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Symmetry and Exact Solutions for Systems of Nonlinear Reaction-Diffusion Equations

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Lie symmetry reduction of systems of nonlinear reaction-diffusion equation with respect to one-dimensional algebras is carried out. Some classes of exact solutions of the investigated equations are found.

1 Introduction

Nonlinear reaction-diffusion equations are widely used in mathematical physics, chemistry and biology. In the present paper we consider the system of nonlinear diffusion equations of the following general form

$$\frac{\partial u_1}{\partial t} - \frac{\partial^2}{\partial x^2}(a_{11}u_1 + a_{12}u_2) = f^1(u_1, u_2),$$
$$\frac{\partial u_2}{\partial t} - \frac{\partial^2}{\partial x^2}(a_{21}u_1 + a_{22}u_2) = f^2(u_1, u_2), \tag{1}$$

where u_1 and u_2 are functions dependent on t and x; a_{11}, a_{12}, a_{21}, a_{22} are constant parameters and $a_{11}a_{22} - a_{21}a_{12} \neq 0$.

In [1] a constructive algorithm was proposed for investigation of conditional and classical Lie symmetries of partial differential equations and classical symmetries of systems of two nonlinear diffusion equations with $1 + m$ independent variables $t, x_1, \ldots, x_m$ were described. Namely, all possible non-linearities f^1, f^2 and the corresponding group generators were found. We notice that symmetry properties of nonlinear multidimensional systems of reaction-diffusion equations were also investigated in papers [2, 3]. In the present paper using the results obtained in [1] we carry out symmetry reduction of equation (1) with respect to one-dimensional symmetry algebras. We restrict ourselves to such non-linearities f^1 and f^2 found in [1] which are defined up to arbitrary functions.

2 Symmetry reduction of equation (1)

We will not give the detailed calculations but present the operators, ansatzes and corresponding reduced systems for some nonlinearities f^1, f^2 found in [1, 3]. We use the following notation:

$$X_0 = \alpha\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial x}, \qquad D_1 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - \frac{2}{k}\hat{B}, \qquad \hat{B} = B^{ab}u_b\frac{\partial}{\partial u_a},$$
$$D_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - \frac{2}{k}\left(\frac{\partial}{\partial u_1} - 2nu_1\frac{\partial}{\partial u_2}\right), \qquad D_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - \frac{2}{k}p_\alpha\frac{\partial}{\partial u_\alpha},$$

where α and β are arbitrary real coefficients, B^{ab} are elements of the 2×2 matrix B which will be specified in the following.

1. Consider the following system of type (1)

$$\frac{\partial u_1}{\partial t} - a\frac{\partial^2 u_1}{\partial x^2} = \exp\left(k\frac{u_2}{u_1}\right)\varphi_1 u_1,$$
$$\frac{\partial u_2}{\partial t} - b\frac{\partial^2 u_1}{\partial x^2} - a\frac{\partial^2 u_2}{\partial x^2} = \exp\left(k\frac{u_2}{u_1}\right)(\varphi_1 u_2 + \varphi_2), \tag{2}$$

where φ_1 and φ_2 are arbitrary (but fixed) functions of u_1, $a_{11} = a_{22} = a$, $a_{12} = 0$, $a_{21} = b$.

This system admits the symmetry operator

$$X = X_0 + \nu D_1, \qquad \text{where} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The corresponding ansatz be obtained using the Lie algorithms is

$$u_1 = \omega_1(z), \qquad u_2 = -\frac{2}{k}\ln(\nu x + \beta)\omega_1(z) + \omega_2(z), \qquad z = \frac{2(\nu x + \beta)^2}{2\nu t + \alpha}. \tag{3}$$

Substituting the ansatz (3) into (2) we come to the following reduced equations

$$2\nu z^2\dot\omega_1 + 2\nu^2 az\dot\omega_1 + 8\nu^2 az^2\ddot\omega_1 = -\exp\left(k\frac{\omega_2}{\omega_1}\right)\varphi_1\omega_1,$$
$$2\nu z^2\dot\omega_2 + \frac{2\nu^2 a}{k}\dot\omega_1 - \frac{8\nu^2 a}{k}z\dot\omega_1 + 2\nu^2 bz\dot\omega_1 + 2\nu^2 az\dot\omega_2 + 8\nu^2 bz^2\ddot\omega_1 + 8\nu^2 az^2\ddot\omega_2$$
$$= -\exp\left(k\frac{\omega_2}{\omega_1}\right)(\varphi_1\omega_2 + \varphi_2).$$

In other words the ansatz (3) reduces (2) to the system of ordinary differential equations.

