EXPLICITLY SOLVABLE SYSTEMS OF FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH POLYNOMIAL RIGHT-HAND SIDES, AND THEIR PERIODIC VARIANTS

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Abstract

In this Letter we identify *special* systems of (an *arbitrary* number) N of first-order Ordinary Differential Equations with *homogeneous* polynomials of *arbitrary* degree M on their right-hand sides, which feature *very simple explicit* solutions; as well as variants of these systems—with right-hand sides no more homogeneous—which feature *periodic* solutions. A novelty of these findings is to consider *special* systems characterized by *constraints* involving both their parameters and their initial data.

The general system of an *arbitrary* number N of first-order Ordinary Differential Equations (ODEs) with *homogeneous polynomials* of *arbitrary* degree M on their right-hand sides reads as follows:

$$\begin{aligned} \dot{z}_n(t) &= \sum_{m_\ell}{}^{(M)} \left\{ c_{nm_1m_2\cdots m_N} [z_1(t)]^{m_1} [z_2(t)]^{m_2} \cdots [z_N(t)]^{m_N} \right\} , \\ n &= 1, 2, ..., N , \end{aligned} \tag{1a}$$

where (above and below) the symbol $\sum_{m_\ell}{}^{(M)}$ denotes a sum running over *all nonnegative* values of the N indices m_ℓ subject to the restriction

$$\sum_{\ell=1}^{N} (m_\ell) = M , \tag{1b}$$

implying that the polynomials in N variables $z_n(t)$ in the right-hand sides of the N ODEs (1a) are *all homogeneous* of degree M.

Notation. Throughout this paper M and N are *positive* integers larger than *unity*; the index n takes *positive integer* values; indices and exponents such as m_1, m_2,... take *all* the *nonnegative integer* values consistent with the restriction (1b); the independent variable t can be considered as playing the role of "time", taking *all nonnegative real* values (but it shall also be eventually convenient to replace it formally with the *complex* variable τ, see below); a superimposed dot indicates a t-differentiation; the coefficients $c_{nm_1m_2\cdots m_M}$ are (t-independent) parameters; while of course the dependent variables $z_n \equiv z_n(t)$ are functions of the independent variable t and ascertaining their t-evolution from the set of N *initial data* $z_n(0)$ is our main task. The coefficients $c_{nm_1m_2\cdots m_M}$ and the dependent variables $z_n(t)$ might be restricted to be *real*; but in the last part of this paper we shall assume that they are instead *complex*, setting

$$c_{nm_1m_2\cdots m_M} = a_{nm_1m_2\cdots m_M} + \mathbf{i} b_{nm_1m_2\cdots m_M} ; \tag{2}$$

and we shall as well replace the independent variable t with a *complex* variable τ, see below eq. (5); here and below of course $\mathbf{i}$ is the *imaginary unit*, $\mathbf{i}^2 = -1$. Finally: below ω denotes an *arbitrary nonvanishing real* parameter. ■

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The system (1) has being investigated over time in an enormous number of mainly mathematical, or mainly applicative, papers (more than it is possible to report in an adequate manner: for a seminal paper see, for instance, [1]); although generally for specific, relatively small, values of N and M. The mathematics behind the findings reported in the present paper is rather elementary; yet these developments may have some interest—perhaps mainly in applicative contexts—because they are based on a somewhat *unconventional* approach: to identify *explicitly solvable* cases of the system (1) by introducing *constraints* involving, in addition to the coefficients $c_{nm_1m_2\cdots m_M}$, also the initial data $z_n(0)$ (which, in many applicative contexts, may well play the role of *control* elements, determining the time evolution of the system).

