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Invariant Linear Spaces and Exact Solutions of Nonlinear Evolution Equations

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1 Introduction

The paper presents a survey of some new results concerning the approach to construction of explicit solutions for nonlinear evolution equations

$$\frac{\partial u}{\partial t} = F[u], \tag{1}$$

proposed in [1, 2]. Here we consider real scalar functions u of two variables $x, t \in R^1$ and differential operators F of the form $F[u] \equiv F\Big(x, u, \dfrac{\partial u}{\partial x}, \dots, \dfrac{\partial^p u}{\partial x^p}\Big)$. Let $f_i(x)$, $i = \overline{1,k}$, be a set of linearly independent functions and W_k denote the linear span of them:

$$W_k = L\{f_1(x), \dots, f_k(x)\}. \tag{2}$$

The space is said to be invariant with respect to F if $F[W_k] \subseteq W_k$, i.e., if there exist some functions $\widetilde{F}_i$, such that $F\Big[\sum\limits_{i=1}^{k} C_i f_i(x)\Big] = \sum\limits_{i=1}^{k} \widetilde{F}_i(C_1, \dots, C_k) f_i(x) \quad \forall C_i \in R^1$.

The idea of the approach is very simple: *if the linear space W_k is invariant under the operator F, then equation (1) possesses solutions of the form*

$$u(x,t) = \sum_{i=1}^{k} \varphi_i(t) f_i(x), \tag{3}$$

where coefficients $\varphi_1(t), \dots, \varphi_k(t)$ satisfy the dynamical system

$$\frac{d\varphi_i(t)}{dt} = \widetilde{F}_i(\varphi_1(t), \dots, \varphi_k(t)), \quad i = \overline{1,k}.$$

Examples of solutions of type (3) for different problems are well-known in the literature, see, e.g., [3–8]. A rich variety of new examples was obtained in [2] and in succeeding papers [9–11].

The main problem arising in this context is **the problem "$F \to W_k$"**: *to construct for a given operator F all invariant spaces W_k.* Here we consider some results concerning this problem as well as the inverse one. Note that every linear space (2) can be defined as the space of solutions of some linear ordinary differential equation

$$L[y] \equiv a_0(x)y^{(k)} + a_1(x)y^{(k-1)} + \ldots + a_{k-1}(x)y' + a_k(x)y = 0, \tag{4}$$

for which the functions $f_i(x)$, $i = \overline{1,k}$, form a fundamental system of solutions (FSS). Then the invariance condition of W_k takes the form:

$$L[F[y]]\Big|_{L[y]=0} \equiv 0. \tag{5}$$

This condition leads to an overdetermined system for the coefficients of equation (4) and provides the description of all invariant spaces of given order k (see examples in [12, 14]). Note also that the condition (5) is actually the invariance criterion of equation (4) under the Lie-Bäcklund operator $X = F[y]\partial/\partial y$ and therefore we can interpret all results obtained in terms of symmetries of linear ODEs [12–14].

2 The problem "$W_k \to F$"

Along with the original problem the inverse one is of interest: given W_k, find all F (possibly from some special class). Precisely this approach was applied in [2] to construct all possible quadratic differential operators of the second order possessing fixed invariant spaces of power, exponential or trigonometrical type. The next theorem provides the complete solution of this problem. In the following we consider only the operators

$$F[y] = F(x, y, y', \ldots, y^{(p)}), \tag{6}$$

of the order $p \le k-1$, because the higher derivatives can be excluded by virtue of equation (4) defining the invariant space.

Theorem 1. *Every operator (6) possessing the invariant space (2) is given by*

$$F[y] = \sum_{i=1}^{k} A^i(I_1, \ldots, I_k) f_i(x), \tag{7}$$

where $A^i(I_1, \ldots, I_k)$, $i = \overline{1,k}$, are arbitrary functions of the first integrals of the corresponding equation (4).

The proof is given in [12]. To complete the discussion of the inverse problem, we should notice only that the full set of the functionally independent first integrals can be easily found (without any integrations) if a FSS of equation (4) is known. We consider here only one illustrating example, for more examples see [12].

