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Separation of variables of a generalized porous medium equation with nonlinear source

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Abstract

This paper considers a general form of the porous medium equation with nonlinear source term: $u_t = (D(u)u_x^n)_x + F(u)$, $n \neq 1$. The functional separation of variables of this equation is studied by using the generalized conditional symmetry approach. We obtain a complete list of canonical forms for such equations which admit the functional separable solutions. As a consequence, some exact solutions to the resulting equations are constructed, and their behavior are also investigated.

Keywords: Separation of variable; Symmetry group; Generalized conditional symmetry; Nonlinear diffusion equation; Exact solution

1. Introduction

This paper is devoted to the study on the functional separation of variables of a generalized porous medium equation with nonlinear source term

$$u_t = \left(D(u)u_x^n\right)_x + F(u), \quad n \neq 1, \tag{1}$$

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for a single function u of two independent variables t and x; $D(u)$ and $F(u)$ are respectively the diffusion and source term. This type of equation has a wide range of applications in physics, diffusion process and engineering sciences (see [1–4] and references therein).

Equation (1) with $n = 1$ has been used to model successfully physical situations in a wide variety of fields involving diffusion process, plasma physics and transport phenomena. It has been studied via many effective methods in a great number of publications. These methods include the classical method [5–10]; the nonclassical method [11–19]; the generalized conditional symmetry (GCS) method [20–24]; the nonlocal symmetry method [25,26]; the truncated Painlevé approach [27–29], the sign-invariant and invariant space methods [30–33]; the transformation method [34–37] and the ansatz-based method [38–44], etc. There are many different directions of the mathematical theory to consider the existence, uniqueness, regularity, asymptotic behavior and other aspects of the qualitative properties of solutions in this case [47–49]. But for case $n \neq 1$, there are very few papers to consider this equation, only the transformation method [3] and the classical symmetry method [4] were used to study the exact solutions and symmetry reductions of Eq. (1) without source term. In [50], the existence, uniqueness, asymptotic behavior and regularity of solutions for Cauchy problem of (1) without source term are discussed.

A solution of (1) is said to be functional separable [51] if there exist some functions q, ϕ and ψ of the indicated single variable such that

$$q(u) = \phi(t) + \psi(x). \tag{2}$$

The classical additive separable solution $u = \phi(t) + \psi(x)$ and product separable solution $u = \phi(t)\psi(x)$ are special cases of the general functional separable solution. There are very few papers coping with the direct method based on some ansatz [52–55] to consider the functional separable solutions of nonlinear partial differential equations (PDEs). From point of view of symmetry group, two methods have been proposed to study the compatibility of (2) and the considered equations. One is the nonclassical method [45] in which three different invariant-surface conditions are used to characterize the functional separable solutions. The other is the GCS approach based on the ansatz (2) [46]. We note that an $(1 + 1)$-dimensional PDE admits functional separable solution (2) if and only if it has the GCS

$$V = \eta \frac{\partial}{\partial u}, \tag{3}$$

where

$$\eta = u_{xt} + \tilde{q}(u) u_x u_t$$

and $\tilde{q}(u) = q''(u)/q'(u)$.

It is interesting to note that if Eq. (1) has functional separable solutions, then equations

$$v_t = B(v)v_x^{n-1}v_{xx} + A(v)v_x^{n+1} + G(v) \tag{4}$$

and

$$w_t = K(w)w_x^{n-1}w_{xx} + Q(w) \tag{5}$$

also have functional separable solutions. In fact, Eqs. (1), (4) and (5) are related as follows:

Relation between (1) and (4) If we put $u = u(v)$ in (1), we obtain

$$v_t = nD\dot{u}^{n-1}v_x^{n-1}v_{xx} + \left(D_u\dot{u}^n + nD\dot{u}^{n-2}\ddot{u}\right)v_x^{n+1} + \frac{F}{\dot{u}},$$

where $\dot{} = d/dv$.

