + f_k(x)g_k(y) = 0 \tag{8}$$

where $f_n = f_n(x)$ and $g_n = g_n(y)$ are unknown quantities ($n = 1, \ldots, k$).

It can be shown that the bilinear functional equation (8) has $k - 1$ different solutions:

$$\begin{aligned} &f_i(x) = A_{i,1} f_{m+1}(x) + A_{i,2} f_{m+2}(x) + \cdots + A_{i,k-m} f_k(x), \quad && i = 1, \ldots, m; \\ &g_{m+j}(y) = -A_{1,j} g_1(y) - A_{2,j} g_2(y) - \cdots - A_{m,j} g_m(y), && j = 1, \ldots, k - m; \\ &\quad m = 1, 2, \ldots, k - 1; \end{aligned} \tag{9}$$

where the $A_{i,j}$ are arbitrary constants. The functions $f_{m+1}(x)$, …, $f_k(x)$, $g_1(y)$, …, $g_m(y)$ on the right-hand sides of equations (9) are defined arbitrarily. It is apparent that for fixed m, solution (9) contains $m(k - m)$ arbitrary constants.

For fixed m, solution (9) contains $m(k - m)$ arbitrary constants $A_{i,j}$. Given k, the solutions having the maximum number of arbitrary constants are defined by

Solution number	*Number of arbitrary constants*	*Conditions on k*
$m = \frac{1}{2}k$	$\frac{1}{4}k^2$	if k is even,
$m = \frac{1}{2}(k \pm 1)$	$\frac{1}{4}(k^2 - 1)$	if k is odd.

Remark 1. It follows from formulas (9) that equation (8) can be satisfied only if the functions f_n (and the g_n) are linearly independent.

Remark 2. The bilinear functional equation (8) and its solutions (9) play an important role in the method of functional separation of variables for the nonlinear PDEs.

1.3. Nonlinear functional-differential equations reducible to the bilinear functional equation

Consider a functional-differential equation of the form

$$f_1(x)g_1(y) + f_2(x)g_2(y) + \cdots + f_k(x)g_k(y) = 0, \tag{10}$$

where the functionals $f_i(x)$ and $g_j(x)$ are prescribed and have the form, respectively,

$$\begin{aligned} f_j(x) &\equiv F_j\bigl(x, \varphi_1, \varphi_1', \varphi_1'', \ldots, \varphi_n, \varphi_n', \varphi_n''\bigr), \\ g_j(y) &\equiv G_j\bigl(y, \psi_1, \psi_1', \psi_1'', \ldots, \psi_m, \psi_m', \psi_m''\bigr). \end{aligned} \tag{11}$$

The functions $\varphi_i = \varphi_i(x)$ and $\psi_j = \psi_j(y)$, dependent on different arguments, are to be found. Here, for simplicity, an equation involving second derivatives is considered; in the general case, the right-hand sides of relations (11) will contain higher-order derivatives of $\varphi_i = \varphi_i(x)$ and $\psi_j = \psi_j(y)$.

The functional-differential equation (10)–(11) is solved by the splitting method. At the first stage, we treat equation (10) as a purely functional equation that depends on two variables x and y, where $f_i = f_i(x)$ and $g_j = g_j(y)$ are unknown quantities. Solutions of this equation are given by formulas (9). At the second stage, we successively substitute the $f_i(x)$ and $g_j(y)$ of (11) into all solutions (9) to obtain systems of ordinary differential equations* for the unknown functions $\varphi_j(x)$ and $\psi_j(y)$. Solving these systems, we get solutions of the functional-differential equation (10)–(11).

* Such systems are usually overdetermined.

Example 1. We will illustrate how functional-differential equations of the form (10) arise. Consider the nonlinear partial differential equation

$$\frac{\partial^2 w}{\partial x \partial y} + \left(\frac{\partial w}{\partial y}\right)^2 - w\frac{\partial^2 w}{\partial y^2} = \nu \frac{\partial^3 w}{\partial y^3}, \qquad (12)$$

which arises in hydrodynamics [2].

