The general solutions of some nonlinear second order PDEs. I. Two independent variables, constant parameters

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Abstract

In the first part of planned series of papers the formal general solutions to selection of 80 examples of different types of second order nonlinear PDEs in two independent variables with constant parameters are given. The main goal here is to show on examples the types of solvable PDEs and what their general solutions look like. The solving strategy, used here, as a rule is the order reduction. The order reduction method is implemented in Maple procedure, which applicable to PDEs of different order with different number of independent variables. Some of given PDEs are solved by order lifting to PDEs, which are solvable by the subsequent order reduction.

1 Introduction

Nonlinear partial differential equations (PDEs) play very important role in many fields of mathematics, physics, chemistry, and biology, and numerous applications. Despite the fact that various methods for solving nonlinear PDEs have been developed in 19-20 centuries [1]-[8], there exists a very disadvantageous opinion that only a very small minority of nonlinear second- and higher-order PDEs admit general solutions in closed form (see, e.g., percentage of PDEs with general solutions in fundamental handbook [9]).

Nevertheless there exist some extensive nontrivial families for different types of nonlinear PDEs which general solutions can be expressed in closed form and which seemingly are not described in literature.

In the first part of planned series of papers the formal general solutions to selection of 80 examples of different types of second order nonlinear PDEs in two independent variables with constant parameters are given. The main goal here is to show on examples the types of solvable PDEs and what their general solutions look like.

The solving strategy, used here, as a rule is the order reduction. The order reduction method is implemented in Maple procedure (see Appendix), which applicable to PDEs of different order with different number of independent variables [10]. Some of given PDEs are solved by order lifting to PDEs, which are solvable by the subsequent order reduction.

I try to to follow the tradition when solvable PDEs are filed up to point transformations, deciding between equivalent PDEs variants the most short one.

I would like to thank Prof. A.D. Polyanin for valuable advices in results presentation.

2 Equations of the Form $\frac{\partial^2 w}{\partial t \partial x} = F(w, \frac{\partial w}{\partial t}, \frac{\partial w}{\partial x})$

2.1 $\frac{\partial^2 w}{\partial t \partial x} = \frac{2}{w}\left[\frac{\partial w}{\partial t} + 1\right]\frac{\partial w}{\partial x}$.

General solution

$$w(x,t) = -\frac{F(t)}{F'(t)} + \left\{[F'(t)]^2\left[G(x) - \int \frac{F''(t)}{F(t)\,[F'(t)]^2}\,dt\right]\right\}^{-1},$$

where $F(t)$ and $G(x)$ are arbitrary functions.

2.2 $\left(\frac{\partial^2 w}{\partial t\,\partial x}\right)^2 = \left(\frac{\partial w}{\partial x}\right)^2 \frac{\partial w}{\partial t}$.

General solution

$$w(t,x) = -\frac{4F'(t)}{F(t)+G(x)} + \int \left(\frac{F''(t)}{F'(t)}\right)^2 dt\,,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

2.3 $\left(\frac{\partial^2 w}{\partial t \partial x} - w\frac{\partial w}{\partial x}\right)^2 + \left(2\frac{\partial w}{\partial t} - w^2\right)\left(\frac{\partial w}{\partial x}\right)^2 = 0$.

General solution

$$w(t,x) = \left(-\frac{1}{2}\int (F'(t))^2\, e^{F(t)}\,dt + G(x)\right) e^{-F(t)}\,,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

2.4 $$\boldsymbol{\left(\frac{\partial w}{\partial t}+w\right)^2+\left(\frac{\partial^2 w}{\partial t\partial x}+\frac{\partial w}{\partial x}+\frac{\partial w}{\partial t}+w\right)\times}$$

$$\boldsymbol{\exp\left(\frac{\frac{\partial^2 w}{\partial t\partial x}+\frac{\partial w}{\partial x}+\frac{\partial w}{\partial t}+w}{\frac{\partial w}{\partial t}+w}\right)=0\,.}$$

General solution

$$w(t,x)=\left(e^{-x}\int e^t\,F(t)\exp[F(t)\,e^{-x}]\,dt+G(x)\right)e^{-t}\,,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

2.5 $$\boldsymbol{\frac{\partial^2 w}{\partial t\partial x}=a\left(\frac{\partial w}{\partial t}+w\right)^n-\frac{\partial w}{\partial x}\,,}$$

where $a\neq 0$, and $n\neq 1$ are constants.

General solution

$$w(t,x)=\left(a^{\frac{1}{1-n}}\int e^t\,[F(t)+(1-n)x]^{\frac{1}{1-n}}\;dt+G(x)\right)e^{-t}\,,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

2.6 $$\boldsymbol{\frac{\partial^2 w}{\partial t\partial x}-\frac{1}{w}\frac{\partial w}{\partial t}\frac{\partial w}{\partial x}-\frac{c}{w}\frac{\partial w}{\partial t}-kw=0\,,}$$

where c,k are constants.

General solution

$$w(t,x)=\frac{\{-c\displaystyle\int\exp[x(1-kt)]G(x)\,dx+F(t)\}\exp[-x(1-kt)]}{G(x)}\,,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

$$\textbf{2.7} \quad \frac{\partial^2 w}{\partial t \partial x} = \frac{1}{w}\left(\frac{\partial w}{\partial x} + a\right)\frac{\partial w}{\partial t} + bw\frac{\partial w}{\partial x}\,,$$

where a and $b \neq 0$ are constants.

General solution

$$w(t,x) = -\frac{1}{b}\frac{\displaystyle\int_{-\infty}^{\infty}\frac{1}{\omega}\left[F(\omega)ab\exp\left(\frac{abt+\omega^2 x}{\omega}\right) + G(\omega)\omega^2\exp\left(\frac{abx+\omega^2 t}{\omega}\right)\right]d\omega}{\displaystyle\int_{-\infty}^{\infty}\left[F(\omega)\exp\left(\frac{abt+\omega^2 x}{\omega}\right) + G(\omega)\exp\left(\frac{abx+\omega^2 t}{\omega}\right)\right]d\omega}\,,$$

where $F(\omega)$ and $G(\omega)$ are arbitrary functions.

$$\textbf{2.8} \quad \frac{\partial^2 w}{\partial t \partial x} - \left(\frac{1}{w}\frac{\partial w}{\partial t} + b\right)\frac{\partial w}{\partial x} - \frac{c}{w}\frac{\partial w}{\partial t} - cb = 0\,,$$

where b, c are constants.

