ON NEW DIRECTIONS OF GROUP ANALISYS
OF ORDINARY DIFFERENTIAL EQUATIONS

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In this paper we elucidated the activity of the Saint-Petersburg school of group analysis of ordinary differential equations (ODE) for the last decade. This step was preceded by creation of bases of the theory and the techniques of an application of discrete group analysis [1,2]. Let's recall some statements of this direction.

1. Discrete group analysis. It is well known [1,2] that a discrete metagroup of transformations (DMT) of an ODE class D is called any set of transformations G such that

1) The set G contains a unity E (the identity transformation and also transformations equivalent to it up to the accuracy of insignificant parameters).
2) The set G contains together with any element $g \in G$ also the inverse element $g^{-1} \in G$.
3) The action of any element $g \in G$ is closed on the class D.

Clearly Lie groups of transformations admitted by the class D and continuous groups of equivalence on parameters defining the class D satisfy this definition. Therefore in discrete group analysis transformations of continuous groups of equivalence are usually identified with the identity transformation E.

It is quite justified as there exists a well developed infinitesimal techniques of searching of finite transformations and their associated admissible operators for search of continuous groups [9,11]. Thus discrete group analysis study DMT with discrete orbits. Any DMT admitted by a class ODE divides it into equivalence subclasses with respect to this metagroup. If in some subclass there is an element with known solution then it is possible to find solutions of all remaining equations from the same subclass using the known transformations – generators of the DMT. In other words DMT "multiplies" solvable equations as well Lie group admitted by some equation "multiplies" its solutions on the basis of some known one (if this solution is not invariant with respect to this group).

Methods of group analysis were used for search of tens before unknown integrated in quadratures or in special functions of the Abel equations of the second kind

$$yy' - y = R(x),$$

the generalized Emden-Fowler equation

$$y'' = Ax^n y^m y'^l,$$

the third order equation

$$y''' = Ax^\alpha y^\beta y'^\gamma y''^\delta$$

which special case is a boundary layer equation and a number of their generalizations [13]. Nevertheless the problem of a proof of a maximality of DMT admitted by a class of ODE is unsolved. The defining system for search of a DMT is an overdetermined system of nonlinear partial differential equations. The obtained DMT will be maximality if all solutions of the system are found. Note that in practice it has not always been possible to construct all solutions. Discrete group analysis differs essentially from classical group analysis in this feature. Recall that in classical group analysis a defining system is linear and the set of its solutions forms a linear vector space. However Ja.V.Delyukova [4] proved that a nonlinear defining system for search of DMT "inherits" continuous symmetries of the ODE class. Therefore a oneparameter Lie group admitted by the ODE class allows to reduce the dimension of the defining system on a unity. In particular a defining system for search of point DMTs becomes an overdetermined system of ordinary differential equations and, as a rule, is solved easily. It has been this method which is use for the proof of maximality of DMTs admitted by the generalized Emden-Fowler equation (1).

This example of "interpenetration" of two directions of group analysis lead to the investigation of a wide generalizations of the concept of an admitted group. For example, it is always possible to find a pair of invariants – the universal invariant I_0 and the first differential invariant I_1 of the given (point) operator

$$X = \xi(x, y)\partial_x + \eta(x, y)\partial_y .$$

The converse is not true as there does not always exist a point operator (or in general a local operator) having the given invariants $I_0(x, y)$ and $I_1(x, y, y')$ [9,14]. Therefore it is necessary to introduce a definition of a universal operator admitted by an arbitrary ordinary differential equation. The advantages of such approach are a uniform description of any continuous symmetries of equations. This description allows a priori to extend bases principles to wider classes of investigated objects in particular to functional differential equations [12].

2. Formal operators. We consider nth order differential equations solved with respect to higher derivative

$$y^{(n)} = F(x, y, y', \dots, y^{(n-1)}) \tag{2}$$

and specifying n-dimensional smooth (differentiable) manifolds $[F]$.