We look for exact solutions of the form

$$w = \varphi(x)\theta(y) + \psi(x). \qquad (13)$$

Substituting (13) into (12) yields

$$\varphi'_x\theta'_y - \varphi\psi\theta''_{yy} + \varphi^2\left[(\theta'_y)^2 - \theta\theta''_{yy}\right] - \nu\varphi\theta'''_{yyy} = 0.$$

This functional-differential equation can be reduced to a bilinear functional equation of the form (5) by setting

$$\begin{aligned} &f_1 = \varphi'_x, \quad f_2 = \varphi\psi, \quad f_3 = \varphi^2, \quad f_4 = \nu\varphi, \\ &g_1 = \theta'_y, \quad g_2 = -\theta''_{yy}, \quad g_3 = (\theta'_y)^2 - \theta\theta''_{yy}, \quad g_4 = -\theta'''_{yyy}. \end{aligned} \qquad (14)$$

On substituting expressions (14) into (6), we obtain the system of ordinary differential equations

$$\begin{aligned} &\varphi'_x = A_1\varphi^2 + A_2\nu\varphi, && \varphi\psi = A_3\varphi^2 + A_4\nu\varphi, \\ &(\theta'_y)^2 - \theta\theta''_{yy} = -A_1\theta'_y + A_3\theta''_{yy}, && \theta'''_{yyy} = A_2\theta'_y - A_4\theta''_{yy}. \end{aligned}$$

Substituting (14) into (7) yields another two systems of ordinary differential equations. For details, see the book [2, pages 708–709].

2. Nonlinear Functional Equations

2.1. Equations in two variables contaning complex argument of the form $z = \varphi(x) + \psi(t)$

Here, we discuss some functional equations two variables that arise most frequently in the functional separation of variables in nonlinear equations of mathematical physics.

1°. Consider a functional equation of the form

$$f(t) + g(x) + h(x)Q(z) + R(z) = 0, \qquad \text{where} \quad z = \varphi(x) + \psi(t). \qquad (15)$$

Here, one of the two functions $f(t)$ and $\psi(t)$ is prescribed and the other is assumed unknown, also one of the functions $g(x)$ and $\varphi(x)$ is prescribed and the other is unknown, and the functions $h(x)$, $Q(z)$, and $R(z)$ are assumed unknown.*

Differentiating the equation (15) with respect to x yields the two-argument equation

$$g'_x + h'_xQ + h\varphi'_xQ'_z + \varphi'_xR'_z = 0. \qquad (16)$$

Such equations were discussed in Subsection 1.1, their solutions are given by formulas (6) and (7). Hence, the following system of ordinary differential equations hold [see formulas (6)]:

$$\begin{aligned} g'_x &= A_1h\varphi'_x + A_2\varphi'_x, \\ h'_x &= A_3h\varphi'_x + A_4\varphi'_x, \\ Q'_z &= -A_1 - A_3Q, \\ R'_z &= -A_2 - A_4Q, \end{aligned} \qquad (17)$$

* In similar equations with a composite argument, it is assumed that $\varphi(x) \not\equiv \text{const}$ and $\psi(y) \not\equiv \text{const}$.

where $A_1, \ldots, A_4$ are arbitrary constants. By integrating system ODEs (17) and substituting the resulting solutions into the original functional equation, one obtains the results given below.

Case 1. If $A_3 = 0$ in (17), the corresponding solution of the functional equation is given by

$$\begin{aligned} f &= -\tfrac{1}{2}A_1A_4\psi^2 + (A_1B_1 + A_2 + A_4B_3)\psi - B_2 - B_1B_3 - B_4, \\ g &= \tfrac{1}{2}A_1A_4\varphi^2 + (A_1B_1 + A_2)\varphi + B_2, \\ h &= A_4\varphi + B_1, \\ Q &= -A_1z + B_3, \\ R &= \tfrac{1}{2}A_1A_4z^2 - (A_2 + A_4B_3)z + B_4, \end{aligned} \tag{18}$$

where the A_k and B_k are arbitrary constants and $\varphi = \varphi(x)$ and $\psi = \psi(t)$ are arbitrary functions.

Case 2. If $A_3 \neq 0$ in (17), the corresponding solution of the functional equation is

$$\begin{aligned} f &= -B_1B_3e^{-A_3\psi} + \left(A_2 - \frac{A_1A_4}{A_3}\right)\psi - B_2 - B_4 - \frac{A_1A_4}{A_3^2}, \\ g &= \frac{A_1B_1}{A_3}e^{A_3\varphi} + \left(A_2 - \frac{A_1A_4}{A_3}\right)\varphi + B_2, \\ h &= B_1e^{A_3\varphi} - \frac{A_4}{A_3}, \\ Q &= B_3e^{-A_3z} - \frac{A_1}{A_3}, \\ R &= \frac{A_4B_3}{A_3}e^{-A_3z} + \left(\frac{A_1A_4}{A_3} - A_2\right)z + B_4, \end{aligned} \tag{19}$$

where the A_k and B_k are arbitrary constants and $\varphi = \varphi(x)$ and $\psi = \psi(t)$ are arbitrary functions.