General solution

$$w(t,x) = \left\{-c\int \exp[-e^{bt}G(x)]\,dx + F(t)\right\}\exp[e^{bt}G(x)]\,,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

$$\textbf{2.9} \quad \frac{\partial^2 w}{\partial t \partial x} = \left[\frac{(2w-a-1)}{(w-1)(w-a)}\frac{\partial w}{\partial x} + \frac{b(w-a)}{w-1}\right]\frac{\partial w}{\partial t}\,,$$

where $a \neq 1$, and b are constants.

General solution

$$w(t,x) = \frac{G'(x) + abG(x) + aF(t)}{G'(x) + bG(x) + F(t)}\,,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

$$\textbf{2.10} \quad \frac{\partial^2 w}{\partial t \partial x} = \frac{(2kw-ak-c)}{(w-a)(kw-c)}\frac{\partial w}{\partial t}\frac{\partial w}{\partial x} + \frac{2b(kw-c)^2}{ak-c}\,,$$

where a, b, c, and k are constants.

General solution

$$w(t,x) = \frac{aF'(t)G'(x) - c[F(t) - bG(x)]^2}{F'(t)G'(x) - k[F(t) - bG(x)]^2}\,,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

2.11 $\left(\frac{\partial^2 w}{\partial t \partial x} + \frac{\partial w}{\partial x}\right)^2 + a^2\left[\left(\frac{\partial w}{\partial t} + w\right)^2 - b^2\right]\left(\frac{\partial w}{\partial t} + w\right)^2 = 0\,,$

where $a \neq 0$, and $b \neq 0$ are constants.

General solution

$$w(t,x) = \left(2b\int \frac{F(t)\exp(abx+t)}{1+F^2(t)\exp(2abx)}\,dt + G(x)\right)e^{-t}\,,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

2.12 $\frac{\partial^2 w}{\partial t \partial x} + \frac{\partial w}{\partial x} + a\left[\left(\frac{\partial w}{\partial t} + w\right)^2 + b^2\right]\left(\frac{\partial w}{\partial t} + w\right) = 0\,,$

where $a \neq 0$, and $b \neq 0$ are constants.

General solution

$$w(t,x) = \left(\pm b\int \frac{e^t\,dt}{\sqrt{F(t)\exp[(2ab^2x) - 1}} + G(x)\right)e^{-t}\,,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

2.13 $\left(w\frac{\partial^2 w}{\partial t \partial x} - \frac{\partial w}{\partial t}\frac{\partial w}{\partial x}\right)\exp\left[\frac{a\frac{\partial^2 w}{\partial t \partial x}}{\frac{\partial w}{\partial x}}\right] = b\frac{\partial w}{\partial x}\,,$

where a, and b are constants.

General solution

$$w(t,x) = \left(-b \int \exp\left[aF'(t) + F(t)\right]\, dt + G(x)\right) e^{-F(t)}\,,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

2.14 $\displaystyle \boldsymbol{\left(\frac{w\frac{\partial^2 w}{\partial t \partial x}}{\frac{\partial w}{\partial x}} - \frac{\partial w}{\partial t}\right)^2 = w\left(a\frac{\partial w}{\partial x} - bw\right)\,,}$

where $a \neq 0$, and b are constants.

General solution

$$w(t,x) = F(t) \exp\left\{\frac{b}{a}\int \left(cos\left[\frac{t\sqrt{b}}{2} + G(x)\right]\right)^{-2} dx\right\}\,,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

2.15 $\displaystyle \boldsymbol{\frac{\partial^2 w}{\partial t \partial x} + b\frac{\partial w}{\partial x} = a \exp\left[-\frac{\partial w}{\partial t} - bw\right]\,,}$

where a and b are constants.

General solution

$$w(t,x) = \left(\int \ln\left[ax + F(t)\right] e^{bt}\, dt + G(x)\right) e^{-bt}\,,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

2.16 $\displaystyle \boldsymbol{(aw+1)\left(\frac{\partial w}{\partial t} + b\right)\frac{\partial^2 w}{\partial t \partial x} =}$

$$\boldsymbol{a\frac{\partial w}{\partial x}\left(\frac{\partial w}{\partial t} + b\right)^2 - (aw+1)^3\,,}$$

where a, b are constants.

General solution

$$w(x,t) = -\left\{ \int \left(b \pm \sqrt{F(t) - 2x} \right) \exp \left[\pm a \int \sqrt{F(t) - 2x} \, dt \right] dt + G(x) \right\} \times$$

$$\exp \left[\mp a \int \sqrt{F(t) - 2x} \, dt \right] ,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

2.17 $$\left[\left(\frac{\partial w}{\partial t} \right)^2 - 2aw \right] \left(w \frac{\partial^2 w}{\partial t \partial x} - \frac{\partial w}{\partial t} \frac{\partial w}{\partial x} \right)^2 =$$
$$\left[w \frac{\partial^2 w}{\partial t \partial x} \frac{\partial w}{\partial t} + aw \frac{\partial w}{\partial x} - \left(\frac{\partial w}{\partial t} \right)^2 \frac{\partial w}{\partial x} \right]^2 ,$$

where $a \neq 0$ is a constant.

General solution

$$w(t,x) = \left(-\frac{a}{2} \int \frac{e^{F(t)}}{F'(t)} \, dt + G(x) \right) e^{-F(t)} ,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

2.18 $$\frac{\partial^2 w}{\partial t \partial x} = \frac{w+1}{w} \frac{\partial w}{\partial t} \frac{\partial w}{\partial x} + a w^{(1-m)} e^{w(1-m)} ,$$

where a, and $m \neq 1$ are constants.