Let $X : \mathbf{C}^\infty \to \mathbf{C}^\infty$ be a linear (nonlocal) operator of the form

$$X = \Phi(x, y, y', \dots)\frac{\partial}{\partial y}, \tag{3}$$

where Φ is a coordinate of canonical operator and depends generally on any order derivatives.

Definition 1. Operator (3) is said to be admitted by equation (2) if

$$\underset{n}{X}[y^{(n)} - F(x, y, y', \dots, y^{(n-1)})]\Big|_{[F]} = 0, \tag{4}$$

where

$$\underset{n}{X} = \sum_{k=0}^{n} D_x^k[\Phi]\frac{\partial}{\partial y^{(k)}}, \tag{5}$$

D_x is the total derivative operator

$$D_x = \frac{\partial}{\partial x} + \sum_{k=0}^{\infty} y^{(k+1)}\frac{\partial}{\partial y^{(k)}} .$$

Expression (5) is called an nth prolongation of operator (3). Symbol $[F]$ means that equality (4) is fulfilled on the manifold $[F]$. In this case one also says that (4) holds on the set of solutions of (2) or by virtue of equation (2).

Definition 2. The function $H(x, y, y', \dots, y^{(k)})$, $\partial H / \partial y^{(k)} \neq 0$ is called a differential invariant of order k or a kth differential invariant of operator (3) if H satisfies the equation

$$\underset{k}{X}[H] = 0. \tag{6}$$

3. Existence of a admissible operator.

Theorem 1. An arbitrary nth order differential equation (2) determining a smooth manifold admits at least n different formal operators (3).

Proof. Actually we prove a stronger statement. Namely, any equation (2) is a differential invariant of order n for at least n different (non-zero) formal operators (3). Condition

(4) can only lead to extension of the set of admissible formal operators. According to Definition 2 the differential invariant of the nth order satisfies the equation

$$\sum_{k=0}^{n} D_x^k[\Phi]\frac{\partial H}{\partial y^{(k)}} = 0. \tag{7}$$

Let us substitute the expression

$$H = y^{(n)} - F(x, y, y', \dots, y^{(n-1)})$$

for H in (7) and consider equation (7) as an equation for the unknown canonical coordinate Φ. This equation is an nth order linear equation with variable coefficients in total derivatives and its general solution is

$$\Phi = C_1\Phi_1 + C_2\Phi_2 + \dots + C_n\Phi_n, \tag{8}$$

where the Φ_k, $k = \overline{1,n}$ are n linearly independent functions depending on arbitrary derivatives of y because, in general, solutions of such equations do not admit representation in terms of elementary functions and even in quadratures (however the latter are integral expressions and are representable in local variables, but only as infinite series).

The functions Φ_k, $k = \overline{1,n}$ satisfy equation (7) identically, i.e., on an arbitrary manifold; consequently they satisfy condition (4). The theorem is proved.

Theorem 2 [3]. Any first order equation

$$y' = F(x, y) \tag{9}$$

admits a unique formal operator (up to identity transformations on the manifold) with a coordinate that does not depend explicitly on derivatives

$$X = \exp\left(\int \frac{\partial F}{\partial y} dx\right)\partial_y. \tag{10}$$

Proof. For equation (9) condition (4) reads

$$\underset{1}{X}[y' - F(x.y)]\big|_{[F]} = 0,$$

i.e.,

$$D_x[\Phi] - \frac{\partial F}{\partial y}\Phi = 0,$$

whence

$$\Phi = \exp\left(\int \frac{\partial F}{\partial y} dx\right).$$

The theorem is proved.

Definition 3. A formal operator of the form

$$X = \exp\left[\int \zeta(x, y, y', \dots, y^{(k)})dx\right]\partial_y \tag{10}$$

is called an exponential nonlocal operator (ENO) in ultracanonical form. It can be shown that, in general, the coordinate of the formal operator (10) depends on the derivatives of y of any order.