The following results (related to equations found in [1]) are presented more briefly.

2. Equations:

$$\frac{\partial u_1}{\partial t} - a\frac{\partial^2 u_1}{\partial x^2} + b\frac{\partial^2 u_2}{\partial x^2} = \varphi_1 u_2 + \varphi_2 u_1, \qquad \frac{\partial u_2}{\partial t} - b\frac{\partial^2 u_1}{\partial x^2} - a\frac{\partial^2 u_2}{\partial x^2} = -\varphi_1 u_1 + \varphi_2 u_2,$$

where φ_1 and φ_2 are arbitrary functions of $\sqrt{u_1^2 + u_2^2}$, $a_{11} = a_{22} = a$, $a_{21} = -a_{12} = b$.

Symmetry:

$$X = X_0 + \mu\hat B, \qquad \text{where} \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Ansatz:

$$u_1 = \cos\left(\frac{\mu}{\alpha}t\right)\omega_1(z) - \sin\left(\frac{\mu}{\alpha}t\right)\omega_2(z), \qquad u_2 = \sin\left(\frac{\mu}{\alpha}t\right)\omega_1(z) + \cos\left(\frac{\mu}{\alpha}t\right)\omega_2(z),$$
$$z = \beta t - \alpha x.$$

Reduced equations:

$$-\frac{\mu}{\alpha}\omega_2 + \beta(a\dot\omega_1 - b\dot\omega_2) - \alpha^2(a\ddot\omega_1 - b\ddot\omega_2) = \varphi_1\omega_2 + \varphi_2\omega_1,$$
$$\frac{\mu}{\alpha}\omega_1 + \beta(b\dot\omega_1 + a\dot\omega_2) - \alpha^2(b\ddot\omega_1 + a\ddot\omega_2) = -\varphi_1\omega_1 + \varphi_2\omega_2,$$

where φ_1 and φ_2 are functions of $\omega_1^2 + \omega_2^2$.

3. Equations:

$$\frac{\partial u_1}{\partial t} - a\frac{\partial^2 u_1}{\partial x^2} = u_1\varphi_1, \qquad \frac{\partial u_2}{\partial t} - b\frac{\partial^2 u_2}{\partial x^2} = u_2\varphi_2,$$

where φ_1 and φ_2 are arbitrary functions of $\frac{u_2}{u_1^d}$, $a_{11} = a$, $a_{12} = a_{21} = 0$, $a_{22} = b$.

Symmetry:

$$X = X_0 + \mu\hat{B}, \qquad \text{where} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}.$$

Ansatz:

$$u_1 = \exp\left(\frac{\mu}{\beta}x\right)\omega_1(z), \qquad u_2 = \exp\left(\frac{\mu d}{\beta}x\right)\omega_2(z), \qquad z = \beta t - \alpha x.$$

Reduced equations:

$$\beta\dot{\omega}_1 - a\left(\frac{\mu}{\beta}\right)^2\omega_1 + 2\alpha a\frac{\mu}{\beta}\dot{\omega}_1 - \alpha^2 a\ddot{\omega}_1 = \omega_1\varphi_1,$$
$$\beta\dot{\omega}_2 - b\left(\frac{\mu d}{\beta}\right)^2\omega_2 + 2\alpha b\frac{\mu d}{\beta}\dot{\omega}_2 - \alpha^2 b\ddot{\omega}_2 = \omega_2\varphi_2,$$

where φ_1 and φ_2 are functions of $\frac{\omega_2}{\omega_1^d}$.