General solution in implicit form

$$\int_1^\infty \frac{e^{-\xi w(t,x)}}{\xi} \, d\xi = G(x) - \int \left[a(1-m)x + F(t) \right]^{\frac{1}{1-m}} \, dt ,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

2.19 $$w \left(\frac{\partial^2 w}{\partial t \partial x} + a \frac{\partial w}{\partial t} \right)^2 - \frac{\partial w}{\partial t} \left(\frac{\partial^2 w}{\partial t \partial x} + a \frac{\partial w}{\partial t} \right) \times$$

$$\left(2aw + \frac{\partial w}{\partial x} \right) + b \left(2aw + \frac{\partial w}{\partial x} \right)^2 = 0 ,$$

where a and b are constants.

General solution

$$w(t,x) = \left(b \int \frac{\exp\left[-ax - e^{ax}F(t)\right]}{F'(t)}\, dt + G(x) \right) \exp\left[e^{ax}F(t)\right] \,,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

2.20 $\boldsymbol{\left(b^2 - 4aw\frac{\partial w}{\partial x}\right)\left(aw^2\frac{\partial^2 w}{\partial t \partial x} - aw\frac{\partial w}{\partial t}\frac{\partial w}{\partial x} + b^2\frac{\partial w}{\partial x}\right)^2 =}$

$$\boldsymbol{b^2\left(aw^2\frac{\partial^2 w}{\partial t \partial x} - 3aw\frac{\partial w}{\partial t}\frac{\partial w}{\partial x} + b^2\frac{\partial w}{\partial x}\right)^2 \,,}$$

where a, and $b \neq 0$ are constants.

General solution

$$w(t,x) = \pm\left(b \int \sqrt{F'(t)}\, e^{aF(t)}\, dt + G(x) \right) e^{-aF(t)} \,,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

2.21 $\boldsymbol{w\frac{\partial^2 w}{\partial t \partial x} = \left(\frac{\partial w}{\partial x} - aw\right)\frac{\partial w}{\partial t} - b\frac{\partial w}{\partial x} - cw^2 + abw \,,}$

where $a \neq 0$, $b \neq 0$, and c are constants.

General solution

$$w(t,x) = \left(b \int \exp\left\{\frac{ct}{a} + e^{-ax}F(t)\right\} dt + G(x) \right) \exp\left\{-\frac{ct}{a} - e^{-ax}F(t)\right\} \,,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

2.22 $\boldsymbol{\frac{\partial^2 w}{\partial t \partial x} = \left(\frac{1}{w}\frac{\partial w}{\partial t} + b\right)\frac{\partial w}{\partial x} + \frac{c}{w}\frac{\partial w}{\partial t} + kw + cb \,,}$

where $b \neq 0$, c, and $k \neq 0$ are constants.

General solution

$$w(t,x) = \left\{ -c \int \exp\left(\frac{k}{b^2}\left[e^{bt}G(x) + bx\right]\right) dx + F(t) \right\} \exp\left\{-\frac{k}{b^2}\left[e^{bt}G(x) + bx\right]\right\} \,,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

2.23 $$\frac{\partial^2 w}{\partial t \partial x} = \frac{a}{w}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{w}\frac{\partial w}{\partial t}\frac{\partial w}{\partial x} + \left(b + \frac{c}{w}\right)\frac{\partial w}{\partial x} +$$

$$\frac{c}{2aw}\frac{\partial w}{\partial t} + \frac{(bw+c)^2}{4aw},$$

where $a \neq 0$ and b, c are constants.

General solution

$$w(t,x) = \left\{-\frac{c}{2a}\int \exp\left[\frac{1}{2a}\int \frac{2\,dx}{t+G(x)} + bx\right] dx + F(t)\right\} \exp\left[-\frac{1}{2a}\int \frac{2\,dx}{t+G(x)} + bx\right],$$

where $F(t)$ and $G(x)$ are arbitrary functions.

2.24 $$\frac{a^2}{w^4}\left(c\frac{\partial w}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial w}{\partial x} - w\frac{\partial^2 w}{\partial t \partial x}\right)^2 =$$

$$\frac{ac}{w} + b + \frac{a}{w}\frac{\partial w}{\partial x},$$

where $a \neq 0$, b, and c are constants.

General solution

$$w(t,x) = \left\{-c\int \exp\left[-\frac{1}{4a}\left(-4bx + \int (t+G(x))^2 dx\right)\right] dx + F(t)\right\} \times$$

$$\exp\left[\frac{1}{4a}\left(-4bx + \int (t+G(x))^2 dx\right)\right],$$

where $F(t)$ and $G(x)$ are arbitrary functions.

2.25 $$w(\frac{\partial w}{\partial x} + aw + b)\frac{\partial^2 w}{\partial t \partial x} - \frac{\partial w}{\partial t}\frac{\partial w}{\partial x}(\frac{\partial w}{\partial x} + aw + 2b) +$$

$$b(aw+b)\frac{\partial w}{\partial t} + cw^3,$$

where a, b, c are constants.

General solution

$$w(t,x) = -\left\{ b \int \exp\left[-\int \left(\pm\sqrt{G(x)+2ct} - a \right) dx \right] dx + F(t) \right\} \times$$

$$\exp\left[\int \left(\pm\sqrt{G(x)+2ct} - a \right) dx \right],$$

where $F(t)$ and $G(x)$ are arbitrary functions.

2.26 $\displaystyle \frac{\partial^2 w}{\partial t \partial x} - \frac{a}{w}\left(\frac{\partial w}{\partial x}\right)^2 - \left(\frac{1}{w}\frac{\partial w}{\partial t} + b + \frac{c}{w}\right)\frac{\partial w}{\partial x} - \frac{c}{2aw}\frac{\partial w}{\partial t} - kw - \frac{bc}{2a} - \frac{c^2}{4aw} = 0,$

where $a \neq 0$, b, c, and k are constants.

