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EXACT DIFFERENCE SCHEMES FOR MULTIDIMENSIONAL HEAT CONDUCTION EQUATIONS

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Abstract — The initial boundary-value problem for the two-dimensional heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_1}\left(k_1(u)\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(k_2(u)\frac{\partial u}{\partial x_2}\right)$$

is considered. A new difference scheme approximating the above equation is constructed. The error of approximation of the considered scheme is $O((\sigma - 0.5)(h_1 + h_2 + \tau) + h_1^2 + h_2^2 + \tau^2)$, where σ is a weight. The main difference between the method proposed in the present paper and the other difference schemes is that for the traveling wave solutions

$$u(x,t) = U\left(\frac{1}{2a}x_1 + \frac{1}{2b}x_2 - t\right), \quad 0 \leqslant x_1 \leqslant l_1, \quad 0 \leqslant x_2 \leqslant l_2, \quad 0 \leqslant t \leqslant T,$$

the considered scheme is exact if the grid steps satisfy definite conditions $\gamma_1 = a\tau/h_1 = 1/2$, $\gamma_2 = b\tau/h_2 = 1/2$. The iteration method is used to solve a nonlinear difference equation.

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1. Introduction

Various schemes have been constructed to approximate the boundary-value problem for the heat conduction equation [8, 24]. Let us consider the one-dimensional parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t).$$

The simplest difference scheme approximating the above problem is explicit (backward) difference scheme involving four points of the grid

$$\frac{y_i^{n+1} - y_i^n}{\tau} = \frac{y_{i-1}^n - 2y_i^n + y_{i+1}^n}{h^2} + \varphi_i^n.$$

The other basic schemes approximating this equation are the Richardsons scheme, the Crank — Nicolsons scheme, the Saulevs schemes or the scheme with weight [1–3, 8, 24].

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In investigating difference schemes, one of the most important problem is the approximation order, which is desired to be as high as possible. For example, the scheme with a weight

$$\frac{y_i^{n+1}-y_i^n}{\tau}=\sigma\frac{y_{i-1}^{n+1}-2y_i^{n+1}+y_{i+1}^{n+1}}{h^2}+(1-\sigma)\frac{y_{i-1}^n-2y_i^n+y_{i+1}^n}{h^2}+\varphi_i^n$$

has the order of approximation $O((\sigma-0.5)\tau+\tau^2+h^2)$, where σ is a weight. For $\sigma=0.5-h^2/(12\tau)$ and $\varphi_i^n=(5/6)f_i^{n+1/2}+(1/12)(f_{i-1}^{n+1/2}+f_i^{n+1/2})$, if $u\in C_3^6$, the error can be reduced to $O(\tau^2+h^4)$. In this case the scheme with a weight is usually termed a higher-accuracy scheme [24].

Difference schemes of a high order of approximation arouse interest among authors [17, 21,26,27]. In the article of Mohanty [18], two-level implicit difference methods of $O(\tau^2+h^4)$ using 19-spatial grid points for the solving the three space dimensional heat conduction equation is proposed. Radwan in [22] gives a comparison of the numerical solutions obtained from higher-order accurate difference schemes for the two-dimensional Burgers equation.

Definition 1.1. A difference scheme is *exact* if the truncation error is equal to zero or $y=u$ at the grid nodes.

It is widely known that there exist exact difference schemes approximating transport equations or some ordinary differential equations. For example, for the Cauchy problem

$$du/dt=f_1(t)f_2(u),\quad 0<t\leqslant T,\quad u(0)=u_0,$$

the difference scheme

$$\frac{y^{n+1}-y^n}{\tau}=\frac{1}{\tau}\int\limits_{t_n}^{t_{n+1}}f_1(t)\,dt\left(\frac{1}{y^{n+1}-y^n}\int\limits_{y_n}^{y_{n+1}}\frac{du}{f_2(u)}\right)^{-1},\quad t\in\omega_\tau,\quad y^0=u_0,$$

where $f_2(u)\neq 0$, is exact [9]. In [24], the exact difference scheme is constructed for the equation

$$(k(x)u'(x))'-q(x)u(x)=-f(x),\quad 0<x<L,\quad u(0)=\mu_1,\quad u(L)=\mu_2.\tag{1.1}$$

It turned out that it is possible to construct other exact difference schemes for some ordinary or partial differential equations .

In the last few years, such schemes have been considered in papers [5–7, 9, 10, 23]. It is worth to mention here paper [7], in which under natural conditions the authors proved the existence of a two-point exact difference scheme for systems of first-order boundary value problems, or papers of Mickens [12–16], in which a nonstandard finite difference schemes were introduced. Mickens [11] gives certain rules for constructing nonstandard finite difference schemes and emphasizes that an important feature of nonstandard schemes is that they often can provide numerical integration techniques, for which elementary numerical instabilities do not occur. Nowadays, other authors are applying techniques initiated by Mickens to obtain exact numerical solutions or numerical solutions of a highorder of accuracy to a wide class of differential equations.

Let us introduce on the domain $\overline{Q}_T=[0,l_1]\times[0,l_2]\times[0,T]$ with boundary Γ a uniform grid

$$\begin{aligned}\overline{\omega}=\overline{\omega}_{h_1}\times\overline{\omega}_{h_2}\times\overline{\omega}_\tau=\{(x_{1,i},x_{2,j},t_n):\ x_{1,i}=ih_1,\ i=\overline{0,N_1},\ hN_1=l_1,\\ x_{2,j}=jh_2,\ j=\overline{0,N_2},\ hN_2=l_2,\ t_n=n\tau,\ n=\overline{0,N_0},\ \tau N_0=T\},\end{aligned}\tag{1.2}$$

$$\Gamma_h = \{(x_{1i}, x_{2j}, t_{n+1}) \in \Gamma: \ x_{1i} = ih_1, \ i = \overline{0, N_1}, \ hN_1 = l_1,$$
$$x_{2j} = jh_2, \ j = \overline{0, N_2}, \ hN_2 = l_2, \ t_n = n\tau, \ n = \overline{0, N_0}, \ \tau N_0 = T\}. \tag{1.3}$$

In this paper, we will use the following notation: $y_{ij} = y_{ij}^n = y(x_{1i}, x_{2j}, t_n)$, $\hat{y} = y^{n+1}$, $\overline{y} = y^{n+1/2}$, $y_t = (\hat{y}_{i,j} - y_{i,j})/\tau$, $y_{\overline{x}_1} = (y_{i,j} - y_{i-1,j})/h_1$, $y_{\overline{x}_2} = (y_{i,j} - y_{i,j-1})/h_2$, $\hat{y}_{x_1} = (\hat{y}_{i+1,j} - \hat{y}_{i,j})/h_1$, $\hat{y}_{x_2} = (\hat{y}_{i,j+1} - \hat{y}_{i,j})/h_1$.