By comparison with (4), we have

$$B = nD(u)(\dot{u})^{n-1}, \qquad A = \dot{D}(u)(\dot{u})^{n-1} + nD(u)\ddot{u}\dot{u}^{n-2}, \qquad G = \frac{F}{\dot{u}},$$

that can be combined as

$$D = \frac{B}{n}\dot{u}^{1-n}, \qquad \frac{\dot{D}}{D} + n\frac{\ddot{u}}{\dot{u}} = n\frac{A}{B}, \qquad G = \frac{F}{\dot{u}}. \tag{6}$$

Using the first equation of (6), the second one can be integrated as

$$u(v) = \int^{v} \left(\frac{n}{B(v)}\exp\left[\int n\frac{A(v)}{B(v)}\,dv\right]\right)dv.$$

Consequently, we have the following relation between (1) and (4):

$$u(v) = \int^{v} \left(\frac{n}{B(v)}\exp\left[\int n\frac{A(v)}{B(v)}\,dv\right]\right)dv, \qquad D(u) = \frac{B(v)}{n}\dot{u}^{1-n},$$
$$F(u) = G(v)\dot{u}, \tag{7}$$

Relation between (5) and (4) If we do $w = w(v)$ in Eq. (5), we obtain

$$v_t = K\dot{w}^{n-1}v_x^{n-1}v_{xx} + K\dot{w}^{n-2}\ddot{w}v_x^{n+1} + \frac{Q}{\dot{w}}.$$

Proceeding as above, the comparison with (4) yields to

$$w(v) = \int^{v} \left(\exp\left[\int \frac{A(v)}{B(v)}\,dv\right]\right)dv, \qquad K(w) = B(v)\dot{w}^{1-n},$$
$$Q(w) = G(v)\dot{w}. \tag{8}$$

Relation between (5) and (1) In the same way, it is possible to relate (1) and (5) as

$$w(u) = \int^{u} \big[D(u)\big]^{1/n}\,du, \qquad K(w) = n\big[D(u)\big]^{1/n},$$
$$Q(w) = F(u)\big[D(u)\big]^{1/n}. \tag{9}$$

Moreover, if (4) admits functional separable solutions, for any arbitrary function $q = q(v)$, then equation

$$q_t = \tilde{B}(q)q_x^{n-1}q_{xx} + \tilde{A}(q)q_x^{n+1} + \tilde{G}(q)$$

also possesses separable solutions, where

$$A(v) = \tilde{A}(q)(\dot{q})^n + \tilde{B}(q)(\dot{q})^{n-2}\ddot{q}, \qquad B(v) = \tilde{B}\dot{q}^{n-1},$$
$$G(v) = \frac{\tilde{G}(q)}{\dot{q}}.$$

So it is sufficient to study the additive separable solution to Eq. (4).

The outline of this paper is as follows. In Section 2, we will classify Eqs. (1), (4) and (5) which admit functional separable solutions. Some exact solutions to the resulting equations are obtained in Section 3. Section 4 is a concluding remarks on this work.

2. Equations with separable solutions

As for group-invariant solutions, the functional separable solutions are also useful to reflect some important properties of PDEs [56]. Here we apply the GCS method to the study of the functional separation of variables of (1), (4) and (5). Indeed, the GCS is a natural generalization of the nonclassical method in a similar way that the generalized symmetry method is a generalization of the Lie point symmetry.

Consider the general mth-order $(1+1)$-dimensional evolution equation

$$u_t = E(x, t, u, u_1, u_2, \ldots, u_m), \tag{10}$$

where $u_k = \partial^k/\partial x^k$, $1 \leqslant k \leqslant m$, and E is a smooth function of the indicated variables. Let

$$V = \eta(t, x, u, u_1, \ldots, u_j)\frac{\partial}{\partial u} \tag{11}$$

be an evolutionary vector field and η its characteristic.

Definition 1. The evolutionary vector field (11) is said to be a generalized symmetry of (10) if and only if

$$V^{(m)}(u_t - E)|_L = 0,$$

where L is the solution set of (10), and $V^{(m)}$ is an mth prolongation of V.