General solution for $b^2 - 4ka \neq 0$

$w(t,x) =$

$$-\frac{c}{2a}\left\{ \int \exp\left[\frac{1}{2a} \int \frac{\exp(t\sqrt{b^2-4ak})G(x)(b+\sqrt{b^2-4ak}) - \sqrt{b^2-4ak} + b}{1+\exp(t\sqrt{b^2-4ak})G(x)} dx \right] dx + F(t) \right\} \times$$

$$\times \exp\left[-\frac{1}{2a} \int \frac{\exp(t\sqrt{b^2-4ak})G(x)(b+\sqrt{b^2-4ak}) - \sqrt{b^2-4ak} + b}{1+\exp(t\sqrt{b^2-4ak})G(x)} dx \right],$$

where $F(t)$ and $G(x)$ are arbitrary functions.

2.27 $\displaystyle \frac{\partial^2 w}{\partial x \partial t} - \left(\frac{1}{w}\frac{\partial w}{\partial t} + b\right)\frac{\partial w}{\partial x} - \frac{c}{w}\frac{\partial w}{\partial t} - aw^2\left(c + kw + \frac{\partial w}{\partial x}\right)^{-1} - sw - bc = 0,$

where a, b, c, k, s are constants.

General solution

$$w(x,t) = e^{-kx} \exp\left[\int W(x,t)\,dx\right] \left\{-c \int e^{kx} \exp\left[-\int W(x,t)\,dx\right]\,dx + F(t)\right\},$$

where $F(t)$ and $G(x)$ are arbitrary functions, and the $W = W(x,t)$ is any root of the transcendental equation

$$\int_0^W \frac{\xi\,d\xi}{(s-bk)\xi + b\xi^2 + a} = t + G(x).$$

2.28 $\boldsymbol{(aw^n + b)\frac{\partial^2 w}{\partial t \partial x} - k(aw^n + b)^{(2-m)}\left(-\frac{\partial w}{\partial t}\right)^m =}$ $\boldsymbol{\left[anw^{(n-1)}\frac{\partial w}{\partial x} - c(aw^n + b)\right]\frac{\partial w}{\partial t},}$

where a, b, c, k, n, and $m \neq 1$ are constants.

General solution in implicit form ($w = w(t,x)$)

$$\int_s^w \frac{d\xi}{a\xi^n + b} + \int \left[F(t)e^{c(m-1)x} - \frac{k}{c}\right]^{\frac{1}{1-m}} dt + G(x) = 0,$$

where $F(t)$ and $G(x)$ are arbitrary functions, s is a constant.

2.29 $\boldsymbol{\frac{\partial^2 w}{\partial t \partial x} = \frac{w-n}{w}\frac{\partial w}{\partial t}\frac{\partial w}{\partial x} + aw^{n(m-1)}e^{w(1-m)},}$

where $a \neq 0$, $n \neq -1$, and $m \neq 1$ are constants.

General solution in implicit form

$$w(t,x)^{\frac{n}{2}} \exp\left[-\frac{w(t,x)}{2}\right] M_{\frac{n}{2},\frac{n+1}{2}}(w(t,x)) = \tag{1}$$

$$(n+1)[a(1-m)]^{\frac{1}{1-m}} \int [x + F(t)]^{\frac{1}{1-m}}\,dt + G(x),$$

where $M_{p,q}(z)$ is the Whittaker M function, $F(t)$ and $G(x)$ are arbitrary functions.

2.30 $\boldsymbol{(aw^2 + bw + cb - ak)\dfrac{\partial^2 w}{\partial t \partial x} =}$

$$\boldsymbol{\left[(2aw + b)\frac{\partial w}{\partial t} + w^2 + 2cw + k\right] \frac{\partial w}{\partial x}\,,}$$

where a, b, c, and k are constants.

General solution

$$w(x,t) = F(t) - W(t)\left\{G(x) + \int \frac{W(t)\,[aF'(t) + F(t) + c]}{aF^2(t) + bF(t) - ak + cb}\,dt\right\}^{-1},$$

where $F(t)$ and $G(x)$ are arbitrary functions, and

$$W(t) = \exp\left\{\int \frac{[2aF(t) + b]\,F'(t) + F^2(t) + 2cF(t) + k}{aF^2(t) + bF(t) - ak + bc}\,dt\right\}.$$

2.31 $\boldsymbol{(aw+b)\dfrac{\partial^2 w}{\partial t \partial x} = \left(\dfrac{\partial w}{\partial t}\right)^2 + \left(a\dfrac{\partial w}{\partial x} + 2cw + 2k\right)\dfrac{\partial w}{\partial t} +}$

$$\boldsymbol{(ak - bc)\frac{\partial w}{\partial x} + (cw + k)^2\,,}$$

where a, b, c, and k are constants.

General solution

$$w(x,t) = \left\{-k^2 \int \frac{F(t) + x}{k[F(t) + x] - b} \exp\left[\int \frac{ck[F(t) + x] + ak - bc}{k[F(t) + x] - b}\,dt\right] dt + G(x)\right\} \times$$

$$\exp\left[-\int \frac{ck[F(t) + x] + ak - bc}{k[F(t) + x] - b}\,dt\right]$$

where $F(t)$ and $G(x)$ are arbitrary functions.

$$\mathbf{2.32} \qquad \boldsymbol{w(aw-b)\left(\frac{\partial^2 w}{\partial t \partial x}\right)^2 = (2aw-b)\frac{\partial w}{\partial t}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial t \partial x} +}$$

$$\boldsymbol{c(2aw-b)^2\left(\frac{\partial w}{\partial x}\right)^2 \,,}$$

where a, b, and c are constants.