4. Equation:

$$\frac{\partial u_1}{\partial t} - a\frac{\partial^2 u_1}{\partial x^2} = \varphi_1, \qquad \frac{\partial u_2}{\partial t} - b\frac{\partial^2 u_1}{\partial x^2} - a\frac{\partial^2 u_2}{\partial x^2} = \frac{u_2}{u_1}\varphi_1 + nu_2 + \varphi_2,$$

where φ_1 and φ_2 are arbitrary functions of u_1, $a_{11} = a_{22} = a$, $a_{12} = 0$, $a_{21} = b$.

Symmetry:

$$X = X_0 + \mu\exp(nt)\hat{B}, \qquad \text{where} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Ansatz:

$$u_1 = \omega_1(z), \qquad u_2 = \frac{\mu}{\alpha n}\omega_1(z)\exp(nt) + \omega_2(z), \qquad z = \beta t - \alpha x.$$

Reduced equations:

$$\beta\dot{\omega}_1 - \alpha^2 a\ddot{\omega}_1 = \varphi_1,$$
$$\beta\dot{\omega}_2 - \alpha^2 b\ddot{\omega}_1 - \alpha a\ddot{\omega}_2 = \frac{\omega_2}{\omega_1}\varphi_1 + n\omega_2 + \varphi_2,$$

where φ_1 and φ_2 are functions of ω_1.

5. Equation:

$$\frac{\partial u_1}{\partial t} - a\frac{\partial^2 u_1}{\partial x^2} = \varphi_1 u_1^{k+1}, \qquad \frac{\partial u_2}{\partial t} - b\frac{\partial^2 u_1}{\partial x^2} - a\frac{\partial^2 u_2}{\partial x^2} = (\varphi_1\ln u_1 + \varphi_2)u_1^{k+1},$$

where φ_1 and φ_2 are arbitrary functions of $u_1\exp\left(-\frac{u_2}{u_1}\right)$, $a_{11} = a_{22} = a$, $a_{12} = 0$, $a_{21} = b$.

Symmetry:

$$X = X_0 + \nu D_1, \qquad \text{where} \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Ansatz:

$$u_1 = (2\nu t + \alpha)^{-\frac{1}{k}}\omega_1(z), \qquad u_2 = (2\nu t + \alpha)^{-\frac{1}{k}}\left(\omega_2(z) - \frac{1}{k}\ln(2\nu t + \alpha)\omega_1(z)\right),$$
$$z = \frac{2(\nu x + \beta)^2}{2\nu t + \alpha}.$$

Reduced equations:

$$\frac{2\nu}{k}\omega_1 + 2\nu z\dot{\omega}_1 + 8\nu^2 a z\ddot{\omega}_1 = -\omega_1^{k+1}\varphi_1,$$
$$\frac{2\nu}{k}\omega_2 + 2\nu z\dot{\omega}_2 + \frac{2\nu}{k}\omega_1 + 8\nu^2 b z\ddot{\omega}_1 + 8\nu^2 a z\ddot{\omega}_2 = -(\varphi_1 \ln\omega_1 + \varphi_2)\omega_1^{k+1},$$

where φ_1 and φ_2 are functions of $\omega_1 \exp\left(-\frac{\omega_2}{\omega_1}\right)$.

3 Conditional symmetry and exact solutions

Thus we presented reductions of equations (1) using their classical symmetry found in [1]. In this section we present exact solutions of equations (1) found by conditional symmetry reduction. We use the same scheme of presentation as in Section 2.

1. Equation:

$$\frac{\partial u_1}{\partial t} - \frac{\partial^2 u_1}{\partial x^2} = u_1^3\varphi_1, \qquad \frac{\partial u_2}{\partial t} - \frac{\partial^2 u_2}{\partial x^2} = u_2^3\varphi_2, \tag{4}$$

where φ_1 and φ_2 are arbitrary functions of $\frac{u_2}{u_1}$.