Theorem 3. Let $z = z(x, y, y', \dots, y^{(k)})$ be a differential invariant of order k for the operator (2). Then $z' = D_x[z]$ is a differential invariant of order $(k+1)$.

Proof. By Definition 2, z satisfies the equation

$$\Phi\frac{\partial z}{\partial y} + D_x[\Phi]\frac{\partial z}{\partial y'} + \dots + D_x^k[\Phi]\frac{\partial z}{\partial y^{(k)}} = 0.$$

Calculating the total derivative of both parts above yields. Then we write out this expression, taking into account that the operators D_x and ∂_x commute, as well as D_x and ∂_y, and

$$\left\lfloor D_x, \partial_{y^{(i)}} \right\rfloor = -\partial_{y^{(i-1)}}$$

for $i > 0$. Thus, all derivatives of z disappear except for the last one, i.e.,

$$\Phi \frac{\partial z'}{\partial y} + D_x[\Phi] \frac{\partial z'}{\partial y'} + \dots D_x^k[\Phi] \frac{\partial z'}{\partial y^{(k)}} + D_x^{k+1}[\Phi] \frac{\partial z}{\partial y^{(k)}} = 0 .$$

From the condition of the theorem we know that $z = z(x, y, y', \dots, y^{(k)})$; therefore,

$$z' = D_x[z] = \frac{\partial z}{\partial x} + y' \frac{\partial z}{\partial y} + y'' \frac{\partial z}{\partial y'} + \dots + y^{(k+1)} \frac{\partial z}{\partial y^{(k)}} ,$$

We obtain $\frac{\partial z}{\partial y^{(k)}} = \frac{\partial z'}{\partial y^{(k+1)}}$, because $D_x[z]$ depends on $y^{(k+1)}$ linearly. Therefore, z' satisfies the equation that determines a differential invariant of the $(k + 1)$st order.

Remark. From Theorem 3 it follows that an differential invariant of any higher order (of order exceeding 1) can be found by using a differential invariant of the first order, and thus it depends linearly on the highest order derivative.

4. Factorization theorems.

Theorem 4 [3,5]. An nth order differential equation (2) specifying a smooth manifold can be factored to yield a specific system of the form

$$\begin{cases} z^{(n-1)} = G(x, z, z', \dots, z^{(n-2)}), \\ z = H(x, y, y'), \end{cases} \tag{11}$$

if and only if equation (2) admits the ENO

$$X = \exp\left[-\int \frac{H_y}{H_{y'}} dx \right] \partial_y \tag{12}$$

Proof. 1. Suppose equation (1) admits the ENO (12). We write down the equation for the differential invariant of order n,

$$\Phi \frac{\partial z_n}{\partial y} + D_x[\Phi] \frac{\partial z_n}{\partial y'} + \dots + D_x^n[\Phi] \frac{\partial z_n}{\partial y^{(n)}} = 0 ,$$

where $\Phi = \exp\left[-\int \frac{H_y}{H_{y'}} dx \right]$, and introduce the new variable $t = y^{(n)} - F(x, y, y', \dots, y^{(n-1)})$, obtaining

$$\Phi \frac{\partial z_n}{\partial y} + D_x[\Phi] \frac{\partial z_n}{\partial y'} + \dots + D_x^{n-1}[\Phi] \frac{\partial z_n}{\partial y^{(n-1)}} -$$

$$- \frac{\partial z_n}{\partial t} \left\{ \Phi \frac{\partial F}{\partial y} + D_x[\Phi] \frac{\partial F}{\partial y'} + \dots + D_x^{n-1}[\Phi] \frac{\partial F}{\partial y^{(n-1)}} - D_x^n[\Phi] \right\} = 0.$$

Since the expression in braces vanishes on the manifold $[F]$ by the definition of an admissible operator, we have

$$\Phi \frac{\partial z_n}{\partial y} + D_x[\Phi] \frac{\partial z_n}{\partial y'} + \dots + D_x^{n-1}[\Phi] \frac{\partial z_n}{\partial y^{(n-1)}} = 0.$$