General solution

$$w(x,t) = F(t) + W(t)\times$$

$$\left\{G(x) - a\int \frac{W(t)\left[\left(F'(t)\right)^2 + F'(t)H(t) - 2bcF(t) + 2acF^2(t)\right]}{F(t)\left[aF(t)-b\right]\left[F'(t)+H(t)\right]}\,dt\right\}^{-1},$$

where $F(t)$ and $G(x)$ are arbitrary functions, and

$$W(t) = \exp\left\{\int \frac{\left[2aF(t)-b\right]\left[\left(F'(t)\right)^2 + F'(t)H(t) - 2bcF(t) + 2acF^2(t)\right]}{F(t)\left[aF(t)-b\right]\left[F'(t)+H(t)\right]}\,dt\right\},$$

$$H(t) = \pm\sqrt{\left(F'(t)\right)^2 + 4acF^2(t) - 4bcF(t)}.$$

$$\mathbf{2.33} \qquad \boldsymbol{\frac{\partial^2 w}{\partial t \partial x} + akw\exp\left[\frac{1}{aw}\frac{\partial w}{\partial t} + \frac{c}{aw} + b\right] -}$$

$$\boldsymbol{\frac{1}{w}\left[\frac{\partial w}{\partial t} + c\right]\frac{\partial w}{\partial x} + m\frac{\partial w}{\partial t} + abmw + mc = 0\,,}$$

where $a \neq 0$, b, c, k and m are constants.

General solution

$$w(t,x) = \left\{-c\int \exp\left[abt - a\int W(t,x)\,dt\right]\,dt + G(x)\right\}\times$$

$$\exp\left[-abt + a\int W(t,x)\,dt\right],$$

here $W(t,x)$ is any solution of the following transcendental equation

$$\int_s^{W(t,x)} \frac{d\xi}{ke^{\xi}+m\xi} + x + F(t) = 0\,,$$

where $F(t)$ and $G(x)$ are arbitrary functions, s is any constant.

$$\textbf{2.34}\quad \left(\frac{\partial^2 w}{\partial t\partial x}\right)^2 + 2a\frac{\partial^2 w}{\partial t\partial x}\frac{\partial w}{\partial x} + b\left(\frac{\partial w}{\partial t}\right)^3 + (3abw-c^2)\left(\frac{\partial w}{\partial t}\right)^2 + aw(3abw-2c^2)\frac{\partial w}{\partial t} + a^2\left(\frac{\partial w}{\partial x}\right)^2 + a^2w^2(abw-c^2) = 0\,,$$

where a, and $b \neq 0$, $c \neq 0$ are constants.

General solution

$$w(t,x) = \left(\frac{4c^3e^{cx}}{b}\int \frac{F(t)[F(t)-1]e^{at}}{[F(t)(c+e^{cx})-c]^2}\,dt + G(x)\right)e^{-at}\,,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

$$\textbf{2.35}\quad w\left(\frac{\partial^2 w}{\partial t\partial x}\right)^2 - \frac{\partial w}{\partial x}\left(\frac{\partial w}{\partial t} - kw + b\right)\frac{\partial^2 w}{\partial t\partial x} - \left(\frac{\partial w}{\partial x}\right)^2\left(k\frac{\partial w}{\partial t} - a + bk\right) + c\left[\left(\frac{\partial w}{\partial t} + kw\right)^2 + 2b\frac{\partial w}{\partial t} - 2w(2a-bk) + b^2\right]\left(\frac{\partial w}{\partial x} - cw\right) = 0\,,$$

where a, b, c, and k are constants.

General solution

$$w(t,x) = \left[-\int\left(\frac{ae^{cx}}{F'(t)} + b\right)\exp\left[kt + F(t)e^{-cx}\right]dt + G(x)\right]\times \exp\left[-kt - F(t)e^{-cx}\right]\,,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

2.36 $$\left(k\frac{\partial^2 w}{\partial t\partial x} + a\frac{\partial w}{\partial x}\right)^2 \times$$

$$\left[\left(k\frac{\partial w}{\partial t} + aw\right)^2 - 2bm\frac{\partial w}{\partial t} - 2bcw\right] =$$

$$\left[\frac{\partial^2 w}{\partial t\partial x}\left(k^2\frac{\partial w}{\partial t} + akw - bm\right) +\right.$$

$$\left. ak\frac{\partial w}{\partial t}\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x}(a^2 w - cb)\right]^2 ,$$

where a, b, c, k, and m are constants.

General solution

$$w(t,x) = -\frac{b}{2(am-ck)F(t)}\left(\int \frac{[cF(t) - mF'(t)]^2}{aF(t) - kF'(t)}\,dt + G(x)\right),$$

where $F(t)$ and $G(x)$ are arbitrary functions.

2.37 $$\frac{\partial^2 w}{\partial t\partial x} = -\frac{a}{w^2}\left(\frac{\partial w}{\partial t}\right)^3 - \left(\frac{b}{w} + \frac{c}{w^2}\right)\left(\frac{\partial w}{\partial t}\right)^2 +$$

$$\left[\frac{1}{w}\frac{\partial w}{\partial x} + \frac{(ak-b)(3ak+b)}{4a} - \frac{2bc}{3aw} - \frac{c^2}{3aw^2}\right]\frac{\partial w}{\partial t} +$$

$$\frac{c}{3aw}\frac{\partial w}{\partial x} - \frac{k(ak-b^2)w}{4a} + \frac{c(ak-b)(3ak+b)}{12a^2} -$$

$$\frac{bc^2}{9a^2 w} - \frac{c^3}{27a^2 w^2},$$

where $a \neq 0$, b, c, and k are constants.

General solution

$$w(t,x) = \frac{1}{3a}\left(-c\int \exp\left\{-\frac{1}{2a}\int \frac{(ak-b)V(t,x)-2ak}{V(t,x)+1}\,dt\right\}dt + G(x)\right)\times$$

$$\exp\left\{\frac{1}{2a}\int \frac{(ak-b)V(t,x)-2ak}{V(t,x)+1}\,dt\right\},$$

here

$$V(t,x) = W\left(F(t)\exp\left\{-\frac{(3ak-b)^2x+4a}{4a}\right\}\right),$$

where $W(z)$ is the Lambert W function, $F(t)$ and $G(x)$ are arbitrary functions.