Definition 2. The evolutionary vector field (11) is said to be a GCS of (10) if and only if

$$V^{(m)}(u_t - E)|_{L\cap W} = 0, \tag{12}$$

where W is the set of equations $D_x^i \eta = 0$, $i = 0, 1, 2, \ldots$.

It follows from (12) that (10) admits the GCS (11) if and only if

$$D_t \eta = 0, \tag{13}$$

where D_t is the total derivative in t. Moreover, if η does not depend on time t explicitly, then

$$\eta' E|_{L\cap W} = 0,$$

where

$$\eta'(u)E = \lim_{\epsilon\to 0} \frac{d}{d\epsilon}\eta(u + \epsilon E)$$

denotes the Gateaux derivative of η along the direction E.

The following theorem (that has been proved in [46]) is useful.

Theorem 1. *Equation* (10) *possesses the functional separable solution* (2) *if and only if it admits the GCS*

$$V = (u_{xt} + \tilde{q} u_x u_t)\frac{\partial}{\partial u}. \tag{14}$$

Corollary 1. *Equation* (10) *possesses the additive functional separable solution*

$$u = \phi(t) + \psi(x) \tag{15}$$

if and only if it admits the GCS

$$V = u_{xt}\frac{\partial}{\partial u}.$$

Lemma 1. *Equation* (4) *has the additive separable solution* (15) *if and only if the coefficient functions A, B and G satisfy*

$$A' = \alpha B, \qquad (n+1)A + B' = \beta B, \qquad G' = \gamma B, \tag{16}$$

where α, β and γ are arbitrary constants.

Proof. By Corollary 1 and Definition 2, Eq. (4) admits the additive separable solution (15) iff it admits the GCS $\eta = v_{xt}$, i.e., $\eta' E = 0$. It is easy to see that

$$\eta' E|_{L\cap W} = \lim_{\epsilon\to 0} \frac{d}{d\epsilon}(v + \epsilon E)_{xt}|_{L\cap W} = D_x D_t E|_{L\cap W},$$

where $E = B(v)v_x^{n-1}v_{xx} + A(v)v_x^{n+1} + G(v)$. In view of $v_{xt} = 0$, we have

$$D_t E = v_t\big[B'(v)v_x^{n-1}v_{xx} + A'(v)v_x^{n+1} + G'(v)\big]. \tag{17}$$

Differentiating Eq. (4) with respect to x and using $v_{xt} = 0$, we get

$$\big(v_x^{n-1}v_{xx}\big)_x = -\frac{B' + (n+1)A}{B} v_x^n v_{xx} - \frac{A'}{B} v_x^{n+2} - \frac{G'}{B} v_x.$$

Combining this equation with (17), we obtain

$$\begin{aligned} D_x D_t E|_{L\cap W} = v_t&\left[\left(B'' + (n+1)A' - (n+1)\frac{AB'}{B} - \frac{B'^2}{B}\right) v_x^n v_{xx}\right] \\ &+ v_t\left[\left(A'' - \frac{A'B'}{B}\right)v_x^{n+2} + \left(G'' - \frac{B'G'}{B}\right)v_x\right]. \end{aligned} \tag{18}$$

The vanishing of expression (18) leads to

$$\left(\frac{A'}{B}\right)' = 0, \qquad \left(\frac{(n+1)A + B'}{B}\right)' = 0, \qquad \left(\frac{G'}{B}\right)' = 0$$

After integration, (16) is obtained. □

Solving the system (16), we have:

Theorem 2. *Equation* (4) *admits the additive separable solution* (15) *if and only if it is equivalent to one of the following equations*:

(A.1) $$v_t = (b_1 + b_2 v)v_x^{n-1}v_{xx} - \frac{b_2}{n+1}v_x^{n+1} + \frac{\gamma}{2}(b_1 + b_2 v)^2 + g_1,$$
for $\alpha = \beta = 0$.