By Definition 2, the integral basis of this equation is the set of all lower invariants, i.e.,

$$z_n = \overline{G}(x, t, z_1, z_2, \ldots, z_{n-1}),$$
and since the manifold $[F]$ under consideration is given by the relation $t = 0$, we obtain
$$z_n = G(x, z_1, z_2, \ldots, z_{n-1}),$$
with $G(x, z_1, z_2, \ldots, z_{n-1}) = \overline{G}(x, 0, z_1, z_2, \ldots, z_{n-1})$.

Substituting, in accordance with Theorem 3, $z_i = z^{(i-1)}, i = \overline{2, n},$ where $z = z_1 = H(x, y, y')$ is a first order differential invariant, we arrive at the statement of the theorem.

2. Suppose equation (2) can be written as in (11). Since $z = H(x, y, y')$ is a first order differential invariant of the operator (12), we deduce easily that this operator is admitted by equation (2).

Remark 1. Any differential equation admitting an operator can be written in invariants of this operator. It is assumed a priori that an admissible operator has a differential invariant of the first order.

Remark 2. Another version of Theorem 4 can be found in the paper [10] by P.Olver. However, in [10] an absolutely wrong inference was made, namely, there it is claimed that an exponential vector field always results in reduction of the order of the equation.

Lemma. The dimension of the basis of invariants of the formal operator (3) admitted by equation (2), on the n-dimensional smooth manifold (2) is equal to $n - k + 1$, provided that $z_k = H(x, y, y', \ldots, y^{(k)})$ is the lowest differential invariant of order k; moreover, the components of the basis $\mathbf{z}$ can be chosen in such a way that
$$\mathbf{z} = (x, z_k, z_k', \ldots, z^{(n-k-1)}).$$

Proof. On the manifold (2) the invariant $\mathbf{z}$ of the formal operator (3) satisfies the equation
$$\Phi\frac{\partial z}{\partial y} + D_x[\Phi]\frac{\partial z}{\partial y'} + \ldots + D_x^n[\Phi]\frac{\partial z}{\partial y^{(n)}} = 0$$

As in the proof of Theorem 4, we introduce the new variable $t = y^{(n)} - F(x, y, y', \ldots, y^{(n-1)})$ in place of $y^{(n)}$ to obtain
$$\Phi\frac{\partial z}{\partial y} + D_x[\Phi]\frac{\partial z}{\partial y'} + \ldots + D_x^n[\Phi]\frac{\partial z}{\partial y^{(n)}} = 0. \tag{13}$$
Let $z_k = H(x, y, y', \ldots, y^k)$ be the lowest invariant of the formal operator (3) on the manifold (2). Then, by Theorem 3, the functions $z_{k+s} = z_k^{(s)}$, with $s = \overline{1, n-k-1}$ are also invariants of (3) and satisfy (13). If we substitute $p_i = z_r$, $r = \overline{k, n-1}$ for $y^{(k)}, y^{(k+1)}, \ldots, y^{(n-1)}$, then equation (13) can be written as
$$\Phi\frac{\partial z}{\partial y} + D_x[\Phi]\frac{\partial z}{\partial y'} + \ldots + D_x^{k-1}[\Phi]\frac{\partial z}{\partial y^{(k-1)}} = 0. \tag{14}$$

Since the lowest invariant is a differential invariant of order k, the integral basis of equation (14) consists of a single element x, and on the n-dimensional smooth manifold (2) the basis of invariants of the formal operator (3) looks like this: $\mathbf{z} = (x, z_k, z_k', \ldots, z^{(n-k-1)})$.