The main goal of this paper is to construct a new difference scheme approximating the two-dimensional heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_1}\left(k_1(u)\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(k_2(u)\frac{\partial u}{\partial x_2}\right), \quad 0 < x_1 < l_1, \quad 0 < x_2 < l_2, \quad 0 < t \leqslant T. \tag{1.4}$$

The order of approximation of the scheme, which will be introduced in this paper, is $O((\sigma - 0.5)(h_1 + h_2 + \tau) + h_1^2 + h_2^2 + \tau^2)$. The essential difference between the suggested method and the other difference schemes is that for traveling wave solutions with a constant velocity $u(x_1, x_2, t) = U(x_1/(2a) + x_2/(2b) - t)$ satisfying the conditions stated in paper and with specific steps h_1, h_2, τ, the considered scheme is exact.

The paper is organized as follows. In Section 2, the new difference scheme for the boundary-value problem (1.4) is constructed and the error of approximation of this scheme is investigated. The iteration method is used to solve the difference equation. In Section 3, the stability of the linear approach of the nonlinear difference scheme (1.4) is proved. In Section 4, the results of the numerical experiments are presented to confirm the theoretical results investigated in the paper and show that an approach can be applied to different problems.

2. Exact difference schemes for the two-dimensional quasilinear parabolic equation

In this Section, a new difference scheme for boundary-value problem for parabolic equation is constructed. Primary consideration is given to the investigation of the error of approximation of this scheme. The iteration method is used to to solve the nonlinear difference equation.

Let us consider in the domain $\overline{Q}_T$ the boundary-value problem for the two-dimensional quasi-linear parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_1}\left(k_1(u)\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(k_2(u)\frac{\partial u}{\partial x_2}\right), \quad 0 < x_1 < l_1, \quad 0 < x_2 < l_2, \quad 0 < t \leqslant T, \tag{2.1}$$

$$u(x_1, x_2, 0) = u_0(x_1, x_2), \quad 0 \leqslant x_1 \leqslant l_1, \quad 0 \leqslant x_2 \leqslant l_2, \tag{2.2}$$

$$u|_\Gamma = \mu(x_1, x_2, t), \quad (x_1, x_2, t) \in \Gamma, \tag{2.3}$$

where $0 < c_1 \leqslant k_1(u)$, $k_2(u) \leqslant c_2$, $u = u(x_1, x_2, t)$, $(x_1, x_2, t) \in Q_T$, $c_1, c_2 = \text{const}$. We assume that $u \in C^4(\overline{Q}_T)$ and $0 < m_1 \leqslant u(x_1, x_2, t) \leqslant m_2$, $(x_1, x_2, t) \in \overline{Q}_T$, $m_1, m_2 = \text{const}$.

Then, equation (2.1) can be written in the equivalent form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_1}\left(u\frac{\partial \varphi_1(u)}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(u\frac{\partial \varphi_2(u)}{\partial x_2}\right), \quad \varphi_1(u) = \int_{u_0}^{u} \frac{k_1(\xi)}{\xi}\, d\xi, \quad \varphi_2(u) = \int_{u_0}^{u} \frac{k_2(\xi)}{\xi}\, d\xi. \tag{2.4}$$

On the uniform grid (1.2), problem (2.1) – (2.3) is approximated by the difference scheme

$$y_t=\sigma(\hat{y}(\varphi_1(\hat{y}))_{\overline{x}_1})_{x_1}+\sigma\left(\hat{y}(\varphi_2(\hat{y}))_{\overline{x}_2}\right)_{x_2}+(1-\sigma)(y(\varphi_1(y))_{x_1})_{\overline{x}_1}+(1-\sigma)(y(\varphi_2(y))_{x_2})_{\overline{x}_2}, \quad (2.5)$$

$$y^0_{ij}=u_0(x_{1i},x_{2j}),\quad x_{1i}\in\overline{\omega}_{1h},\quad x_{2j}\in\overline{\omega}_{2h},\quad y^{n+1}_{ij}=\mu(x_{1i},x_{2j},t_{n+1}),\quad (x_{1i},x_{2j},t_{n+1})\in\Gamma_h. \quad (2.6)$$

Let us investigate the error of approximation of scheme (2.5), (2.6). We assume that $\varphi_1,\varphi_2\in C^4[m_1,m_2]$. Then the error of approximation satisfies the chain of equalities

$$\psi^{n+1}_{ij}=-u_t+\sigma(\hat{u}(\varphi_1(\hat{u}))_{\overline{x}_1})_{x_1}+\sigma(\hat{u}(\varphi_2(\hat{u}))_{\overline{x}_2})_{x_2}+(1-\sigma)(u(\varphi_1(u))_{x_1})_{\overline{x}_1}+(1-\sigma)(u(\varphi_2(u))_{x_2})_{\overline{x}_2}=$$

$$=-u_t+\sigma\frac{\partial\hat{u}}{\partial t}+(1-\sigma)\frac{\partial u}{\partial t}+\sigma(\hat{u}(\varphi_1(\hat{u}))_{\overline{x}_1})_{x_1}-\sigma\frac{\partial}{\partial x_1}\left(\hat{u}\frac{\partial\varphi_1(\hat{u})}{\partial x_1}\right)+(1-\sigma)(u(\varphi_1(u))_{x_1})_{\overline{x}_1}-$$

$$-(1-\sigma)\frac{\partial}{\partial x_1}\left(u\frac{\partial\varphi_1(u)}{\partial x_1}\right)+\sigma(\hat{u}(\varphi_2(\hat{u}))_{\overline{x}_2})_{x_2}-\sigma\frac{\partial}{\partial x_2}\left(\hat{u}\frac{\partial\varphi_2(\hat{u})}{\partial x_2}\right)+(1-\sigma)(u(\varphi_2(u))_{x_2})_{\overline{x}_2}-$$

$$-(1-\sigma)\frac{\partial}{\partial x_2}\left(u\frac{\partial\varphi_2(u)}{\partial x_2}\right)=\psi_1+\psi_2+\psi_3.$$

We may introduce ψ_1 in the form

$$\psi_1=-u_t+\sigma\frac{\partial\hat{u}}{\partial t}+(1-\sigma)\frac{\partial u}{\partial t}=(\sigma-0.5)\tau\frac{\partial^2\overline{u}}{\partial t^2}+O(\tau^2). \quad (2.7)$$