(A.2) $$v_t = \big(b_1 + b_2 e^{\beta v}\big)v_x^{n-1}v_{xx} + \frac{\beta b_1}{n+1}v_x^{n+1} + \gamma\left(b_1 v + \frac{b_2}{\beta}e^{\beta v}\right) + g_1,$$
for $\alpha = 0,\ \beta \neq 0$.

(A.3) $$\begin{aligned} v_t &= e^{(\beta/2)v}(b_1 + b_2 v)v_x^{n-1}v_{xx} \\ &\quad + \frac{e^{(\beta/2)v}}{n+1}\left(\frac{\beta b_1}{2} + \frac{\beta b_2}{2}v - b_2\right)\big(v_x^{n+1} + g_2\big) + g_1, \end{aligned}$$
for $\beta^2 = 4(n+1)\alpha$.

(A.4)
$$v_t = \big(b_1 e^{p_1 v} + b_2 e^{p_2 v}\big) v_x^{n-1} v_{xx} + \frac{1}{n+1}\big(p_2 b_1 e^{p_1 v} + p_1 b_2 e^{p_2 v}\big)\big(v_x^{n+1} + g_2\big) + g_1,$$
for $\beta^2 > 4(n+1)\alpha$, *where* $\alpha(n+1) = p_1 p_2$ *and* $\beta = p_1 + p_2$.

(A.5)
$$v_t = e^{(\beta/2)v}(b_1 \cos cv + b_2 \sin cv) v_x^{n-1} v_{xx} + \frac{e^{(\beta/2)v}}{n+1}\left[\left(\frac{\beta}{2} b_1 - bc_2\right)\cos cv + \left(\frac{\beta}{2} b_2 + cb_1\right)\sin cv\right]\big(v_x^{n+1} + g_2\big) + g_1,$$
for $\beta^2 < 4(n+1)\alpha$, *where* $c = \sqrt{4(n+1)\alpha - \beta^2}$,

where b_i *and* g_i, $i = 1, 2$, *are arbitrary constants.*

Remark 1. Using the transformations (7) and (8), we can obtain the corresponding equations of the forms (1) and (5), which also admit the functional separable solutions.

It is worthy of noting that the equations in Theorem 2 are new. In Section 3, we will construct some exact solutions of these equations by the approach of functional separation of variables.

3. Exact solutions of Eqs. (1), (4) and (5)

We now construct exact solutions of the equations obtained in Section 2 and give a prescription for properties such as the asymptotics behavior, blow up, etc., for some exact solutions.

Example 3.1. The equation

$$v_t = v_x^{n-1} v_{xx} + v_x^{n+1} + \alpha v \tag{19}$$

possesses the additive separable solution (15), where $\phi(t)$ and $\psi(x)$ satisfy

$$\phi' - \alpha\phi = \lambda,$$
$$\psi'^{n-1}\psi'' + \psi'^{n+1} + \alpha\psi = \lambda, \tag{20}$$

where and hereafter λ always denotes the separation constant. Exact solutions of system (20) are listed in Table 1. In Table 1, $\Psi(x)$ is determined implicitly by

$$\int^{\Psi(x)} \frac{dz}{\big(ce^{-(n+1)z} - \alpha z + \lambda + \frac{1}{n+1}\alpha\big)^{1/(n+1)}} = x + x_0.$$

Table 1
Exact solutions to system (20)

Parameters	$\phi(t)$	$\psi(x)$
$\alpha = c = 0,\ \lambda \neq 0$	λt	$\lambda^{1/(n+1)}x$
$\alpha = 0,\ \lambda, c \neq 0$	λt	$\Psi(x)$
$\alpha \neq 0,\ c = 0$	$-\frac{\lambda}{\alpha} + e^{\alpha t}$	$(-\alpha)^{1/n}\left(\frac{n}{n+1}x\right)^{(n+1)/n} + \frac{1}{n+1} + \frac{\lambda}{\alpha}$
$\alpha, c \neq 0$	$-\frac{\lambda}{\alpha} + e^{\alpha t}$	$\Psi(x)$