Theorem 5 [15]. Any nth order differential equation (2) determining a smooth manifold can be factored to a specific system of the form

$$
\begin{cases} z^{(n-k)} = G(x, z, z', \dots, z^{n-k-1}), & \\ z = H(x, y, y', \dots, y^{(k)}), & \dfrac{\partial z}{\partial y^{(k)}} \neq 0, \end{cases} \tag{15}
$$

if equation (2) admits a formal operator (3) for which $H(x, y, y', \dots, y^{(k)})$ is a lowest differential invariant on the manifold (2).

If equation (2) admits factorization to the system (15), then it admits a formal operator (3) for which $H(x, y, y', \dots, y^{(k)})$ is a differential invariant of order k on the manifold (2).

Proof. 1. Suppose equation (2) admits the nonlocal operator (3). Then on the n-dimensional smooth manifold (2) the nth order differential invariant z_n of the operator (3) satisfies the equation

$$
\Phi \frac{\partial z_n}{\partial y} + D_x[\Phi]\frac{\partial z_n}{\partial y'} + \dots + D_x^n[\Phi]\frac{\partial z_n}{\partial y^{(n)}} = 0 .
$$

Putting $t = y^{(n)} - F(x, y, y', \dots, y^{(n-1)})$, on the manifold $[F]$ we obtain the equation

$$
\Phi \frac{\partial z_n}{\partial y} + D_x[\Phi]\frac{\partial z_n}{\partial y'} + \dots + D_x^{n-1}[\Phi]\frac{\partial z_n}{\partial y^{(n-1)}} = 0.
$$

By the lemma, the integral basis of this equation is the set of all lower invariants. Therefore, the invariant z_n can be written as

$$
z_n = \overline{G}(x, t, z_k, z_{k+1}, \dots, z_{n-1}) .
$$

Since the manifold $[F]$ is question is given by $t = 0$, we have

$$
z_n = G(x, z_k, z_{k+1}, \dots, z_{n-1}) ,
$$

with

$$
G(x, z_k, z_{k+1}, \dots, z_{n-1}) = \overline{G}(x, 0, z_k, z_{k+1}, \dots, z_{n-1}) .
$$

In accordance with the lemma, we substitute $z_i = z^{(i-k)}$, $i = \overline{k+1, n}$, where $z = z_k = H(x, y, y', \dots, y^{(k)})$ is a differential invariant of order k; this yields the required factorization.

2. For the proof of the converse statement, we construct a formal operator having the differential invariant $z = H(x, y, y', \dots, y^{(k)})$ on the manifold (2). For this we need to solve the kth order linear equation in total derivatives

$$
\Phi \frac{\partial z}{\partial y} + D_x[\Phi]\frac{\partial z}{\partial y'} + \dots + D_x^k[\Phi]\frac{\partial z}{\partial y^{(k)}} = 0 .
$$

This equation has k linearly independent solutions. On the manifold (2) the number of such solutions can only be greater. Being differential sequels of the invariant z, functions $z', \dots, z^{(n-k-1)}$ are also invariants of the operators found above. So, equation (2) admits any of the constructed formal operators (3).

Remark. The requirement that $z = H(x, y, y', \dots, y^{(k)})$ be the lowest invariant of the operator (3) is not necessary, because equation (2) may happen to be written in a form involving not all invariants of the admissible operator.

5. Classical and nonclassical reductions. It is easily seen that Theorems 4 and 5 generalize all known ways for reduction of order of ordinary differential equations that are based on properties of continuous symmetries.

1. Point symmetries. The point symmetry operator in the canonical form

$$X = [\eta(x,y) - y'\xi(x,y)]\partial_y$$

is equivalent to the operator (10) with

$$\zeta = \frac{\eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2 - \xi y''}{\eta - y'\xi}$$

If order of an equation is equal to 2, then the second derivative is replaced by the right-hand side of the equation, and the function ζ takes the form

$$\zeta = \frac{\eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2 - \xi F(x,y,y')}{\eta - y'\xi}$$

It should be mentioned that the symmetry defined by (10) is local if ζdx represents a total differential.