Substituting the expansions

$$\begin{aligned}(\varphi_1(\hat{u}))_{x_1,ij}&=\frac{\partial}{\partial x_1}(\varphi_1(\hat{u}))_{ij}+\frac{h}{2}\frac{\partial^2}{\partial x_1^2}(\varphi_1(\hat{u}))_{ij}+\frac{h^2}{3!}\frac{\partial^3}{\partial x_1^3}(\varphi_1(\hat{u}))_{ij}+\\&+\frac{h^3}{4!}\left(\frac{\partial^2}{\partial x_1^2}(\varphi_1(\tilde{u}))\right),\quad \tilde{u}=u(\xi_1,x_{2j},t_{n+1}),\quad \xi_1\in[x_{1i},x_{1i+1}],\\(\varphi_1(\hat{u}))_{\overline{x}_1,ij}&=\frac{\partial}{\partial x_1}(\varphi_1(\hat{u}))_{ij}-\frac{h}{2}\frac{\partial^2}{\partial x_1^2}(\varphi_1(\hat{u}))_{ij}+\frac{h^2}{3!}\frac{\partial^3}{\partial x_1^3}(\varphi_1(\hat{u}))_{ij}-\\&-\frac{h^3}{4!}\left(\frac{\partial^2}{\partial x_1^2}(\varphi_1(\tilde{\tilde{u}}))\right),\quad \tilde{\tilde{u}}=u(\xi_2,x_{2j},t_{n+1}),\quad \xi_2\in[x_{1i-1},x_{1i}],\end{aligned}$$

into ψ_2, it is easy to get the expression

$$\begin{aligned}\psi_2=\sigma(\hat{u}(\varphi_1(\hat{u}))_{\overline{x}_1})_{x_1}-\sigma\frac{\partial}{\partial x_1}\left(\hat{u}\frac{\partial\varphi_1(\hat{u})}{\partial x_1}\right)+(1-\sigma)(u(\varphi_1(u))_{x_1})_{\overline{x}_1}-(1-\sigma)\frac{\partial}{\partial x_1}\left(u\frac{\partial\varphi_1(u)}{\partial x_1}\right)=\\=\sigma\left(\frac{\hat{u}_{i+1,j}-\hat{u}_{ij}}{h_1}-\frac{\partial\hat{u}}{\partial x_1}\right)\frac{\partial}{\partial x_1}(\varphi(\hat{u}))+\sigma\left(\frac{\hat{u}_{i+1,j}+\hat{u}_{ij}}{2}-\hat{u}_{ij}\right)\frac{\partial^2}{\partial x_1^2}(\varphi_1(\hat{u}))+\\+h_1^2\left(\frac{1}{3!}\hat{u}_{x_1,ij}\frac{\partial^3}{\partial x_1^3}(\varphi_1(\hat{u}))+\frac{\hat{u}_{i+1,j}}{4}\frac{\partial^4}{\partial x_1^4}\left(\varphi_1(\hat{\tilde{u}})\right)+\frac{\hat{u}_{ij}}{4}\frac{\partial^4}{\partial x_1^4}\left(\varphi_1(\hat{\tilde{\tilde{u}}})\right)\right)+\\+(1-\sigma)\left(\frac{u_{ij}-u_{i-1,j}}{h_1}-\frac{\partial u}{\partial x_1}\right)\frac{\partial}{\partial x_1}(\varphi(u))+(1-\sigma)\left(\frac{u_{ij}+u_{i-1,j}}{2}-u_{ij}\right)\frac{\partial^2}{\partial x_1^2}(\varphi_1(u))+\\+h_1^2\left(\frac{1}{3!}u_{\overline{x}_1,ij}\frac{\partial^3}{\partial x_1^3}(\varphi_1(u))+\frac{u_{ij}}{4}\frac{\partial^4}{\partial x_1^4}(\varphi_1(\tilde{u}))+\frac{u_{i-1,j}}{4}\frac{\partial^4}{\partial x_1^4}\left(\varphi_1(\tilde{\tilde{u}})\right)\right).\end{aligned}$$

Taking into account the estimates

$$|\hat{u}_{x_1}| = \left| \frac{1}{h_1} \int\limits_{x_{1i}}^{x_{1i+1}} \frac{\partial u(\xi, x_{2j}, t_{n+1})}{\partial \xi} d\xi \right| \leqslant \frac{1}{h_1} \int\limits_{x_{1i}}^{x_{1i+1}} \left| \frac{\partial u(\xi, x_{2j}, t_{n+1})}{\partial \xi} \right| d\xi \leqslant$$

$$\leqslant \frac{1}{h_1} \max_{(x_1,x_2,t)\in\overline{Q}_T} \left| \frac{\partial u}{\partial x_1} \right| \int\limits_{x_{1i}}^{x_{1i+1}} d\xi \leqslant \max_{(x_1,x_2,t)\in\overline{Q}_T} \left| \frac{\partial u}{\partial x_1} \right|,$$

$$|u_{\overline{x}_1}| = \left| \frac{1}{h_1} \int\limits_{x_{1i-1}}^{x_{1i}} \frac{\partial u(\xi, x_{2j}, t_n)}{\partial \xi} d\xi \right| \leqslant \frac{1}{h_1} \int\limits_{x_{1i-1}}^{x_{1i}} \left| \frac{\partial u(\xi, x_{2j}, t_n)}{\partial \xi} \right| d\xi \leqslant$$

$$\leqslant \frac{1}{h_1} \max_{(x_1,x_2,t)\in\overline{Q}_T} \left| \frac{\partial u}{\partial x_1} \right| \int\limits_{x_{1i-1}}^{x_{1i}} d\xi \leqslant \max_{(x_1,x_2,t)\in\overline{Q}_T} \left| \frac{\partial u}{\partial x_1} \right|,$$

we arrive at

$$\psi_2 = h_1(\sigma - 0.5)\frac{\partial^2 u}{\partial x_1^2}\frac{\partial}{\partial x_1}(\varphi_1(u)) + h_1(\sigma - 0.5)\frac{\partial u}{\partial x_1}\frac{\partial^2}{\partial x_1^2}(\varphi_1(u)) + O(\tau^2 + h_1^2). \tag{2.8}$$

Analogously, we write ψ_3 in the form

$$\psi_3 = h_2(\sigma - 0.5)\frac{\partial^2 u}{\partial x_2^2}\frac{\partial}{\partial x_2}(\varphi_2(u)) + h_2(\sigma - 0.5)\frac{\partial u}{\partial x_2}\frac{\partial^2}{\partial x_2^2}(\varphi_2(u)) + O(\tau^2 + h_2^2). \tag{2.9}$$

From equalities (2.7)–(2.9), we find that the difference scheme (2.5), (2.6) has a first order of approximation $O((\sigma - 0.5)(h_1 + h_2 + \tau) + h_1^2 + h_2^2 + \tau^2)$ for $\sigma \neq 0.5$ and a second order for $\sigma = 0.5$.

Lemma 2.1. *If following conditions are satisfied*

$$\frac{\partial}{\partial x_1}(\varphi_1(u)) = -a, \quad \frac{\partial}{\partial x_2}(\varphi_2(u)) = -b, \quad a, b = \text{const} > 0, \tag{2.10}$$

then equality

$$\sigma\left(\hat{u}\,(\varphi_1(\hat{u}))_{\overline{x}_1}\right)_{x_1} + \sigma\left(\hat{u}\,(\varphi_2(\hat{u}))_{\overline{x}_2}\right)_{x_2} + (1-\sigma)\left(u\,(\varphi_1(u))_{x_1}\right)_{\overline{x}_1} + (1-\sigma)\left(u\,(\varphi_2(u))_{x_2}\right)_{\overline{x}_2} =$$
$$= a\sigma\hat{u}_{x_1} + b\sigma\hat{u}_{x_2} + a(1-\sigma)u_{\overline{x}_1} + b(1-\sigma)u_{\overline{x}_2} \tag{2.11}$$

is valid.