Case 3. In addition, the functional equation has the two degenerate solutions [obtainable with formulas (7)]:

$$f = A_1\psi + B_1, \quad g = A_1\varphi + B_2, \quad h = A_2, \quad R = -A_1z - A_2Q - B_1 - B_2, \tag{20}$$

where $\varphi = \varphi(x)$, $\psi = \psi(t)$, and $Q = Q(z)$ are arbitrary functions, A_1, A_2, B_1, and B_2 are arbitrary constants, and

$$f = A_1\psi + B_1, \quad g = A_1\varphi + A_2h + B_2, \quad Q = -A_2, \quad R = -A_1z - B_1 - B_2, \tag{21}$$

where $\varphi = \varphi(x)$, $\psi = \psi(t)$, and $h = h(x)$ are arbitrary functions, A_1, A_2, B_1, and B_2 are arbitrary constants.

Example 2. We will now illustrate how functional equations of the form (15) arise. Consider the nonlinear heat equation with a source

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \mathcal{F}(w). \tag{22}$$

We look for exact solutions of the form

$$w = w(z), \quad z = \varphi(x) + \psi(t). \tag{23}$$

Substituting (23) into (22) and dividing by w'_z yields the functional-differential equation

$$\psi'_t = \varphi''_{xx} + (\varphi'_x)^2\frac{w''_{zz}}{w'_z} + \frac{\mathcal{F}(w(z))}{w'_z}.$$

We rewrite it as the functional equation (15) in which

$$f(t) = -\psi'_t, \quad g(x) = \varphi''_{xx}, \quad h(x) = (\varphi'_x)^2, \quad Q(z) = w''_{zz}/w'_z, \quad R(z) = f(w(z))/w'_z. \tag{24}$$

Substituting (24) into (18)–(21) yields systems of ordinary differential equations for determining $\varphi(x)$, $\psi(t)$, $w(z)$, and $\mathcal{F}(w)$. For details, see the book [2, pages 724–725].

2°. Consider a functional equation of the form

$$f(t) + g(x)Q(z) + h(x)R(z) = 0, \qquad \text{where} \quad z = \varphi(x) + \psi(t). \tag{25}$$

Differentiating with (25) respect to x yields the two-argument functional-differential equation

$$g'_x Q + g\varphi'_x Q'_z + h'_x R + h\varphi'_x R'_z = 0, \tag{26}$$

which coincides with equation (5), up to notation.

Nondegenerate case. Equation (26) can be solved using formulas (6)–(7). In this way, we arrive at the system of ordinary differential equations

$$\begin{aligned} g'_x &= (A_1 g + A_2 h)\varphi'_x, \\ h'_x &= (A_3 g + A_4 h)\varphi'_x, \\ Q'_z &= -A_1 Q - A_3 R, \\ R'_z &= -A_2 Q - A_4 R, \end{aligned} \tag{27}$$

where $A_1, \ldots, A_4$ are arbitrary constants.

The solution of equation (27) is given by

$$\begin{aligned} g(x) &= A_2 B_1 e^{k_1\varphi} + A_2 B_2 e^{k_2\varphi}, \\ h(x) &= (k_1 - A_1)B_1 e^{k_1\varphi} + (k_2 - A_1)B_2 e^{k_2\varphi}, \\ Q(z) &= A_3 B_3 e^{-k_1 z} + A_3 B_4 e^{-k_2 z}, \\ R(z) &= (k_1 - A_1)B_3 e^{-k_1 z} + (k_2 - A_1)B_4 e^{-k_2 z}, \end{aligned} \tag{28}$$

where $B_1, \ldots, B_4$ are arbitrary constants and k_1 and k_2 are roots of the quadratic equation

$$(k - A_1)(k - A_4) - A_2 A_3 = 0. \tag{29}$$

In the degenerate case $k_1 = k_2$, the terms $e^{k_2\varphi}$ and $e^{-k_2 z}$ in (28) must be replaced by $\varphi e^{k_1\varphi}$ and $z e^{-k_1 z}$, respectively. In the case of purely imaginary or complex roots, one should extract the real (or imaginary) part of the roots in solution (28).

On substituting (28) into the original functional equation, one obtains conditions that must be met by the free coefficients and identifies the function $f(t)$, specifically,

$$\begin{aligned} B_2 = B_4 = 0 &\quad\Longrightarrow\quad f(t) = [A_2A_3 + (k_1 - A_1)^2]B_1B_3 e^{-k_1\psi}, \\ B_1 = B_3 = 0 &\quad\Longrightarrow\quad f(t) = [A_2A_3 + (k_2 - A_1)^2]B_2B_4 e^{-k_2\psi}, \\ A_1 = 0 &\quad\Longrightarrow\quad f(t) = (A_2A_3 + k_1^2)B_1B_3 e^{-k_1\psi} + (A_2A_3 + k_2^2)B_2B_4 e^{-k_2\psi}. \end{aligned} \tag{30}$$

Solution (28), (30) involves arbitrary functions $\varphi = \varphi(x)$ and $\psi = \psi(t)$.