2. The first integrals. The symmetry defined by the formal operator (3) is the first integral (the preservation law) if the system (15) is of the form

$$\begin{cases} z' = 0, \\ z = H(x, y, y', \dots, y^{(n-1)}). \end{cases} \tag{16}$$

A simple procedure allows us to find also a conjugate Lie symmetry (if this symmetry is variational, i.e., Noetherian). Consider the following simple example. Obviously, the equation

$$y'' = Ay^m y'^l \tag{17}$$

has the first integral

$$P = \frac{1}{2-l} y'^{2-l} - \frac{A}{m+1} y^{m+1}. \tag{18}$$

The corresponding operator (10) takes the form

$$X = \exp\left(A \int y^m y'^{1-l} dx\right)\partial_y .$$

Using (17), on the set of solutions of the initial equation we can write

$$X = \exp\left(\int \frac{y''}{y'} dx\right)\partial_y = y'\partial_y .$$

Thus, on the manifold the operator corresponding to the first integral (18) turns out to be equivalent to the translation operator ∂_x, which is a variational symmetry of the equation (17).

3. Nonclassical symmetries and Ermakov equation. Clearly, if ζdx fails to represent a total differential, then, generally speaking, the operator (10) determines a nonlocal symmetry. In this case factorization and, ultimately, integration of equations are not predicted by the classical algorithm of S.Lie. Nevertheless, among equations possessing a similar symmetry there are ones having more than 100 years history. One of these is the Ermakov equation.

Theorem 6 [5]. The equation

$$y'' = f(x)y + g'(x)y^{-1} - [g^2(x)]y^{-3} \tag{19}$$

where $f(x)$ and $g(x)$ are arbitrary functions is the only (up to the Kummer–Liouville equivalence transformations) equation of the form $y'' = F(x,y)$ admitting an ENO

$$X = \exp\left(\int \zeta(x,y)dx\right)\eta(x,y)\partial_y ,$$

and it can be factored to the system

$$\begin{cases} z' + z^2 = f(x), \\ y' = z(x,C)y + g(x)y^{-1}. \end{cases}$$

Proof. The second prolongation of the operator X has the form

$$\underset{2}{X} = \exp\left(\int \zeta(x,y)dx\right)\{\eta\partial_y + (\eta_x + \eta_y y' + \eta\zeta)\partial_{y'} +$$

$$+[\eta_{xx} + 2\zeta\eta_x + \eta\zeta_x + \zeta^2\eta + (2\eta_{xy} + 2\zeta\eta_y + \eta\zeta_y)y' + \eta_{yy}y'^2 + \eta_y y'']\partial_{y''}\}.$$

We apply it to the equation $y'' = F(x,y)$ and replace y'' with $F(x,y)$ everywhere; this results in the invariance condition

$$\eta_{xx} + 2\zeta\eta_x + \eta\zeta_x + \zeta^2\eta + (2\eta_{xy} + 2\zeta\eta_y + \eta\zeta_y)y' + \eta_{yy}y'^2 + \eta_y F(x,y) - \eta F_y = 0.$$

Splitting this identify by degrees of the "independent" variable y', we obtain the system of three equations for the components η, ζ of the canonical coordinate ENO:

$$\begin{cases} \eta_{yy} = 0, \\ 2\eta_{xy} + 2\zeta\eta_y + \eta\zeta_y = 0, \\ \eta_{xx} + 2\zeta\eta_x + \eta\zeta_x + \zeta^2\eta + \eta_y F - \eta F_y = 0. \end{cases}$$

The first equation implies that $\eta = a(x)y + b(x)$, and from the second equation we see that

$$\zeta = -\frac{aa'y^2 + 2a'by + c(x)}{(ay+b)^2},$$

where $a(x)$, $b(x)$, $c(x)$ are arbitrary functions. The third equation can be viewed as a linear ordinary first order differential equation for the unknown function $F(x,y)$:

$$\frac{dF}{dy} - \frac{\eta_y}{\eta}F = \frac{1}{\eta}(\eta_{xx} + 2\zeta\eta_x + \eta\zeta_x + \eta\zeta^2).$$

Hence, after the substitution of η and ζ, we get

$$F(x,y) = (ay+b)f(x) + \frac{(aa'' - 2a'^2)b - (ab'' - 2a'b')a}{a^3} -$$

$$-\frac{(aa'' - 3a'^2)b^2 + 2aa'bb' - (ac' - 2a'c)a}{2a^3(ay+b)} - \frac{(a'b - ac)^2}{4a^3(ay+b)^3},$$

where $f(x)$ is an arbitrary function.