Proof. Multiplying equations (2.10) by u and differentiating obtained formulas with respect to x_1 and x_2 respectively, we get following equations

$$\frac{\partial}{\partial x_1}\left(u\frac{\partial \varphi_1(u)}{\partial x_1}\right) = -a\frac{\partial u}{\partial x_1}, \quad \frac{\partial}{\partial x_2}\left(u\frac{\partial \varphi_2(u)}{\partial x_2}\right) = -b\frac{\partial u}{\partial x_2}.$$

Integrating equations (2.10) on the intervals $[x_{1,i-1}, x_{1,i}]$, $[x_{1,i}, x_{1,i+1}]$ and $[x_{2,j-1}, x_{2,j}]$, $[x_{2,j}, x_{2,j+1}]$ respectively, we get equality

$$\sigma\left(\hat{u}\,(\varphi_1\,(\hat{u}))_{\overline{x}_1}\right)_{x_1} + \sigma\left(\hat{u}\,(\varphi_2\,(\hat{u}))_{\overline{x}_2}\right)_{x_2} + (1-\sigma)\left(\hat{u}\,(\varphi_1\,(u))_{x_1}\right)_{\overline{x}_1} + (1-\sigma)\left(\hat{u}\,(\varphi_2\,(u))_{x_2}\right)_{\overline{x}_2} =$$
$$= -a\sigma\hat{u}_{x_1} - b\sigma\hat{u}_{x_2} - a(1-\sigma)u_{\overline{x}_1} - b(1-\sigma)u_{\overline{x}_2}. \tag{2.12}$$

□

Theorem 2.1. *If conditions* (2.10) *of Lemma* 2.1 *are satisfied and* $2a\tau/h_1=1$, $2b\tau/h_2=1$, *then the difference scheme* (2.5), (2.6) *is exact for traveling wave solutions of the form*

$$u(x_1,x_2,t_n)=U\left(\frac{1}{2a}x_1+\frac{1}{2b}x_2-t\right).$$

Proof. It is easy to see that

$$u_{i+1,j}^{n+1}=u(x_{1,i+1},x_{2,j},t_{n+1})=U\left(\frac{1}{2a}x_{1,i+1}+\frac{1}{2b}x_{2,j}-t_{n+1}\right)=$$

$$=U\left(\frac{1}{2a}x_{1,i}+\frac{1}{2b}x_{2,j}-t_n+\frac{1}{2a}h_1-\tau\right)=U\left(\frac{1}{2a}x_{1,i}+\frac{1}{2b}x_{2,j}-t_n\right)=u(x_{1,i},x_{2,j},t_n)=u_{i,j}^n,$$

and similarly $u_{i,j+1}^{n+1}=u_{i,j}^n$. Denote $\gamma_1=a\tau/h_1$ and $\gamma_2=b\tau/h_2$. Then an alternative form of the condition $2a\tau/h_1=2b\tau/h_2=1$ is $\gamma_1=\gamma_2=1/2$.

Applying Lemma 2.1, the error of approximation of the difference scheme (2.5), (2.6) satisfies the chain of equalities

$$\psi_{i,j}^{n+1}=-u_t+\sigma\left(\hat u\left(\varphi_1(\hat u)\right)_{\overline{x}_1}\right)_{x_1}+\sigma\left(\hat u\left(\varphi_2(\hat u)\right)_{\overline{x}_2}\right)_{x_2}+(1-\sigma)\left(u\left(\varphi_1(u)\right)_{x_1}\right)_{\overline{x}_1}+$$

$$+(1-\sigma)\left(u\left(\varphi_2(u)\right)_{x_2}\right)_{\overline{x}_2}=-(\sigma+1-\sigma)u_t-a\sigma\hat u_{x_1}-b\sigma\hat u_{x_2}-a(1-\sigma)u_{\overline{x}_1}-b(1-\sigma)u_{\overline{x}_2}=$$

$$=\sigma\frac{-u_t-a\tau\hat u_{x_1}-b\tau\hat u_{x_2}}{\tau}+(1-\sigma)\frac{-u_t-a\tau u_{\overline{x}_1}-b\tau u_{\overline{x}_2}}{\tau}=$$

$$=\sigma\frac{-\hat u_{i,j}+u_{i,j}-\gamma_1\hat u_{i+1,j}+\gamma_1\hat u_{i,j}-\gamma_2\hat u_{i,j+1}+\gamma_2\hat u_{i,j}}{\tau}+$$

$$+(1-\sigma)\frac{-\hat u_{i,j}+u_{i,j}-\gamma_1 u_{i,j}+\gamma_1 u_{i-1,j}-\gamma_2 u_{i,j}+\gamma_2 u_{i,j-1}}{\tau}=$$

$$=\sigma\frac{(-1+\gamma_1+\gamma_2)\,\hat u_{i,j}+u_{i,j}-\gamma_1\hat u_{i+1,j}-\gamma_2\hat u_{i,j+1}}{\tau}+$$

$$+(1-\sigma)\frac{(1-\gamma_1-\gamma_2)\,u_{i,j}-\hat u_{i,j}+\gamma_1 u_{i-1,j}+\gamma_2 u_{i,j-1}}{\tau}=\sigma\frac{u_{i,j}-\gamma_1\hat u_{i+1,j}-\gamma_2\hat u_{i,j+1}}{\tau}+$$

$$+(1-\sigma)\frac{-\hat u_{i,j}+\gamma_1 u_{i-1,j}+\gamma_2 u_{i,j-1}}{\tau}=\sigma\frac{(1-\gamma_1-\gamma_2)\,u_{i,j}}{\tau}+(1-\sigma)\frac{(-1+\gamma_1+\gamma_2)\,\hat u_{i,j}}{\tau}=0.$$

Thus, the difference scheme (2.5), (2.6) is exact. □

Remark 2.1. The difference scheme (2.5), (2.6) is nonlinear, so there is a need to use the iteration method, for example,

$$\frac{\overset{s+1}{y}-y^n}{\tau}=\sigma\left[\overset{s}{y}\left(\varphi_1(\overset{s}{y})+\varphi_1'\left(\overset{s}{y}\right)\left(\overset{s+1}{y}-\overset{s}{y}\right)\right)_{\overline{x}_1}\right]_{x_1}+\sigma\left[\overset{s}{y}\left(\varphi_2(\overset{s}{y})+\varphi_2'\left(\overset{s}{y}\right)\left(\overset{s+1}{y}-\overset{s}{y}\right)\right)_{\overline{x}_2}\right]_{x_2}+$$

$$+(1-\sigma)\left(y\left(\varphi_1(y)\right)_{x_1}\right)_{\overline{x}_1}+(1-\sigma)\left(y\left(\varphi_2(y)\right)_{x_2}\right)_{\overline{x}_2},\tag{2.13}$$

where the initial approximation $\overset{0}{y}$ is calculated from the explicit difference scheme

$$\frac{\overset{0}{y}-y^n}{\tau}=\left(y\left(\varphi_1(y)\right)_{x_1}\right)_{\overline{x}_1}+\left(y\left(\varphi_2(y)\right)_{x_2}\right)_{\overline{x}_2}.\tag{2.14}$$

The application of the iteration scheme is connected with checking the condition

$$\|\overset{s+1}{y} - \overset{s}{y}\|_C \leqslant \epsilon,$$

where ϵ is the previously given accuracy and the norm $\|\cdot\|_C$ is defined as follows:

$$\|v\|_C = \max_{0<i<N_1, 0<j<N_2} |v_{ij}| .$$

If this condition is satisfied, then we obtain $y_{i,j}^{n+1} = \overset{s+1}{y}_{i,j}$, $i = \overline{1, N_1 - 1}$, $j = \overline{1, N_2 - 1}$.