By the transformation $u = e^v$, we find that the equation

$$u_t = \left(u^{1-n}u_x^n\right)_x + \alpha u \ln u \tag{21}$$

has the product separable solution

$$u = \tilde{\phi}(t)\tilde{\psi}(x), \tag{22}$$

where

$$\tilde{\phi}(t) = e^{\phi(t)}, \qquad \tilde{\psi}(x) = e^{\psi(x)}.$$

In this paper, if $v(x,t)$ is a solution of an equation, then $v(x + x_0, t - t_0)$ for arbitrary constants x_0 and t_0 are also solutions of the equation because of the invariance under time and space translations.

Remark 2. One can readily find that Eq. (19) is invariant under the four-dimensional Lie symmetry group

$$V_1 = \partial_t, \qquad V_2 = \partial_x, \qquad V_3 = \partial_v, \qquad V_4 = (n+1)t\partial_t + x\partial_x,$$

for $\alpha = 0$, and the three-dimensional symmetry group V_1, V_2 and $e^{\alpha t}V_3$ for $\alpha \neq 0$. It is easy to verify that the solutions in Table 1 is invariant under the Lie symmetry groups for any α. So the functional separable solutions of Eq. (19) can be derived by the classical Lie symmetry method. In fact, these solutions can be viewed as a generalization of the known solutions of Eq. (19) with $n = 1$ [48].

Example 3.2. The equation

$$v_t = vv_x^{n-1}v_{xx} - \frac{1}{n+1}v_x^{n+1} + \alpha v^2 + \beta \tag{23}$$

possesses the additive separable solution (15), where $\phi(t)$ and $\psi(x)$ satisfy

$$\begin{aligned} &\phi' = \alpha\phi^2 + c\phi + \lambda + \beta, \\ &\psi'^{n+1} = -(n+1)\alpha\psi^2 + (n+1)c\psi - (n+1)\lambda. \end{aligned} \tag{24}$$

Exact solutions of system (24) are listed in Table 2. In Table 2, $\Delta = c^2 - 4\alpha(\lambda + \beta)$, $a = \sqrt{\Delta}/(2\alpha)$, $\tilde{a} = \sqrt{-\Delta}/(2\alpha)$, and $\Psi(x)$ is determined implicitly by

Table 2
Exact solutions to system (24)

Parameters	$\phi(t)$	$\psi(x)$
$\alpha = c = 0$	$(\lambda+\beta)t$	$[-(n+1)\lambda]^{1/(n+1)}x$
$\alpha = 0,\ c \neq 0$	$\frac{1}{c}(e^{ct} - \lambda - \beta)$	$\frac{1}{n+1} n^{-(n+1)/n} c^{1/n} (x+x_0)^{(n+1)/n} + \frac{\lambda}{c}$
$\alpha \neq 0,\ \Delta = 0$	$-\frac{1}{\alpha}\left(\frac{1}{t} + \frac{c}{2}\right)$	$\Psi(x)$
$\alpha \neq 0,\ \Delta > 0$	$-\frac{c}{2\alpha} - a\coth(a\alpha t)$	$\Psi(x)$
$\alpha \neq 0,\ \Delta < 0$	$-\frac{c}{2\alpha} + \tilde{a}\tan(\tilde{a}\alpha t)$	$\Psi(x)$

$$\int^{\Psi(x)} \left[(n+1)(-\alpha z^2 + cz - \lambda)\right]^{-1/(n+1)} dz = x.$$

In terms of the transformation

$$v = \left(-\frac{n+1}{u}\right)^{(n+1)/n},$$

we find that the equation

$$u_t = \left(u^{-2n} u_x^n\right)_x + \alpha u^{-1/n} + \beta u^{(2n+1)/n} \tag{25}$$

admits the separable solutions

$$u = -(n+1)\big(\phi(t) + \psi(x)\big)^{-n/(n+1)},$$

where $\phi(t)$ and $\psi(x)$ are given in Table 2. For $\Delta \geqslant 0$, the solution has the property

$$u \to -(n+1)\left(\Psi(x) - \frac{c}{2\alpha} - \frac{\sqrt{\Delta}}{2\alpha}\right)^{-n/(n+1)}, \quad \text{as } t \to \infty.$$