2.38 $bw\frac{\partial^2 w}{\partial t\partial x} = a\left(\frac{\partial w}{\partial t}\right)^2 + \left(b\frac{\partial w}{\partial x} + cw + 2af\right)\frac{\partial w}{\partial t} + bf\frac{\partial w}{\partial x} + \frac{c^2g(1-g)}{a}w^2 + cfw + af^2,$

where $a \neq 0, b \neq 0, c \neq 0$, and g, f are constants.

General solution

$$w(x,t) = -\frac{1}{c} + \frac{W(t,x)}{a}\times$$

$$\left\{G(x) + \int \frac{\left[a(af+g-1)\exp\left(\frac{c}{b}[x+2gF(t)]\right) - (af-g)\exp\left(\frac{c}{b}[2gx+F(t)]\right)\right]}{W(t,x)\left[-a\exp\left(\frac{c}{b}[x+2gF(t)]\right) + \exp\left(\frac{c}{b}[2gx+F(t)]\right)\right]}\,dt\right\},$$

where $F(t)$ and $G(x)$ are arbitrary functions, and

$$W(t,x) = \exp\left\{\frac{c}{a}\int \frac{\left[a(g-1)\exp\left(\frac{c}{b}[x+2gF(t)]\right) + g\exp\left(\frac{c}{b}[2gx+F(t)]\right)\right]}{\left[a\exp\left(\frac{c}{b}[x+2gF(t)]\right) - \exp\left(\frac{c}{b}[2gx+F(t)]\right)\right]}\,dt\right\}.$$

$$\mathbf{2.39} \qquad (a^2bw^4+ck^2)\left(\frac{\partial^2 w}{\partial t\partial x}\right)^2 - 2a^2w^3\frac{\partial w}{\partial x}\left(2b\frac{\partial w}{\partial t}+k\right)\frac{\partial^2 w}{\partial t\partial x}$$

$$+\,4a^2kw^2\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial w}{\partial t}+4a^2bw^2\left(\frac{\partial w}{\partial t}\right)^2\left(\frac{\partial w}{\partial x}\right)^2=0\,,$$

where a, b, c, and k are constants.

General solution

$$w(x,t)=F(t)-W(t)\left\{G(x)+a\int\frac{W(t)F'(t)\left[aF^2(t)+H(t)\right]}{a^2F^4(t)+aF^2(t)H(t)-2ckF'(t)}\,dt\right\}^{-1},$$

where $F(t)$ and $G(x)$ are arbitrary functions, and

$$W(t)=\exp\left\{2a\int\frac{F(t)F'(t)\left[aF^2(t)+H(t)\right]}{a^2F^4(t)+aF^2(t)H(t)-2ckF'(t)}\,dt\right\},$$

$$H(t)=\pm\sqrt{a^2F^4(t)-4cb\left[F'(t)\right]^2-4ckF'(t)}.$$

$$\mathbf{2.40} \qquad (aw^6+b)\left(\frac{\partial^2 w}{\partial t\partial x}\right)^3-$$

$$2w^5\frac{\partial w}{\partial x}\left(3a\frac{\partial w}{\partial t}+c\right)\left(\frac{\partial^2 w}{\partial t\partial x}\right)^2$$

$$+\,4w^4\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial w}{\partial t}\left(3a\frac{\partial w}{\partial t}+2c\right)\frac{\partial^2 w}{\partial t\partial x}-$$

$$8w^3\left(\frac{\partial w}{\partial x}\right)^3\left(\frac{\partial w}{\partial t}\right)^2\left(a\frac{\partial w}{\partial t}+c\right)=0\,,$$

where a, and $b\neq 0, c\neq 0$ are constants.

General solution

$$w(x,t) = F(t) - W(t)\times$$

$$\left\{G(x) + \int \frac{W(t)F'(t)\left[12bF'(t)F^2(t) + c^3P^2(t)\right]}{12bF'(t)F^4(t) - 6bcF'(t)P(t) + c^3F^2(t)P^2(t)}\,dt\right\}^{-1},$$

where $F(t)$ and $G(x)$ are arbitrary functions, and

$$W(t) = \exp\left\{2\int \frac{F(t)F'(t)\left[12bF'(t)F^2(t) + c^3P^2(t)\right]}{12bF'(t)F^4(t) - 6bcF'(t)P(t) + c^3F^2(t)P^2(t)}\,dt\right\},$$

$$P^3(t) = -\frac{108b^2}{c^6}\left[F'(t)\right]^2\left[aF'(t) - \frac{\sqrt{3}}{9}H(t) + c\right],$$

$$H(t) = \pm\sqrt{\frac{27a^2b\left[F'(t)\right]^3 + 54abc\left[F'(t)\right]^2 + 27bc^2F'(t) - 4c^3F^6(t)}{bF'(t)}}.$$

2.41 $\boldsymbol{4a^2b^2fg^2(bcgw + ah)\left(a\frac{\partial w}{\partial t} + cw\right)\frac{\partial^2 w}{\partial t\partial x} = \left(bg\frac{\partial w}{\partial t} - h\right)\Big[4a^2b^2g^2k^2\left(\frac{\partial w}{\partial t}\right)^2 + \Big(4a^3b^2cfg^2\frac{\partial w}{\partial x} + 4ab^2cg^2k(g+3k)w + 4a^2bghk(g+k)\Big)\frac{\partial w}{\partial t} + 4a^2b^2c^2g^2fw\frac{\partial w}{\partial x} + b^2c^2g^2(g+3k)w^2 + 2abcgh(g+k)(g+3k)w + a^2h^2(g+k)^2\Big]}$,

where $a \neq 0$, $b \neq 0$, c, $f \neq 0$, $g \neq 0$, h and k are constants.