3. Stability of the linear approach

In the present Section the object of investigation is the stability of the linear approach to the nonlinear difference scheme (2.5), (2.6).

Let us assume that the coefficients of equation (2.1) depend only on x_1, x_2 and the boundary conditions

$$y_t = \sigma(k_1 \hat{y}_{\overline{x}_1})_{x_1} + \sigma(k_2 \hat{y}_{\overline{x}_2})_{x_2} + (1-\sigma)(k_1 y_{x_1})_{\overline{x}_1} + (1-\sigma)(k_2 y_{x_2})_{\overline{x}_2}, \tag{3.1}$$

$$y_{ij}^0 = u_0(x_{1i}, x_{2j}), \quad x_{1i} \in \overline{\omega}_{1h}, \quad x_{2j} \in \overline{\omega}_{2h},$$

$$y_{ij}^{n+1} = 0, \quad (x_{1i}, x_{2j}, t_{n+1}) \in \Gamma_h. \tag{3.2}$$

are homogeneous. We assume that the coefficients k_1 and k_2 are positive and bounded:

$$0 < c_1 \leqslant k_1(x_1, x_2), \quad k_2(x_1, x_2) \leqslant c_2, \quad x_1 \in [0, l_1], \quad x_2 \in [0, l_2], \tag{3.3}$$

where $c_1, c_2 = \text{const}$.

Let us introduce the space H comprising all the functions v defined on the grid $\overline{\omega}$ and satisfying the condition $v_{ij}^n = 0$, $(x_{1i}, x_{2j}, t_n) \in \Gamma_h$, with the inner product $(y, v) = \sum_{i=1}^{N_1-1} \sum_{j=1}^{N_2-1} y_{ij} v_{ij} h_1 h_2$ and norm $\|v\| = \sqrt{(v, v)}$.

Let us define in the space H the grid operators A, B, A_1, A_2, A_1', A_2'

$$(A_1 v) = -\sigma(k_1 v_{\overline{x}_1})_{x_1}, \quad (A_2 v) = -(1-\sigma)(k_1 v_{x_1})_{\overline{x}_1},$$

$$(A_1' v) = -\sigma(k_2 v_{\overline{x}_2})_{x_2}, \quad (A_2' v) = -(1-\sigma)(k_2 v_{x_2})_{\overline{x}_2},$$

$$A = A_1 + A_2 + A_1' + A_2', \quad B = E + \tau\left(A_1 + A_1'\right), \tag{3.4}$$

where E is the identity operator making it possible to rewrite problem (3.1) - (3.2) in the equivalent operator form

$$B y_t + A y = 0, \quad y^0 = u_0. \tag{3.5}$$

It is easy to see that

$$Ay = -\left[\sigma(k_1 y_{\overline{x}_1})_{x_1} + (1-\sigma)(k_1 y_{x_1})_{\overline{x}_1} + \sigma(k_2 y_{\overline{x}_2})_{x_2} + (1-\sigma)(k_2 y_{x_2})_{\overline{x}_2}\right] =$$

$$= -\left[\sigma(k_1 y_{\overline{x}_1})_{x_1} + (1-\sigma)(k_{1,-1} y_{\overline{x}_1})_{x_1} + \sigma(k_2 y_{\overline{x}_2})_{x_2} + (1-\sigma)(k_{2,-1} y_{\overline{x}_2})_{x_2}\right] =$$

$$= -\left[(k_{1(\sigma)} y_{\overline{x}_1})_{x_1} + (k_{2(\sigma)} y_{\overline{x}_2})_{x_2}\right],$$

where $k_{1,-1} = k_{1,i-1,j}$, $k_{2,-1} = k_{2,i,j-1}$, $k_{1(\sigma)} = \sigma k_{1,i,j} + (1-\sigma) k_{1,i-1,j}$, $k_{2(\sigma)} = \sigma k_{2,i,j} + (1-\sigma) k_{2,i,j-1}$.

Let the self-adjoint and positive operator $A : H \to H$ be prescribed. The norm $\|v\|_A = \sqrt{(Av, v)}$ is called the energy norm generated by the operator A. H_A denotes a Hilbert space consisting of elements $v \in H$ provided with the inner product $(y, v)_A = (Ay, v)$ and norm $\|v\|_A$ [25].

Lemma 3.1 [25]. *Let the operator A be a self-adjoint positive operator independent of n. The condition*

$$B \geqslant 0.5\tau A, \quad t \in \omega_\tau,$$

is necessary and sufficient for scheme (3.5) *in H_A to be stable with respect to the initial data*

$$\|y^n\|_A \leqslant \|u_0\|_A\,. \tag{3.6}$$

It is wellknown that the operator A (independent of n) defined by (3.4) is self-adjoint and positive. Thereby, it is sufficient to verify the inequality $B \geqslant 0.5\tau A$. Let us notice that

$$B - 0.5\tau A = E + \tau\left(A_1 + A_1'\right) - 0.5\tau A_1 - 0.5\tau A_1' - 0.5\tau A_2 - 0.5\tau A_2' =$$
$$= E + 0.5\tau(A_1 - A_2) + 0.5\tau(A_1' - A_2').$$

Let $v \in H$ be an arbitrary function, then

$$((B - 0.5\tau A)\, v, v) = ((E + 0.5\tau(A_1 - A_2) + 0.5\tau(A_1' - A_2'))\, v, v) =$$
$$= \|v\|^2 + 0.5\tau\left((A_1 - A_2)\, v, v\right) + 0.5\tau\left((A_1' - A_2')\, v, v\right) = \|v\|^2 - 0.5\tau\sigma\left(\left(k_1 v_{\overline{x}_1}\right)_{x_1}, v\right) +$$
$$+0.5\tau(1-\sigma)\left(\left(k_1 v_{x_1}\right)_{\overline{x}_1}, v\right) - 0.5\tau\sigma\left(\left(k_2 v_{\overline{x}_2}\right)_{x_2}, v\right) + 0.5\tau(1-\sigma)\left(\left(k_2 v_{x_2}\right)_{\overline{x}_2}, v\right).$$