Conditional symmetry:

$$X = \frac{\partial}{\partial t} - \frac{3}{x + k_1}\frac{\partial}{\partial x} - \frac{3}{(x + k_1)^2}\left(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}\right).$$

The ansatz

$$u = (x + k_1)\omega(z), \qquad z = \frac{1}{2}x^2 + k_1 x + 3t$$

reduces equation (4) to the system:

$$\ddot{\omega}_1 + \varphi_1\omega_1^3 = 0, \qquad \ddot{\omega}_2 + \varphi_2\omega_2^3 = 0,$$

where φ_1 and φ_2 are functions of $\frac{\omega_2}{\omega_1}$.

Depending on the form of the functions φ_1, φ_2, we receive different solutions of the system.

1) $\varphi_1 = a > 0$, $\varphi_2 = b < 0$, where a and b are constants:

$$u_1(x,t) = \frac{\sqrt{2a}}{2a}(x + k_1)\,\mathrm{sd}\left(\frac{1}{2}x^2 + k_1 x + 3t; \frac{1}{2}\sqrt{2}\right),$$
$$u_2(x,t) = -\frac{\sqrt{-2b}}{b}(x + k_1)\,\mathrm{ds}\left(\frac{1}{2}x^2 + k_1 x + 3t; \frac{1}{2}\sqrt{2}\right).$$

2) $\varphi_1 = a > 0$, $\varphi_2 = 0$:

$$u_1(x,t) = \frac{\sqrt{2a}}{2a}(x + k_1)\,\mathrm{sd}\left(\frac{1}{2}x^2 + k_1 x + 3t; \frac{1}{2}\sqrt{2}\right),$$
$$u_2(x,t) = (x + k_1)\left[\left(\frac{1}{2}x^2 + k_1 x + 3t\right)C_1 + C_2\right].$$

2. Equation:

$$\frac{\partial u_1}{\partial t} - \frac{\partial^2 u_1}{\partial x^2} = u_1^3\varphi_1 - 2\mu^2 u_1, \qquad \frac{\partial u_2}{\partial t} - \frac{\partial^2 u_2}{\partial x^2} = u_2^3\varphi_2 - 2\mu^2 u_2, \tag{5}$$

where φ_1 and φ_2 are arbitrary functions of $\frac{u_2}{u_1}$.

Conditional symmetry:

$$X = \frac{\partial}{\partial t} + 3\mu\tan(\mu x + k_1)\frac{\partial}{\partial x} - 3\mu^2\sec^2(\mu x + k_1)\left(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}\right).$$

The ansatz

$$u = \cos(\mu x + k_1)\exp\left(-3\mu^2 t\right)\omega(z), \qquad z = \sin(\mu x + k_1)\exp(-3\mu^2 t)$$

reduces equation (5) to the system:

$$\mu^2\ddot{\omega}_1 + \omega_1^3\varphi_1 = 0, \qquad \mu^2\ddot{\omega}_2 + \omega_2^3\varphi_2 = 0,$$

where φ_1 and φ_2 are functions of $\frac{\omega_2}{\omega_1}$.

Setting more particular form for the functions φ_1, φ_2, we get the following solutions of the reduced system.

1) $\varphi_1 = a > 0$, $\varphi_2 = b > 0$:

$$u_1(x,t) = \frac{\mu\sqrt{2a}}{2a}\cos(\mu x + k_1)\exp\left(-3\mu^2 t\right)\operatorname{sd}\left[\sin(\mu x + k_1)\exp\left(-3\mu^2 t\right); \frac{1}{2}\sqrt{2}\right],$$
$$u_2(x,t) = \frac{\mu\sqrt{2b}}{2b}\cos(\mu x + k_1)\exp\left(-3\mu^2 t\right)\operatorname{sd}\left[\sin(\mu x + k_1)\exp\left(-3\mu^2 t\right); \frac{1}{2}\sqrt{2}\right].$$