Example 1. Choosing a 2-dimensional invariant space in the form

$$W_2 = L\{e^x, e^{-x}\} \quad (\Leftrightarrow y'' - y = 0),$$

we find in accordance with (7) the general expression for the operators F:

$$F[y] = A^1(I_1, I_2)e^x + A^2(I_1, I_2)e^{-x},$$

where

$$I_1 = (y' - y)e^x, \quad I_2 = (y' + y)e^{-x}.$$

One can use this expression to describe operators satisfying some additional conditions. For instance, operators which do not depend explicitly on x have the form

$$F[y] = y' A(J) + y B(J),$$

where A and B are arbitrary functions of $J = {y'}^2 - y^2$. Special cases of the last expression are

$$F_1[y] = 1/(y' + y), \quad F_2[y] = y'({y'}^2 - y^2),$$

$$F_3[y] = y({y'}^2 - y^2) \quad \sim \quad F_4[y] = y''({y'}^2 - y^2),$$

where "tilde" denotes equivalence on the invariant space. These operators allow to construct various evolution equations possessing solutions of the given form

$$u(x, t) = \varphi_1(t) e^x + \varphi_2(t) e^{-x},$$

for example, the following "diffusion" equations:

$$u_t = u_{xx} + (u_x + u)^{-1}, \quad u_t = u_{xx} + u(u_x^2 - u^2), \quad u_t = (u_x^2 - u^2) u_{xx},$$

etc. Here we take into account that the space is also invariant under the operator $F[u] = u_{xx}$.

3 The problem "$F \to W_k$"

Let us return to the original problem: to construct all invariant spaces for a given operator. The following statement establishes the upper bound on the dimension of an invariant space.

Theorem 2 [13, 14]. *If a linear space W_k is invariant under a nonlinear operator (6) of the order p, then*

$$k \leq 2p + 1.$$

Note that for an arbitrary p there exist nonlinear operators possessing invariant spaces of the maximal dimension $2p + 1$ (*for example*: $F[y] = (y^{(p)})^2$, $W_{2p+1} = L\{1, x, \ldots, x^{2p}\}$). The natural problem is *to describe all operators of this kind*. For *nonlinear* operators of the first and the second orders this problem was considered in [15]. It was shown that every such operator should necessarily be quadratic in y and in its derivatives. Classification of the operators was given with respect to some equivalence transformations. To illustrate these results, we restrict ourselves to nonlinear operators of the first order, which do not depend on x. The operators are considered up to constant factors. Denoting by $\mathcal{F}_3$ the set of such operators possessing three-dimensional invariant spaces, we obtain the following result.

Proposition 1. The set $\mathcal{F}_3$ is exhausted by the operators

$$F_1[y] = {y'}^2 + N y^2 + A y' + B y + C$$

and

$$F_2[y] = (y' + M y)^2 + A y' + B y + C,$$

where A, B, C, M, N are arbitrary constants, $M \neq 0$. The corresponding invariant spaces are, in the case of F_1:

$$W_3 = L\{1, \sin(\sqrt{N}x), \cos(\sqrt{N}x)\} \quad (N > 0), \quad W_3 = L\{1, x, x^2\} \quad (N = 0),$$

$$W_3 = L\{1, sh(\sqrt{-N}x), ch(\sqrt{-N}x)\} \quad (N < 0),$$

and in the case of F_2:

$$W_3 = L\{1, \exp(-Mx/2), \exp(-Mx)\}.$$

Note that these spaces are also invariant under an arbitrary *linear* differential operator with constant coefficients. This fact is used in the following example taken from [7].