Let us calculate a differential invariant of the operator X:

$$u = \frac{y'}{ay+b} - \frac{a'b - ab'}{a^2(ay+b)} + \frac{a'b^2 - ac}{2a^2(ay+b)^2}.$$

Now, having found du/dx, we can find y''. Since the type of the function $F(x,y)$ is already known, we arrive at the following factorization of the initial equation:

$$\begin{cases} \dfrac{du}{dx} + au^2 + \dfrac{a'}{a}u = f, \\ (ay+b)\dfrac{dy}{dx} = (ay+b)^2 u + \dfrac{a'b - ab'}{a^2}(ay+b) - \dfrac{a'b^2 - ac}{2a^2}. \end{cases}$$

The familiar equivalence transformation $ay + b \to y$, accompanied by appropriate replacement of the independent variable and renaming leads to the system

$$\begin{cases} \dfrac{du}{dx} + u^2 = f(x), \\ \dfrac{dy}{dx} = uy + g(x)y^{-1}, \end{cases}$$

and the initial equation takes the form (29). This proves the theorem.

Thus the first equation of the system turns out to be Riccati, and it can be solved independently of the second equation. In the second line we have the Bernoulli equation, which admits integration in quadratures for arbitrary coefficient $z(x, C)$, i.e., for a general solution of the first equation.

Corollary. Equation (19) is a direct generalization of the Ermakov equation

$$y'' = f(x)y + Ay^{-3}$$

reshapes back to it for g = const, and has all properties of it except for admitting a 3-dimensional Lie algebra. For arbitrary f and g equation (19) admits no point group at all (we exclude the trivial group). The general solution of the above equation is

$$y = \left[C_1 + 2\int \frac{g(x)dx}{u^2}\right]^{1/2},$$

where u is a general solution of the "shortened" linear equation $u'' = f(x)u$. Surely, one of the three arbitrary constants in the solution is not independent. Equation (19) admits one-dimensional (!) point algebra only (!) if

$$f = \frac{1}{2}\frac{g'''}{g'} + \frac{3}{4}\left(\frac{g''}{g'}\right)^2 - \frac{1}{2}\frac{g''}{g} + \frac{k}{4}\left(\frac{g'}{g}\right)^2,$$

with is an arbitrary constant k a 3-dimensional algebra is admitted only by the classical Ermakov equation (i.e., in the case where g = const).

The mechanism described above allows us to solve a problem posed more than 100 years ago, namely, the problem of constructing analogs of the Ermakov equation for any order. Largely, the previous attempts were based on properties of an admissible point 3-dimensional Lie algebra, so that the starting point for the search was the equation

$$y^{(n)} = Ay^{\frac{1+n}{1-n}},$$

which admits, like the Ermakov equation, a 3-dimensional unsolvable Lie algebra (the latter is isomorphic to $\mathbf{SL}(2,\mathbf{R})$ and is variational for n even). Evidently, this search could not be a success, because the properties of the Ermakov equation result from extremely nonlocal symmetry. In the case of the third order, the analog of the Ermakov equation is of the form

$$y''' + f(x)y' + g(x)y = (n+2)h(x)y^{n-1}y'' + (n-1)(n+1)hy^{n-2}y'^2 +$$
$$+\left[(2n+1)h' - 3nh^2y^{n-1}\right]y^{n-1}y' + (h'' + fh)y^n - 3hh'y^{2n-1} + h^3y^{3n-2}.$$