2) $\varphi_1 = a < 0$, $\varphi_2 = b > 0$:

$$u_1(x,t) = -\frac{\mu\sqrt{-2a}}{a}\cos(\mu x + k_1)\exp\left(-3\mu^2 t\right)\operatorname{ds}\left[\sin(\mu x + k_1)\exp\left(-3\mu^2 t\right); \frac{1}{2}\sqrt{2}\right],$$
$$u_2(x,t) = \frac{\mu\sqrt{2b}}{2b}\cos(\mu x + k_1)\exp\left(-3\mu^2 t\right)\operatorname{sd}\left[\sin(\mu x + k_1)\exp\left(-3\mu^2 t\right); \frac{1}{2}\sqrt{2}\right].$$

3. Equation:

$$\frac{\partial u_1}{\partial t} - \frac{\partial^2 u_1}{\partial x^2} = u_1^3\varphi_1 + 2\mu^2 u_1, \qquad \frac{\partial u_2}{\partial t} - \frac{\partial^2 u_2}{\partial x^2} = u_2^3\varphi_2 + 2\mu^2 u_2, \tag{6}$$

where φ_1 and φ_2 are arbitrary functions of $\frac{u_2}{u_1}$.

Conditional symmetry:

$$X = \frac{\partial}{\partial t} - 3\mu\coth(\mu x + k_1)\frac{\partial}{\partial x} - 3\mu^2\csc h^2(\mu x + k_1)\left(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}\right).$$

The ansatz

$$u = \sinh(\mu x + k_1)\exp\left(3\mu^2 t\right)\omega(z), \qquad z = \cosh(\mu x + k_1)\exp\left(3\mu^2 t\right)$$

reduces equation (6) to the system:

$$\mu^2\ddot{\omega}_1 + \omega_1^3\varphi_1 = 0, \qquad \mu^2\ddot{\omega}_2 + \omega_2^3\varphi_2 = 0,$$

where φ_1 and φ_2 are functions of $\frac{\omega_2}{\omega_1}$.

We present the obtained results for some functions φ_1 and φ_2.

1) $\varphi_1 = a < 0$, $\varphi_2 = b < 0$:

$$u_1(x,t) = -\frac{\mu\sqrt{-2a}}{a}\sinh(\mu x + k_1)\exp\left(3\mu^2 t\right)\,\mathrm{ds}\left[\cosh(\mu x + k_1)\exp\left(3\mu^2 t\right); \frac{1}{2}\sqrt{2}\right],$$
$$u_2(x,t) = -\frac{\mu\sqrt{-2b}}{b}\sinh(\mu x + k_1)\exp\left(3\mu^2 t\right)\,\mathrm{ds}\left[\cosh(\mu x + k_1)\exp\left(3\mu^2 t\right); \frac{1}{2}\sqrt{2}\right].$$

2) $\varphi_1 = 0$, $\varphi_2 = b > 0$:

$$u_1(x,t) = \sinh(\mu x + k_1)\exp\left(3\mu^2 t\right)\left[C_1\cosh(\mu x + k_1)\exp\left(3\mu^2 t\right) + C_2\right],$$
$$u_2(x,t) = \frac{\mu\sqrt{2b}}{2b}\sinh(\mu x + k_1)\exp\left(3\mu^2 t\right)\,\mathrm{sd}\left[\cosh(\mu x + k_1)\exp\left(3\mu^2 t\right); \frac{1}{2}\sqrt{2}\right].$$

Besides for equation

$$u_t - u_{xx} = -u^2,$$

we got the following solutions

$$u = \frac{(48 - 12\sqrt{6})x^2 + (48 - 12\sqrt{6})k_1 x + 40(36 - 15\sqrt{6})t + (24 - 12\sqrt{6})k_2 + 6k_1^2}{[x^2 + k_1 x + 2(15 - 5\sqrt{6})t + k_2]^2},$$

and

$$u = \frac{(48 + 12\sqrt{6})x^2 + (48 + 12\sqrt{6})k_1 x + 40(36 + 15\sqrt{6})t + (24 + 12\sqrt{6})k_2 + 6k_1^2}{[x^2 + k_1 x + 2(15 + 5\sqrt{6})t + k_2]^2}.$$

Thus we presented reduced equations and exact solutions for some of nonlinear reaction-diffusion equations whose symmetry was studied in [1, 3]. We plan to extend our results to all systems described in [3].

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