For $\Delta \leqslant 0$, the solution has the behavior

$$u \to -(n+1)\left(\Psi(x) - \frac{c}{2\alpha}\right)^{-n/(n+1)}, \quad \text{as } t \to t_0.$$

This solution blows up along a curve $\Psi(x) = c/(2\alpha)$. Exact solutions to the equation

$$w_t = w^2 w_x^{n-1} w_{xx} + \alpha w^{2+1/n} + \beta w^{-1/n} \tag{26}$$

are obtained by the solutions of Eq. (23) and the transformation $w = -u^{-1}$. Equation (26) is generalization of the curve shortening equation [57,58]

$$w_t = w^2 w_{xx} + w^3.$$

Example 3.3. The equation

$$v_t = e^v v_x^{n-1} v_{xx} + e^v v_x^{n+1} + \alpha e^v + \beta \tag{27}$$

Table 3
Exact solutions to system (28)

Parameters	$\phi(t)$	$\psi(x)$
$\alpha=\beta=c=0,\ \lambda\neq 0$	$-\ln(-\lambda t)$	$\ln\left(\frac{n+1}{n}\lambda\right)+(n+1)\ln\frac{x}{n+1}$
$\alpha=\lambda=0,\ \beta,c\neq 0$	βt	$\ln x+\frac{1}{n+1}\ln c$
$\lambda=c=0,\ \alpha,\beta\neq 0$	βt	$(-\alpha)^{1/(n+1)}x$
$\alpha c\neq 0,\ \lambda\neq 0,\ \beta=0$	$-\ln(-\lambda t)$	$\Psi(x)$
$\alpha,c,\lambda,\ \beta\neq 0$	$-\ln\left[\frac{\lambda}{\beta}(e^{-\beta t)}-1)\right]$	$\Psi(x)$
$\alpha=c=0,\ \lambda,\beta\neq 0$	$-\ln\left[\frac{\lambda}{\beta}(e^{-\beta t}-1)\right]$	$\ln\left(\frac{n+1}{n}\lambda\right)+(n+1)\ln\frac{x}{n+1}$

admits the additive separable solution (15), where $\phi(t)$ and $\psi(x)$ satisfy

$$\begin{aligned}&\phi'-\beta=\lambda e^{\phi},\\&\psi'^{n-1}\psi''+\psi'^{n+1}+\alpha=\lambda e^{-\psi}.\end{aligned}\tag{28}$$

Exact solutions to system (28) are listed in Table 3. In Table 3, $\Psi(x)$ is determined implicitly by

$$\int^{\Psi(x)}\left[\frac{n+1}{n}\lambda e^{-z}-\alpha+ce^{-(n+1)z}\right]^{-1/(n+1)}dz=x.$$

By the transformation

$$v=\frac{1}{n-1}\ln\frac{n-1}{n}u$$

and the exact solutions of (27), we can obtain exact solutions to the equation

$$u_t=\left(e^u u_x^n\right)_x+\alpha e^u+\beta.$$

Example 3.4. The equation

$$v_t=e^v v_x^{n-1}v_{xx}+\sigma e^v v_x^{n+1}+\alpha e^v+\beta,\qquad \sigma\neq 1,\tag{29}$$

admits the additive separable solution (15), where $\phi(t)$ and $\psi(x)$ satisfy

$$\begin{aligned}&\phi'-\beta=\lambda e^{\phi},\\&\psi'^{n-1}\psi''+\sigma\psi'^{n+1}+\alpha=\lambda e^{-\psi}.\end{aligned}\tag{30}$$

Exact solutions to system (30) are listed in Table 4. In Table 4, $\Psi(x)$ is determined implicitly by

$$\int^{\Psi(x)}\left[ce^{-(n+1)z}+\frac{\lambda(n+1)}{\sigma(n+1)-1}e^{-z}-\frac{\alpha}{\sigma}\right]^{-1/(n+1)}dz=x,$$