General solution

$$w(t,x) = -\frac{1}{bgE(t,x)}\left[h(g+k)\int \frac{E(t,x)[V(t,x)+1]}{2kV(t,x) - g - k}\,dt + G(x)\right],$$

here

$$E(t,x) = \exp\left[\frac{c(g+3k)}{a}\int \frac{V(t,x)}{2kV(t,x)-g-k}\,dt\right],$$

$$V(t,x) = W\left(-\frac{g+k}{2bf}\exp\left\{-\frac{(g+3k)^2[F(t)+x]}{4a^2 f}\right\}\right),$$

where $W(z)$ is the Lambert W function, $F(t)$ and $G(x)$ are arbitrary functions.

2.42 $\quad V\left(\frac{\partial w}{\partial t} + bw\right)\left(\frac{\partial^2 w}{\partial t \partial x} + b\frac{\partial w}{\partial x}\right) + a = 0\,,$

where a, b are constants and $V(z)$ is any function.

General solution

$$w(x,t) = \left(\int W(t,x)e^{bt}\,dt + G(x)\right)e^{-bt},$$

where $W(t,x)$ is any solution of the following transcendental equation

$$\int_s^{W(t,x)} V(\xi)\,d\xi + ax = F(t)\,,$$

$F(t)$ and $G(x)$ are arbitrary functions, and s is an arbitrary constant.

2.43 $\quad V\left(\frac{\frac{\partial^2 w}{\partial t \partial x}}{(2aw+b)\frac{\partial w}{\partial x}}\right) + \frac{\partial w}{\partial t} = \frac{w(aw+b)\frac{\partial^2 w}{\partial t \partial x}}{(2aw+b)\frac{\partial w}{\partial x}}\,,$

where a, b are constants, and $V(z)$ is any function.

General solution

$$w(x,t) = F(t) - \exp\left\{\int W(t)\,[2aF(t)+b]\,dt\right\}\times$$

$$\left\{a\int W(t)\exp\left[\int W(t)\,[2aF(t)+b]\,dt\right]dt + G(x)\right\}^{-1},$$

where $W(t)$ is any solution of the following transcendental equation

$$V\,[W(t)] + F'(t) = W(t)F(t)\,[aF(t)+b]\,,$$

$F(t)$ and $G(x)$ are arbitrary functions.

2.44 $$\boldsymbol{\frac{\partial^2 w}{\partial t \partial x} + akwV\left(\frac{1}{aw}\frac{\partial w}{\partial t} + \frac{c}{aw} + b\right) -}$$

$$\boldsymbol{\frac{1}{w}\left[\frac{\partial w}{\partial t} + c\right]\frac{\partial w}{\partial x} = 0\,,}$$

where $a \neq 0$, b, c are constants, and $V(z) \neq 0$ is any function.

General solution

$$w(t,x) = \left\{-c\int \exp\left[abt - a\int W(t,x)\,dt\right]\,dt + G(x)\right\} \times$$

$$\exp\left[-abt + a\int W(t,x)\,dt\right]\,,$$

here $W(t,x)$ is any solution of the following transcendental equation

$$\int_s^{W(t,x)} \frac{d\xi}{V(\xi)} + x + F(t) = 0\,,$$

where $F(t)$ and $G(x)$ are arbitrary functions, s is any constant.

2.45 $$\boldsymbol{V\left(\frac{w\frac{\partial w}{\partial x}}{\frac{\partial^2 w}{\partial t \partial x} + b\frac{\partial w}{\partial x}}\right) + \frac{2aw\frac{\partial w}{\partial x}\left(\frac{\partial w}{\partial t} + bw\right)}{\frac{\partial^2 w}{\partial t \partial x} + b\frac{\partial w}{\partial x}} = aw^2\,,}$$

where $a \neq 0$, b are constants, and $V(z)$ is any function.

General solution

$$w(x,t) = F(t) - \exp\left\{\int \frac{2aF(t)}{W(t)}dt - bt\right\} \times$$

$$\left\{a\int \frac{1}{W(t)} \exp\left[\int \frac{2aF(t)}{W(t)}dt - bt\right]dt + G(x)\right\}^{-1}\,,$$

where $W(t)$ is any solution of the following transcendental equation

$$V\left[\frac{W(t)}{2a}\right] + W(t)[F'(t) + bF(t)] = aF^2(t)\,,$$

$F(t)$ and $G(x)$ are arbitrary functions.

2.46 $\left[(a_2b_1 - a_1b_2)w - a_1b_3 + b_1a_3\right] \dfrac{\partial^2 w}{\partial t \partial x} =$

$$\left[(a_2b_1 - a_1b_2)\frac{\partial w}{\partial t} + a_2b_3 - a_3b_2\right] \frac{\partial w}{\partial x}$$

$$- \left(b_1\frac{\partial w}{\partial t} + b_2 w + b_3\right)^2 V\left(\frac{a_1\frac{\partial w}{\partial t} + a_2 w + a_3}{b_1\frac{\partial w}{\partial t} + b_2 w + b_3}\right),$$

where $V \neq 0$ is an arbitrary function, a_i and b_i are constants.

General solution

$$w(t,x) = \exp\left[\int \frac{-a_2 + b_2 Y(t,x)}{a_1 - b_1 Y(t,x)}\,dt\right]\left\{\int \frac{-a_3 + b_3 Y(t,x)}{a_1 - b_1 Y(t,x)}\right.$$

$$\left.\exp\left[-\int \frac{-a_2 + b_2 Y(t,x)}{a_1 - b_1 Y(t,x)}\,dt\right]\,dt + G(x)\right\},$$

where the function $Y(t,x)$ is determined by the transcendental equation

$$\int_s^Y \frac{dz}{V(z)} = x + F(t)$$

and $F(t)$ and $G(x)$ are arbitrary functions, s is any constant.

3 Equations of the Form $\dfrac{\partial^2 w}{\partial t \partial x} = F(w, \dfrac{\partial w}{\partial t}, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2})$

3.1 $\dfrac{\partial^2 w}{\partial t\,\partial x} = w\dfrac{\partial^2 w}{\partial t^2}$.