Our main result is the following

Proposition. The system (1) features the special solution

$$z_n(t) = z_n(0)(1+Kt)^{1/(1-M)} \quad , \quad n = 1, 2, ..., N \; , \tag{3a}$$

provided there hold the following N explicit *algebraic constraints* on the *a priori arbitrary* parameter K, the coefficients $c_{nm_1m_2\cdots m_M}$ and the N initial data $z_n(0)$:

$$\begin{aligned} Kz_n(0) &= (1-M)\left[\sum_{m_\ell}{}^{(M)}\left\{c_{nm_1m_2\cdots m_M}[z_1(0)]^{m_1}[z_2(0)]^{m_2}\cdots[z_N(0)]^{m_N}\right\}\right] \; , \\ n &= 1, 2, ..., N \; . \; \blacksquare \end{aligned} \tag{3b}$$

Remark 1. The proof that (3) satisfies the system of ODEs (1) is elementary: just insert (3a) in (1a) and verify that, thanks to (1b) and (3b), the N ODEs (1a) are satisfied. ■

Remark 2. The system of N *algebraic* equations (3b) generally determines—for any given assignment of the *a priori arbitrary* coefficients $c_{nm_1m_2\cdots m_M}$—N out of the $N+1$ quantities K and $z_n(0)$; but it is also possible to select *ad libitum* N elements out of the *complete* set of data K, $c_{nm_1m_2\cdots m_M}$ and $z_n(0)$, and to then consider these *selected* elements as those to be determined—by the N conditions (3b)—in terms of the remaining *arbitrarily assigned* elements in the *complete* set of these data. If one chooses to satisfy these N conditions by solving the N equations (3b) for N of the coefficients $c_{nm_1m_2\cdots m_M}$—or for the parameter K and $N-1$ of the coefficients $c_{nm_1m_2\cdots m_M}$—then this task can be generally performed *explicitly*, since the relevant *algebraic* equations to be solved are then *linear* in the unknown quantities; otherwise these determinations require the solution of *nonlinear* equations, a task which can be performed *explicitly* only rarely in an *algebraic* setting; but which can generally be performed, with *arbitrary* approximation, in a *numerical* context. ■

Example 1. Assume for instance $N=2$ and $M=4$, so that the system (1) reads as follows (note below the notational simplification):

$$\dot{z}_n(t) = \sum_{m=0}^{4} c_{nm}[z_1(t)]^{4-m}[z_2(t)]^m \quad , \quad n = 1, 2 \; , \tag{4a}$$

featuring 2 dependent variables $z_n(t)$ and 10 *a priori arbitrary* coefficients c_{nm} ($n=1,2$; $m=0,1,2,3,4$). Then the solution (3a) reads as follows:

$$z_n(t) = z_n(0)[1+Kt]^{-1/3} \quad , \quad n = 1, 2 \; , \tag{4b}$$

and the 2 conditions (3b) read as follows:

$$Kz_n(0) = -3\sum_{m=0}^{4} c_{nm}[z_1(0)]^{4-m}[z_2(0)]^m \quad , \quad n = 1, 2 \; . \tag{4c}$$

These algebraic constraints can of course be *explicitly* solved for any 2 of the 10 coefficients c_{nm} in terms of the other 8 coefficients c_{nm} and of the 3 *arbitrary* data K, $z_1(0)$, $z_2(0)$; or alternatively for K and only 1 of the 10 coefficients c_{nm} in terms of the other 9 coefficients c_{nm} and of the 2 *arbitrary* initial data $z_1(0)$, $z_2(0)$; with many other possibilities left to the imagination of the interested reader. ■

The *periodic* variant obtains from the previous results—where we now assume all quantities to be *complex* and we formally replace the independent variable t with the *complex* variable τ—via the following well-known trick (amounting to a simple change of dependent and independent variables: see, for instance, [2]):

$$x_n(t) + \mathbf{i}y_n(t) = \{\exp[\mathbf{i}\omega t/(M-1)]\}\, z_n(\tau) \quad , \quad \tau = [\exp(\mathbf{i}\omega t) - 1]/(\mathbf{i}\omega) \; , \tag{5}$$

implying $\dot{\tau}(t) = \exp(\mathbf{i}\omega t)$ and transforming the system (1a) into the following (still *autonomous*!) system involving now the $2N$ *real* variables $x_n(t)$ and $y_n(t)$ (depending of course on the *real* independent variable t: "time"):