Table 4
Exact solutions to system (30)

Parameters	$\phi(t)$	$\psi(x)$
$\alpha=\beta=c=0,\ \sigma\neq 1/(n+1)$	$-\ln(-\lambda t)$	$\ln\frac{(n+1)\lambda}{\sigma(n+1)-1}+(n+1)\ln\frac{x}{n+1}$
$\alpha=\beta=c=0,\ \sigma\neq 1/(n+1)$	$-\ln(-\lambda t)$	$\Psi(x)$
$\alpha=\lambda=0,\ \beta,c\neq 0$	βt	$\frac{1}{\sigma}\left[\ln\sigma x+\frac{1}{n+1}\ln c\right]$
$\lambda=c=0,\ \alpha,\beta\neq 0$	βt	$\left(-\frac{\alpha}{\sigma}\right)^{1/(n+1)}x$
$\alpha c\neq 0,\ \lambda\neq 0,\ \beta=0$	$-\ln(-\lambda t)$	$\Psi(x)$
$\alpha,c,\lambda,\beta\neq 0$	$-\ln\left[\frac{\lambda}{\beta}(e^{-\beta t}-1)\right]$	$\Psi(x)$
$\alpha=c=0,\ \lambda,\beta\neq 0$	$-\ln\left[\frac{\lambda}{\beta}(e^{-\beta t}-1)\right]$	$\ln\frac{(n+1)}{\sigma(n+1)-1}+(n+1)\ln\frac{x}{n+1}$

for $\sigma\neq 1/(n+1)$, and

$$\int^{\Psi(x)}\left[\lambda(n+1)ze^{-z}-\alpha(n+1)+ce^{-z}\right]^{-1/(n+1)}dz=x,$$

for $\sigma=1/(n+1)$.

In terms of the transformations

$$u=\frac{n}{n\sigma-1}e^{(n\sigma-1)v}\quad\text{and}\quad u=w^{1-p},\qquad p=m/(m+n),$$

and the exact solutions of (29), we can get exact solutions to the equations

$$u_t=\left(u^m u_x^n\right)_x+\alpha u+\beta u^{m+n},\qquad m=\frac{1}{n\sigma-1}+1-n, \tag{31}$$

and

$$w_t=w^p w_x^{n-1}w_{xx}+\alpha w^{p+n}+\beta w. \tag{32}$$

Equations (31) and (32) are quasilinear parabolic equations with the "critical exponent" [48,49], which play a particular role in the study of the qualitative property of quasilinear parabolic equations, such as the existence, blow-up and asymptotic behavior. Using the approach of functional separation of variables, we can get some exact blow-up solutions. From Table 4, we find that Eq. (31) has an exact solution

$$u=n(m+n-1)\left[\frac{\lambda}{\beta}\left(e^{-\beta(t-t_0)}-1\right)\right]^{1/(m+n-1)}e^{1/(m+n-1)\Psi(x)}.$$

It possesses the following properties:

(i) For $m<1-n$, it blows up at finite time t_0; namely,

$$\lim_{t\to t_0}u(t,x)=\infty.$$

Table 5
Exact solutions to system (34)

Parameters	$\phi(t)$	$\psi(x)$
$\alpha=\lambda=0,\ c\neq 0,-\beta$	$(c+\beta)t$	$c^{1/(n+1)}x$
$\alpha\neq 0,\ \lambda=0$	$\frac{1}{\alpha}(e^{\alpha t}-c-\beta)+\frac{1}{n+1}$	$\left[-\alpha\left(\frac{n}{n+1}x\right)^{n+1}\right]^{1/n}+\frac{c}{\alpha}$
$\alpha=c=0,\ \beta,\lambda\neq 0$	$-\frac{1}{n+1}\ln\left[\frac{1}{\beta}(e^{-(n+1)\beta t}-1)\right]$	$\ln[(-\lambda)^{1/(n+1)}x]$
$\alpha=c=\beta=0,\ \lambda\neq 0$	$-\frac{1}{n+1}\ln[-(n+1)\lambda t]$	$\ln[(-\lambda)^{1/(n+1)}x]$
$\alpha=0,\ c=-\beta,\ \lambda\neq 0$	$-\frac{1}{n+1}\ln[-(n+1)\lambda t]$	$\Psi(x)$
$\alpha=0,\ c\neq 0,\ -\beta,\lambda\neq 0$	$-\frac{1}{n+1}\ln\left[\frac{\lambda}{c+\beta}(e^{-(n+1)(c+\beta)t}-1)\right]$	$\Psi(x)$