Applying the first Green formula with regard to the operator $\left(k_\alpha v_{\overline{x}_\alpha}\right)_{x_\alpha}$, $\alpha = 1, 2$, [24], we have

$$((B - 0.5\tau A)\, v, v) = \|v\|^2 + 0.5\tau\sigma\left(k_1 v_{\overline{x}_1}, v_{\overline{x}_1}\right]_1 - 0.5\tau(1-\sigma)\left(k_{1,-1} v_{\overline{x}_1}, v_{\overline{x}_1}\right]_1 +$$
$$+0.5\tau\sigma\left(k_2 v_{\overline{x}_2}, v_{\overline{x}_2}\right]_2 - 0.5\tau(1-\sigma)\left(k_{2,-1} v_{\overline{x}_2}, v_{\overline{x}_2}\right]_2 =$$
$$= \|v\|^2 + 0.5\tau\left(\left(\sigma k_1 - (1-\sigma)k_{1,-1}\right) v_{\overline{x}_1}, v_{\overline{x}_1}\right]_1 + 0.5\tau\left(\left(\sigma k_2 - (1-\sigma)k_{2,-1}\right) v_{\overline{x}_2}, v_{\overline{x}_2}\right]_2 =$$
$$= \|v\|^2 + 0.5\tau\left(\sigma k_1 - (1-\sigma)k_{1,-1}, v_{\overline{x}_1}^2\right]_1 + 0.5\tau\left(\sigma k_2 - (1-\sigma)k_{2,-1}, v_{\overline{x}_2}^2\right]_2,$$

where

$$(u, v]_1 = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2-1} u_{i,j} v_{i,j} h_1 h_2, \quad ||v]|_1 = \sqrt{(v, v]_1},$$
$$(u, v]_2 = \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2} u_{i,j} v_{i,j} h_1 h_2, \quad ||v]|_2 = \sqrt{(v, v]_2}.$$

Since $\|v\|^2 \geqslant h_1^2\, ||v_{\overline{x}_1}]|_1^2/4$ and $\|v\|^2 \geqslant h_2^2\, ||v_{\overline{x}_2}]|_2^2/4$, then

$$((B-0.5\tau A)v, v) \geqslant \left(\frac{h_1^2}{8} + \frac{\sigma\tau k_1}{2} - \frac{(1-\sigma)\tau k_{1,-1}}{2}, v_{\overline{x}_1}^2\right]_1 + \left(\frac{h_2^2}{8} + \frac{\sigma\tau k_2}{2} - \frac{(1-\sigma)\tau k_{2,-1}}{2}, v_{\overline{x}_2}^2\right]_2. \tag{3.7}$$

It is easy to see that $((B - 0.5\tau A)v, v) \geqslant 0$, if $h_1^2/8 + \sigma\tau k_{1,i,j}/2 - (1-\sigma)\tau k_{1,i-1,j}/2 \geqslant 0$ and $h_2^2/8 + \sigma\tau k_{2,i,j}/2 - (1-\sigma)\tau k_{2,i,j-1}/2 \geqslant 0$ for $i = \overline{1, N_1}$, $j = \overline{1, N_2}$. These conditions are equivalent to the following one

$$\sigma \geqslant \frac{3}{4} - \frac{c_1}{4c_2} - \frac{1}{8\tau c_2}\min\left\{h_1^2, h_2^2\right\}. \tag{3.8}$$

If condition (3.8) is satisfied, then scheme (3.1), (3.2) is stable. By contrast, the well-known scheme with weight

$$y_t = \sigma(k_1 \hat{y}_{\overline{x}_1})_{x_1} + \sigma(k_2 \hat{y}_{\overline{x}_2})_{x_2} + (1-\sigma)(k_1 y_{\overline{x}_1})_{x_1} + (1-\sigma)(k_2 y_{\overline{x}_2})_{x_2},$$

is stable under the condition [24]

$$\sigma \geqslant 1/2 - (8\tau c_2)^{-1}\min\left\{h_1^2, h_2^2\right\}.$$

4. Numerical experiments

In this Section, the results of the numerical experiments presented to show that the approach, presented in Section 2 can be used for different problems and confirm the theoretical results investigated in the paper.

4.1. Example of the exact difference scheme for the two-dimensional linear parabolic equation. In the domain $\overline{Q}_T$, let us consider the two-dimensional linear parabolic problem [19]

$$\frac{\partial u}{\partial t} = a_1^2 \frac{\partial^2 u}{\partial x_1^2} + b_1^2 \frac{\partial^2 u}{\partial x_2^2}, \quad 0 < x_1 < l_1, \quad 0 < x_2 < l_2, \quad 0 < t \leqslant T, \tag{4.1}$$

where $a_1, b_1 = const > 0$. The exact solution of problem (4.1) given by $u(x_1, x_2, t) = \exp\{t/2 - x_1/(2a_1) - x_2/(2b_1)\}$ satisfies conditions (2.10) with constants $a = a_1/2$, $b = b_1/2$. The boundary and initial conditions are consistent with the exact solution.

Problem (4.1) is approximated by the difference scheme

$$y_{t,i,j} = \sigma(\hat{y}[\varphi_1(\hat{y})]_{\overline{x_1}})_{x_1,i,j} + (1-\sigma)(y[\varphi_1(y)]_{x_1})_{\overline{x_1},i,j} + \sigma(\hat{y}[\varphi_2(\hat{y})]_{\overline{x_2}})_{x_2,i,j} +$$

$$+ (1-\sigma)(y[\varphi_2(y)]_{x_2})_{\overline{x_2},i,j}, \quad x_{1i} \in \omega_{h_1}, \quad x_{2j} \in \omega_{h_2}, \quad t_n \in \omega_\tau, \tag{4.2}$$

where $\varphi_1(y) = a_1^2 \ln y$, $\varphi_2(y) = a_2^2 \ln y$. It is worth to emphasize here that scheme (4.2) is nonlinear as compared to a lot of other schemes approximating the linear problem (4.1).

In all numerical experiments, the accuracy is $\epsilon = 1.0E - 17$ (see Figs. 4.1, 4.2).

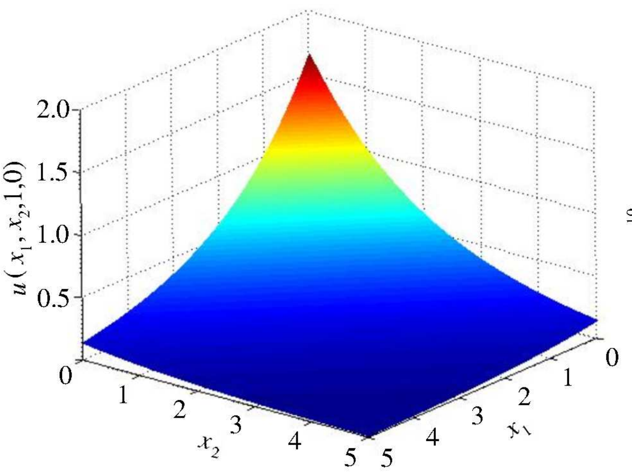

Fig. 4.1. Exact solution of problem (4.1) for $t = 1$, $l_1 = 5$, $l_2 = 5$, $T = 1$, $a_1 = 1$ and $a_2 = 1$