6. Example. The equation

$$yy''' + y''^2 - y'y'' - F(x)y^2 = 0 \qquad (20)$$

admits two operators

$$X_1 = y\partial_y, \qquad X_2 = \left(y\int y^{-2}dx\right)\partial_y. \qquad (21)$$

Their canonical coordinates satisfy the equation in total derivatives

$$yD_x^2[\Phi] - y''\Phi = 0$$

The first operator in (21) (the operator X_1) is a usual point operator of dilation along the y-axis (equation (20) is homogeneous), and the second operator X_2 is nonlocal, but it is not an ENO.

Equation (20) admits factorization to the system

$$\begin{cases} z' + z^2 = F(x), \\ y'' - yz = 0. \end{cases}$$

We show that z is the lowest differential invariant of the second operator in (21). We calculate the first prolongation of the operator X_2 and try to find a differential invariant depending on the first derivative. By Definition 2, this invariant must satisfy the equation

$$\left(y\int y^{-2}dx\right)\frac{\partial I_1}{\partial y}+\left(y^{-1}+y'\int y^{-2}dx\right)\frac{\partial I_1}{\partial y'}=0. \tag{22}$$

Since the nonlocal expression $\int y^{-2}dx$ depends on derivatives of any order, it can be regarded as an independent variable. Then equation (22) splits up to the system

$$\begin{cases} y\dfrac{\partial I_1}{\partial y}+y'\dfrac{\partial I_1}{\partial y'}=0, \\ y^{-1}\dfrac{\partial I_1}{\partial y'}=0. \end{cases}$$

The second equation of the above system implies that $\partial I_1/\partial y'=0$.

Finally we make some remarks about search algorithms for admitted formal operators. The procedure of determination of a canonical coordinate is the same as for the classical symmetries. Having written the invariance condition, we split it by degrees of “independent” variables to obtain a overdetermined system of equations. If this system is compatible, then we can find functions η and ζ. However, the formal operator can be written in different forms, and the number of such forms exceeds the number of forms for the point and contact operators. If we seek the formal operator in the general form, then, as a rule, we arrive at a trivial result (like the “empty” operator $\xi\partial/\partial_x+\xi F\partial/\partial_y$ admitted by any first order equation $y'=F(x,y)$ for any $\xi=\xi(x,y)$). Therefore inverse problems become important, besides the direct ones. Solutions of inverse problems give descriptions of the classes of ordinary differential equations that admit operators of a given form (necessary conditions). The solution of the Ermakov equation problem was obtained precisely by this method. It should be noted that the presence of a nonlocal variable, which is involved in the canonical coordinate as a multiplier rather than as an exponential, results in the appearance of an additional “independent” variable $I=\int\zeta(x,y,y',\ldots)dx$ in the initial equation.

In general, we may ask how the nonlocal variable I can enter the coordinate of the canonical operator. Let us look for an operator with the coordinate $\Phi=\eta(x,y)F(I)$. It turns out that a nontrivial operator exists if functions the $F,F',F'',\ldots$ are linearly independent or are equal to zero starting with F''. Otherwise the system in question becomes too overdetermined, because all $F,F',F'',\ldots$ play the role of “independent” variables. Thus possible classes of nonlocal operators are exhausted by two types:

$$X=\left(\eta_1\exp\int\zeta dx+\eta_2\right)\partial_y \quad \text{and}$$

$$X=\left(\eta_1\int\zeta dx+\eta_2\right)\partial_y.$$

Plurality of forms arises because of essential differences in the implementation of the search algorithm: we can choose either ultracanonical, or canonical, or “geometrical” form of the operator.

The universal factorization principle applies not only to ordinary differential equations, but also to functional differential equations. For these equations we do not know any general and efficient method for finding analytical solutions. Elements of the theory of point and nonlocal operators admissible by functional differential equations, together with a series of important practical results, can be found in [6-8,12].

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