$$\dot{x}_n(t) = -[\omega/(M-1)]\, y_n(t) + Re[Z_n(t)] \ ,$$
$$\dot{y}_n(t) = [\omega/(M-1)]\, x_n(t) + Im[Z_n(t)] \ , \tag{6a}$$

where (see (5) and (2))

$$Z_n(t) = \sum_{m_\ell}{}^{(M)} \{(a_{nm_1m_2\cdots m_N} + \mathbf{i}b_{nm_1m_2\cdots m_N}) \cdot$$
$$\cdot [x_1(t) + \mathbf{i}y_1(t)]^{m_1} \cdots [x_N(t) + \mathbf{i}y_N(t)]^{m_N}\} \ . \tag{6b}$$

Remark 3. The fact that *all* solutions $x_n(t)$, $y_n(t)$ of the system (6) obtained via the definition (5) with $z_n(\tau)$ defined by (3a) (of course with t replaced there by τ, see (5)) are *periodic* with a period T which is an (easily identifiable on a case-by-case basis) *integer* multiple of the basic period $2\pi/|\omega|$ is rather *obvious*; in case of *doubt*, see [2]. ■

Example 2. As an example of *solvable* system featuring *periodic* solutions let us display the findings reported in the *special* case with $N = 2$ and $M = 4$. Then the system (6) of 4 ODEs reads as follows:

$$\dot{x}_n(t) = -(\omega/3)\, y_n(t) + Re[Z_n(t)] \ , \quad n = 1, 2 \ ,$$
$$\dot{y}_n(t) = (\omega/3)\, x_n(t) + Im[Z_n(t)] \ , \quad n = 1, 2 \ , \tag{7a}$$

$$Z_n(t) = \sum_{m=0}^{4} \left\{(a_{nm} + \mathbf{i}b_{nm})\, [x_1(t) + \mathbf{i}y_1(t)]^{4-m}\, [x_1(t) + \mathbf{i}y_1(t)]^m\right\} \ ; \tag{7b}$$

its *explicit* solutions read as follows:

$$x_n(t) = Re[\zeta_n(t)] \ , \quad y_n(t) = Im[\zeta_n(t)] \ , \quad n = 1, 2 \ , \tag{8a}$$

$$\zeta_n(t) = [x_n(0) + \mathbf{i}y_n(0)] \exp(\mathbf{i}\omega t/3) \cdot$$
$$\cdot \left\{[1 + (K_R + \mathbf{i}K_I)\, [\exp(\mathbf{i}\omega t) - 1]\, / \,(\mathbf{i}\omega)]^{-1/3}\right\} \ , \quad n = 1, 2 \ , \tag{8b}$$

provided the 2 (*a priori arbitrary*) *real* parameters K_R and K_I, the 4 (*a priori arbitrary*) *real initial data* $x_n(0)$ and $y_n(0)$ and the 20 (*a priori arbitrary*) *real* coefficients a_{nm} and b_{nm} ($n = 1, 2;\ m = 0, 1, 2, 3, 4$) are related to each other by the following 2 *complex* (i. e., 4 *real*) *constraints*:

$$(K_R + \mathbf{i}K_I)\, [\mathbf{x}_n(0) + \mathbf{i}y_n(0)]$$
$$= -3 \sum_{m=0}^{4} \left\{(a_{nm} + \mathbf{i}b_{nm})\, [x_1(0) + \mathbf{i}y_1(0)]^{4-m}\, [x_1(0) + \mathbf{i}y_1(0)]^m\right\} \ , \ n = 1, 2 \ . \ \blacksquare \tag{8c}$$

Final Remark. As already noted above, the mathematics behind the results reported above is rather *elementary*. Yet these findings do not seem to have been advertised so far, while their *applicable* potential is clearly vast; so—especially among *applied* mathematicians and *practitioners* of the various scientific disciplines where systems of ODEs such as those discussed above play a key role—a wider knowledge of them seems desirable; for instance via their inclusion in standard compilations of *solvable* ODEs such as [3]. ■

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