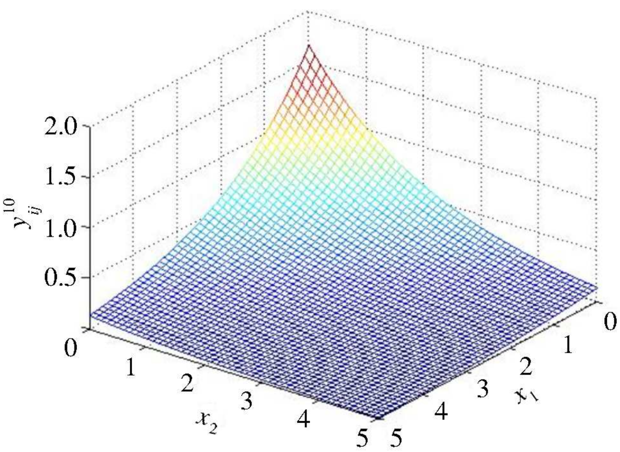

Fig. 4.2. Solution of the difference scheme (4.2) for $\sigma = 0.5$, $l_1 = 5$, $l_2 = 5$, $T = 1$, $a_1 = 1$, $a_2 = 1$, $N_1 = 50$, $N_2 = 50$ and $N_0 = 10$

Here ξ is the number of iterations. Let us stress here that the iteration method (2.13) converges after two-three iterations performed. This is due to the calculation of the initial approximation $\overset{0}{y}_{i,j}$ from the exact but unstable explicit scheme (2.14) [9]. Tables 4.1 and 4.2, in which the results of the numerical experiments for different parameter values are presented, confirm the theoretical results stated in Theorem 2.1.

Table 4.1. $\boldsymbol{\sigma = 0.5,\ a_1 = 1,\ b_1 = 1,\ l_1 = 1,\ l_2 = 1,\ T = 1}$

h_1	h_2	τ	$\max\limits_{0\leqslant n\leqslant N_0} \|y^n - u(t_n)\|_{\overline{C}}$	ξ
0.2	0.2	0.2	$8.67E-19$	2
0.1	0.1	0.1	$9.76E-19$	2
0.05	0.05	0.05	$1.95E-18$	2
0.02	0.02	0.02	$6.94E-18$	2
0.01(6)	0.01(6)	0.01(6)	$6.94E-18$	2

Table 4.2. $\boldsymbol{\sigma = 0.5,\ a_1 = 2,\ b_1 = 4,\ l_1 = 10,\ l_2 = 10,\ T = 5}$

h_1	h_2	τ	$\max\limits_{0\leqslant n\leqslant N_0} \|y^n - u(t_n)\|_{\overline{C}}$	ξ
1.0	2.0	0.5	$9.98E-18$	3
0.5	1.0	0.25	$1.30E-17$	3
0.25	0.5	0.125	$1.86E-17$	3
0.2	0.4	0.1	$2.13E-17$	3

The results of the numerical experiments presented in Figs. 4.3 and 4.4 show that the new scheme resolves the boundary-value problem (4.1) retaining the accuracy of the order $O((\sigma-0.5)(h_1+h_2+\tau)+h_1^2+h_2^2+\tau^2)$.

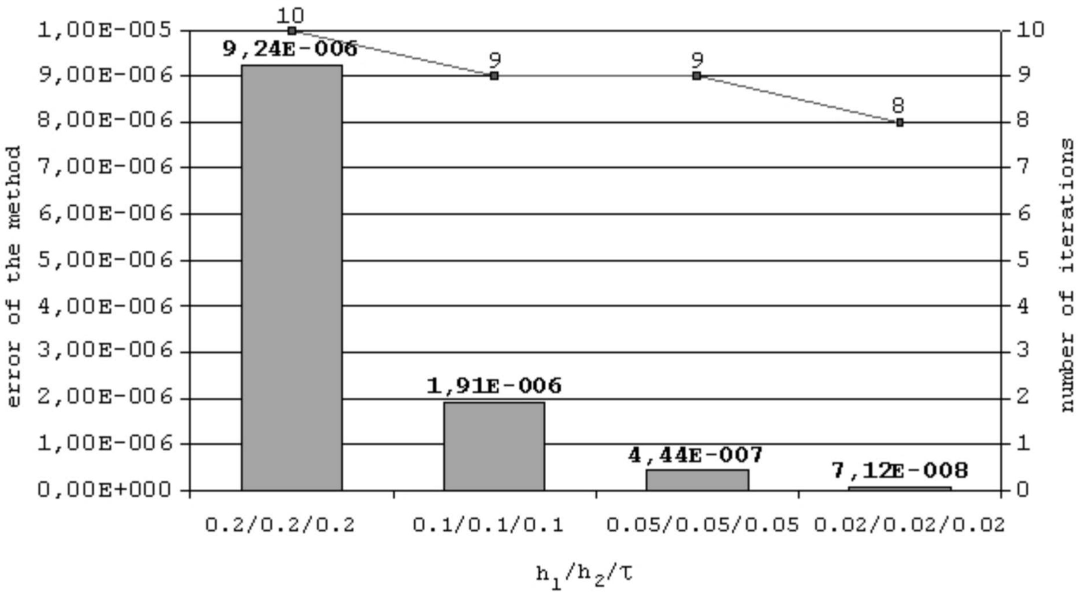

Fig. 4.3. Numerical results for $\sigma = 0.5$, $l_1 = 1$, $l_2 = 1$, $T = 1$, $a_1 = 0.5$ and $b_1 = 4$

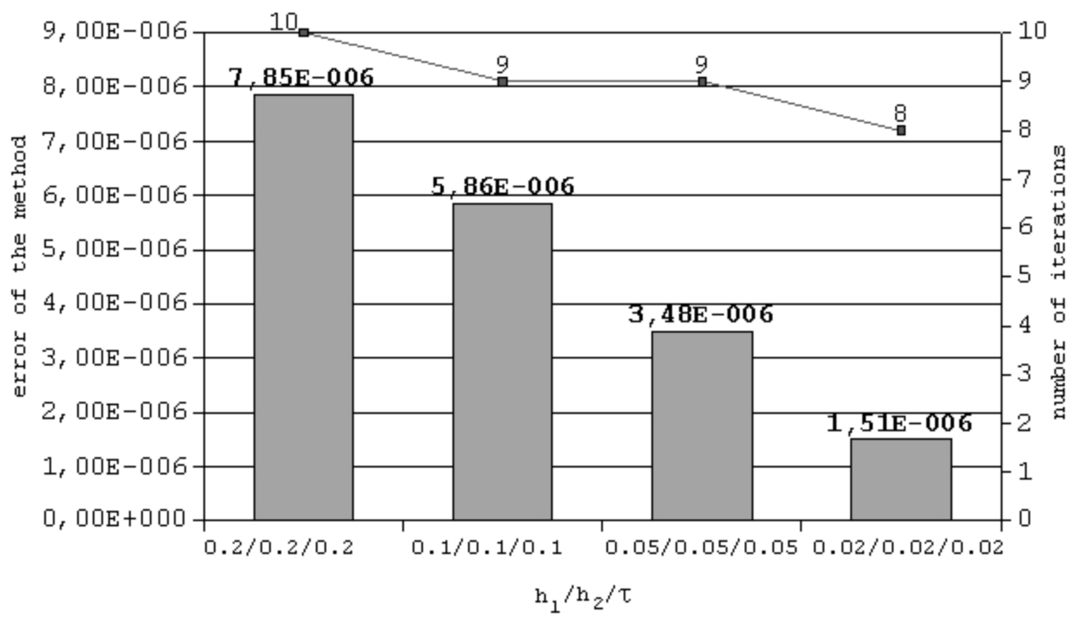

Fig. 4.4. Numerical results for $\sigma = 0.75$, $l_1 = 1$, $l_2 = 1$, $T = 1$, $a_1 = 0.5$ and $b_1 = 4$