Example 2. Consider the parabolic equation

$$u_t = u_{xx} + u_x^2 + \alpha u^2 + \beta u + \gamma \tag{8}$$

connected via the transformation $u = \ln v$ with the equation

$$v_t = v_{xx} + v(\alpha \ln^2 v + \beta \ln v + \gamma) \tag{9}$$

describing certain properties of combustion in blow-up processes. For the sake of definiteness, suppose that $\alpha = 1$. In accordance with Proposition 1 Eq. (8) admits solutions of the form

$$u(x,t) = \varphi_1(t) + \varphi_2(t)\sin x + \varphi_3(t)\cos x. \tag{10}$$

The coefficients $\varphi_i(t)$ satisfy the system

$$\dot{\varphi}_1 = \varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \beta\varphi_1 + \gamma, \quad \dot{\varphi}_2 = \varphi_2(2\varphi_1 + \beta - 1), \quad \dot{\varphi}_3 = \varphi_3(2\varphi_1 + \beta - 1).$$

4 Generalizations

The following remarks illustrate some possible generalizations of this approach.

4.1. The trivial observation is that the approach can be easily extended to equations (1) with arbitrary linear differential operators (in t) on the left-hand side. For instance, instead of Galaktionov–Posashkov's equation (8) one can consider a more general equation of the second order

$$a(t)u_{tt} + b(t)u_t = u_{xx} + u_x^2 + \alpha u^2 + \beta u + \gamma$$

(with arbitrary $a(t)$, $b(t)$) also possessing solutions (10) but, of course, with different $\varphi_i(t)$. For other examples see [1, 2].

It is interesting that a more general situation, when the operator on the left-hand side becomes nonlinear, does not exclude the existence of solutions of type (3). For more details see [16], where a method of generalized separation of variables is proposed, and also [17].

4.2. There are many examples, where the original equation has no invariant spaces, but the spaces appear after some change of variables and the problem reduces to constructing of such transformations, see [2], [7–10]. The illustration was actually given by Example 2, because it is easily shown that Eq. (9) has no invariant spaces. It is interesting to note that up to translations and dilations of variables equation (9) is the most general semilinear diffusion equation of the form $v_t = v_{xx} + q(v)$ gaining three-dimensional invariant spaces after some transformation $v = f(u)$ (the higher dimensional invariant spaces are impossible in this case).

4.3. Applications of the method to the systems of equations were discussed in [12], where in particular an analog of Theorem 1 was established. Here is an illustrating example taken from that work.

Example 3. Let us consider the system

$$u_t = -v_{xx} - \nu\, v\, (u^2 + v^2), \qquad v_t = u_{xx} + \nu\, u\, (u^2 + v^2),$$

being a real representation of the cubic Schrödinger equation

$$i\, z_t + z_{xx} + \nu\, |z|^2 z = 0,$$

with $z = u + i\, v$ ($\nu =$ const). The (vector) operator on the right–hand side has the two-dimensional invariant space $W_2 = L\{\mathbf{f}_1,\ \mathbf{f}_2\}$, where $\mathbf{f}_1 = (\cos x,\ \sin x)^T$, $\mathbf{f}_2 = (-\sin x, \cos x)^T$. Therefore we can look for solutions of the form

$$\begin{pmatrix} u \\ v \end{pmatrix} = \varphi_1(t) \begin{pmatrix} \cos x \\ \sin x \end{pmatrix} + \varphi_2(t) \begin{pmatrix} -\sin x \\ \cos x \end{pmatrix}.$$

Integrating the system on the coefficients $\varphi_1(t)$, $\varphi_2(t)$ and rewriting solution in the complex form, one can obtain the travelling wave solution

$$z = u + i\, v = C_1 exp\{i\, [x + (C_1^2 \nu - 1)t + C_2]\},$$

with arbitrary real constants C_1 and C_2.

4.4. In conclusion, we point out the papers [2, 7, 10, 11], where examples of solutions of type (3) for *multidimensional* evolution equations are given.

5 Final remark

Possibly one of the most general methods for determining special solutions to nonlinear PDEs is the method of differential constraints and many other methods can be treated as its particular cases. But, of course, "the main difficulty with this approach is that it appears to be *too general* to practical use" [20]. The method considered above is not so general: it is in fact a method of the simplest linear ordinary differential constraints. But this simplicity allows to obtain more complete results. Along with many applications given in the cited papers, this makes the method very attractive. Here we have no opportunity to discuss the approach in context of other methods and refer the reader to [18] and to the survey papers [19,20].

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