(ii) For $m>1-n$, it satisfies

$$\lim_{t\to\infty}u(t,x)=ce^{1/(m+n-1)\Psi(x)},$$

where c is a constant, $x\in\Omega$, a bounded domain.

Example 3.5. The equation

$$v_t=\left(e^{(n+1)v}+1\right)v_x^{n-1}v_{xx}+v_x^{n+1}+\alpha\left(\frac{1}{n+1}e^{(n+1)v}+v\right)+\beta \tag{33}$$

has the additive separable solution (15), where $\phi(t)$ and $\psi(x)$ satisfy

$$\phi'=\lambda e^{(n+1)\phi}+\alpha\phi+c+\beta-\frac{\alpha}{n+1},$$
$$\psi'^{n+1}=-\lambda e^{-(n+1)\psi}-\alpha\psi+c. \tag{34}$$

Some exact solutions of system (34) are listed in Table 5. In Table 5, $\Psi(x)$ is determined implicitly by

$$\int^{\Psi(x)}\left[-\lambda e^{-(n+1)z}-\alpha z-c\right]^{-1/(n+1)}dz=x.$$

In terms of the transformations

$$v=-\frac{1}{n+1}\ln\left(u^{-(n+1)/n}-1\right)$$

and

$$v=q(w),$$

where $q(w)$ is determined implicitly by

$$\int^{q(w)}\left(1+e^{-(n+1)z}\right)^{-1/(n+1)}dz=w,$$

and Table 5, we can obtain some exact solutions to the following quasilinear heat equations:

$$
\begin{aligned}
u_t = {} & \big(\big(u^{(n-1)/n} - u^2\big)^{-n} u_x^n\big)_x \\
& + u\big[(\alpha - \beta)u^{1+1/n} - \alpha\big(1 - u^{1+1/n}\big)\ln\big(1 - u^{-1-1/n}\big) + \beta\big]
\end{aligned} \tag{35}
$$

and

$$
\begin{aligned}
w_t = {} & \big(e^{(n+1)q} + 1\big)^{2n/(n+1)} e^{(1-n)q} w_x^{n-1} w_{xx} \\
& + \left[\alpha\left(\frac{1}{n+1} e^{(n+1)q} + q\right) + \beta\right]\big(1 + e^{-(n+1)q}\big)^{-1/(n+1)}.
\end{aligned}
$$

Equation (35) can be viewed as a generalization of the Mullin's equation [59].

4. Concluding remarks

In this paper, the generalized conditional symmetry approach has been used to study the functional separation of variables of the generalized porous medium equations with nonlinear source terms (1). We obtained a complete list of canonical forms for such equations, which admit the functional separable solutions. As the consequence, some exact solution to the resulting equations such as the blow-up solutions, time-periodic solutions and global solutions are derived. The results also provide a support to the assertion on the nonlinear diffusion equations with "critical exponent." For such equations, we have obtained the blow-up solutions and global solutions for specific initial conditions. This approach also provides a symmetry group interpretation to the functional separable solutions. It is noteworthy to develop other methods used in Eq. (1) with $n = 1$ to tackle the case $n \neq 1$, and it is of interest to extend this approach to higher-dimensional nonlinear parabolic PDEs and to generalize the ansatz (2) to be more general to include more nonlinear diffusion equations with functional separable solutions.

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