Degenerate case. In addition, the functional equation has two degenerate solutions [obtainable with formulas (7)],

$$f = B_1B_2 e^{A_1\psi}, \quad g = A_2B_1 e^{-A_1\varphi}, \quad h = B_1 e^{-A_1\varphi}, \quad R = -B_2 e^{A_1 z} - A_2 Q,$$

where $\varphi = \varphi(x)$, $\psi = \psi(t)$, and $Q = Q(z)$ are arbitrary functions, A_1, A_2, B_1, and B_2 are arbitrary constants; and

$$f = B_1B_2 e^{A_1\psi}, \quad h = -B_1 e^{-A_1\varphi} - A_2 g, \quad Q = A_2B_2 e^{A_1 z}, \quad R = B_2 e^{A_1 z},$$

where $\varphi = \varphi(x)$, $\psi = \psi(t)$, and $g = g(x)$ are arbitrary functions, and A_1, A_2, B_1, and B_2 are arbitrary constants.

3°. A more general functional equation of the form

$$f(t) + g_1(x)Q_1(z) + \ldots + g_n(x)Q_n(z) = 0, \qquad \text{where} \quad z = \varphi(x) + \psi(t) \tag{31}$$

can be reduced, also by differentiating with respect to x, to a functional-differential equation that may be treated as a bilinear functional equation of the form (8). One can first use (9) to write out its solution in the form of a system of ODEs and then find solutions to the original equation (31).

4°. Consider a functional equation of the form

$$f_1(t)g_1(x)+\ldots+f_m(t)g_m(x)+h_1(x)Q_1(z)+\ldots+h_n(x)Q_n(z)=0, \qquad \text{where} \quad z=\varphi(x)+\psi(t). \tag{32}$$

Assume that $g_m(x) \not\equiv 0$. We divide equation (32) by $g_m(x)$ and differentiate with respect to x. This results in the equation

$$f_1(t)\bar{g}_1(x) + \cdots + f_{m-1}(t)\bar{g}_{m-1}(x) + \sum_{i=1}^{2n} s_i(x)R_i(z) = 0$$

with fewer functions $f_i(t)$. By repeating the above procedure, one can eventually eliminate all functions $f_i(t)$ and obtain a two-variable functional-differential equation of the form (10)–(11), which is further reduced to the standard bilinear functional equation.

2.2. Equations in two variables containing complex argument of the form $z = \varphi(t)\theta(x) + \psi(t)$

Consider a functional equation of the form

$$[\alpha_1(t)\theta(x) + \beta_1(t)]R_1(z) + \cdots + [\alpha_n(t)\theta(x) + \beta_n(t)]R_n(z) = 0, \qquad z = \varphi(t)\theta(x) + \psi(t). \tag{33}$$

Passing on from the variables x and t to the new variables z and t [the function θ is substituted by $(z - \psi)/\varphi$], one arrives at a bilinear equation of the form (8):

$$\sum_{i=1}^{n} \alpha_i(t)zR_i(z) + \sum_{i=1}^{n}[\varphi(t)\beta_i(t) - \psi(t)\alpha_i(t)]R_i(z) = 0.$$

Example 3. We will now show how functional equations of the form (33) arise. Consider again the heat equation with a nonlinear source (22). We look for traveling-wave solutions of the form

$$w = w(z), \quad z = \varphi(t)x + \psi(t). \tag{34}$$

The functions $w(z)$, $\varphi(t)$, $\psi(t)$, and $\mathcal{F}(w)$ in (34) are to be determined.

On substituting (34) into (22) and on dividing by w'_z, we have

$$\varphi'_t x + \psi'_t = \varphi^2 \frac{w''_{zz}}{w'_z} + \frac{\mathcal{F}(w)}{w'_z}. \tag{35}$$

This functional-differential equation can be treated as a functional equation (33) with

$$n = 4, \quad R_1(z) = R_2(z) = 1, \quad R_3(z) = \frac{w''_{zz}}{w'_z}, \quad R_4(z) = \frac{\mathcal{F}(w)}{w'_z},$$
$$\theta(x) = x, \quad \alpha_1(t) = \varphi'_t, \quad \alpha_2(t) = \alpha_3(t) = \alpha_4(t) = 0,$$
$$\beta_1(t) = 0, \quad \beta_2(t) = \psi'_t(t), \quad \beta_3(t) = -\varphi^2(t), \quad \beta_4(t) = -1.$$

For details on solutions (34) to equation (22), see the book [2, pages 714–715].

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