4.2. Example of the exact difference scheme for the two-dimensional quasi-linear heat conduction equation. In the domain $\overline{Q}_T$, let us consider the two-dimensional quasi-linear heat conduction problem [20]

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x_1}\left(k\left(u\right)\frac{\partial u}{\partial x_1}\right)+\frac{\partial u}{\partial x_2}\left(k\left(u\right)\frac{\partial u}{\partial x_2}\right),\quad 0<x_1<l_1,\quad 0<x_2<l_2,\quad 0<t\leqslant T, \tag{4.3}$$

where $k(u)=ue^{-u}$. Function

$$u(x_1,x_2,t)=\begin{cases}-\ln\left[1-\lambda\left(\dfrac{\lambda}{2}t-\dfrac{x_1}{2}-\dfrac{x_2}{2}\right)\right], & x_1+x_2\leqslant\lambda t<x_1+x_2+\dfrac{2}{\lambda},\\ 0, & \lambda t<x_1+x_2,\end{cases}$$

where $\lambda=const>0$, is the solution of problem (4.3) and satisfies condition (2.10) with constants $a=b=\frac{\lambda}{2}$. The boundary and initial conditions are consistent with the exact solution.

Problem (4.3) is approximated by the difference scheme

$$y_{t,i,j}=\sigma\left(\hat{y}\left[\varphi(\hat{y})\right]_{\overline{x_1}}\right)_{x_1,i,j}+(1-\sigma)\left(y\left[\varphi(y)\right]_{x_1}\right)_{\overline{x_1},i,j}+\sigma\left(\hat{y}\left[\varphi(\hat{y})\right]_{\overline{x_2}}\right)_{x_2,i,j}+(1-\sigma)\left(y\left[\varphi(y)\right]_{x_2}\right)_{\overline{x_2},i,j},$$
$$x_{1i}\in\omega_{h_1},\quad x_{2j}\in\omega_{h_2},\quad t_n\in\omega_\tau, \tag{4.4}$$

where $\varphi(y)=\exp\{-y\}$.

In all numerical experiments, the accuracy is $\epsilon=1.0E-17$ (see Figs. 4.5, 4.6).

Here ξ is the number of iterations. Tables 4.3 and 4.4, in which the results of the numerical experiments for different parameter values are presented, confirm the theoretical results stated in Theorem 2.1.

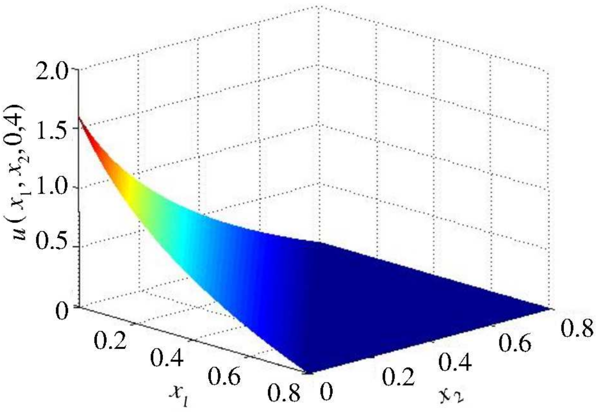

Fig. 4.5. Exact solution of problem (4.3) for $t=0.4$, $l_1=0.8$, $l_2=0.8$, $T=0.4$ and $\lambda=2$

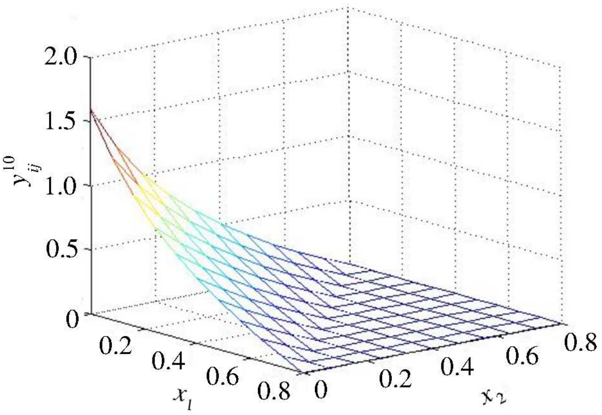

Fig. 4.6. Solution of the difference scheme (4.4) for $\sigma=0.5$, $t=0.4$, $l_1=0.8$, $l_2=0.8$, $T=0.4$, $\lambda=2$, $N_1=10$, $N_2=10$ and $N_0=10$

Table 4.3. $\boldsymbol{\sigma=0.5,\ \lambda=1,\ l_1=1,\ l_2=1,\ T=1}$

h_1	h_2	τ	$\max\limits_{0\leqslant n\leqslant N_0}\Vert y^n-u(t_n)\Vert_{\overline{C}}$	ξ
0.2	0.2	0.2	$1.08E-19$	2
0.1	0.1	0.1	$1.68E-18$	2
0.05	0.05	0.05	$5.10E-18$	2
0.02	0.02	0.02	$3.36E-18$	2

Table 4.4. $\boldsymbol{\sigma=0.5,\ \lambda=2,\ l_1=0.8,\ l_2=0.8,\ T=0.4}$

h_1	h_2	τ	$\max\limits_{0\leqslant n\leqslant N_0}\Vert y^n-u(t_n)\Vert_{\overline{C}}$	ξ
0.2	0.2	0.1	$1.08E-19$	1
0.1	0.1	0.05	$1.14E-18$	1
0.05	0.05	0.025	$4.23E-18$	2
0.02	0.02	0.01	$7.91E-18$	3

5. Conclusions

In this paper, we have constructed a new difference scheme approximating the two-dimensional heat conduction equation. We have proved that for the traveling wave solutions of the form

$$u(x,t) = U\left(\frac{1}{2a}x_1 + \frac{1}{2b}x_2 - t\right), \quad 0 \leqslant x_1 \leqslant l_1, \quad 0 \leqslant x_2 \leqslant l_2, \quad 0 \leqslant t \leqslant T,$$

the considered scheme is exact if the grid steps satisfy definite conditions

$$\gamma_1 = \frac{a\tau}{h_1} = \frac{1}{2}, \quad \gamma_2 = \frac{b\tau}{h_2} = \frac{1}{2}.$$

Numerical results have been presented to confirm the theoretical results stated in the paper.

Future research will focus on multidimensional problems for parabolic equations with a traveling wave solution of the form $u(x_1,x_2,t) = U(z_1,z_2)$, $z_1 = k_1x_1 - \lambda_1 t$, $z_2 = k_2x_2 - \lambda_2 t$, where λ_1/k_1 and λ_2/k_2 are the velocities of wave propagation in the directions x_1 and x_2, respectively.

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