General solution

$$w(t,x) = \int_s^W F(\xi)\,e^{-x/\xi}\,d\xi + G'(x),$$

where $W = W(t,x)$ is a solution of the following transcendental equation

$$t - \int_v^W \xi F(\xi)\, e^{-x/\xi}\, d\xi + G(x) = 0$$

and $F(\xi)$ and $G(x)$ are arbitrary functions, s and v are arbitrary constants.

3.2 $\left(\frac{\partial w}{\partial x} + w^2\right)\frac{\partial^2 w}{\partial t \partial x} = \left(\frac{\partial^2 w}{\partial x^2} + 3w\frac{\partial w}{\partial x} + w^3\right)\frac{\partial w}{\partial t}$

General solution in implicit form ($w = w(t,x)$):

$$\int_s^{\frac{1-xw}{w}} G'(\xi)\xi d\xi + xG\left(\frac{1-xw}{w}\right) = F(t)\,,$$

where $F(t)$ and $G(z)$ are arbitrary functions, s is an arbitrary constant.

3.3 $\frac{\partial^2 w}{\partial t \partial x} + \frac{1}{w}\frac{\partial w}{\partial x} = \frac{1}{\frac{\partial w}{\partial x}}\left(\frac{\partial w}{\partial t} - 1\right)\frac{\partial^2 w}{\partial x^2}$

General solution

$$w(t,x) = G'[W(t,x) + F'(t)] + t\,,$$

where $W(t,x)$ is any solution of the following transcendental equation

$$G[W(t,x) + F'(t)] + tW(t,x) = F(t) - tF'(t) + x\,,$$

and $F(t)$ and $G(z)$ are arbitrary functions.

3.4 $\frac{\partial^2 w}{\partial t \partial x} = w\frac{\partial^2 w}{\partial x^2} + n\left(\frac{\partial w}{\partial x}\right)^2$

where $n \neq 0$ is a constant.

General solution

$$w(t,x) = -\frac{1}{n}\int_s^{W(t,x)} [G(\xi) + t]^{\frac{1-n}{n}}\, d\xi + F'(t)\,,$$

where $W(t,x)$ is any solution of the following transcendental equation

$$\int_s^{W(t,x)} [G(\xi) + t]^{\frac{1}{n}}\, d\xi = x + F(t)\,,$$

and $F(t)$ and $G(\xi)$ are arbitrary functions, s is any constant.

$$\textbf{3.5}\quad \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial t\partial x} = \left[\frac{\partial^2 w}{\partial x^2} + a\left(\frac{\partial w}{\partial x}\right)^2\right]\frac{\partial w}{\partial t}\,,$$

where a is a constant.

General solution in implicit form ($w = w(t,x)$):

$$e^{aw}F(t) = x + G(w)\,,$$

where $F(t)$ and $G(z)$ are arbitrary functions.

$$\textbf{3.6}\quad w\left(\frac{\partial w}{\partial x} + a\right)\frac{\partial^2 w}{\partial t\partial x} =$$

$$\left[w\frac{\partial^2 w}{\partial x^2} - a\frac{\partial w}{\partial x} - a^2\right]\frac{\partial w}{\partial t}\,,$$

where $a \neq 0$ is a constant.

General solution

$$w(t,x) = -ae^{aW(t,x)}F(t) + G'[W(t,x)]\,,$$

where $W(t,x)$ is any solution of the following transcendental equation

$$G[W(t,x)] = e^{aW(t,x)} - x\,,$$

and $F(t)$ and $G(z)$ are arbitrary functions.

$$\textbf{3.7}\quad \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial t^2} - \left(\frac{\partial w}{\partial t} + a\right)\frac{\partial^2 w}{\partial t\partial x} =$$

$$\frac{\partial w}{\partial x}\left[2\left(\frac{\partial w}{\partial t}\right)^2 + 3a\frac{\partial w}{\partial t} + a^2\right]\,,$$

where $a \neq 0$ is a constant.

General solution in implicit form ($w = w(t,x)$)

$$e^{at+2w} + F(at+w)e^{at+w} + G(x) = 0\,,$$

where $F(z)$ and $G(x)$ are arbitrary functions.

3.8 $\boldsymbol{\frac{\partial w}{\partial x}\left(\frac{\partial^2 w}{\partial t\partial x}\right)^2 + a\left(\frac{\partial^2 w}{\partial x^2}\right)^2 = \frac{\partial w}{\partial t}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial t\partial x}}$**,

where a is a constant.

General solution

$$w(t,x) = -a\int \frac{dt}{F'(t)} + G[x - F(t)]\,,$$

where $F(t)$ and $G(z)$ are arbitrary functions.

3.9 $\boldsymbol{w\left(\frac{\partial w}{\partial x} - aw + 1\right)\frac{\partial^2 w}{\partial t\partial x} =}$

$$\boldsymbol{\left[w\frac{\partial^2 w}{\partial x^2} - \frac{\partial w}{\partial x} - (aw-1)^2\right]\frac{\partial w}{\partial t}}$$

where $a \neq 0$ is a constant.

General solution

$$w(t,x) = \frac{1}{a} + e^{ax}G'[W(t,x)]\,,$$

where $W(t,x)$ is any solution of the following transcendental equation

$$W(t,x)e^{-ax} + aG[W(t,x)] = F(t)\,,$$

and $F(t)$ and $G(z)$ are arbitrary functions.

3.10 $\boldsymbol{w\left(\frac{\partial w}{\partial x} + w\right)\frac{\partial^2 w}{\partial t\partial x} =}$

$$\boldsymbol{\left[w\frac{\partial^2 w}{\partial x^2} + \left(\frac{\partial w}{\partial x}\right)^2 + w(a+2)\frac{\partial w}{\partial x} + aw^2\right]\frac{\partial w}{\partial t}}$$

where $a \neq 1$ is a constant.

General solution in implicit form ($w = w(t,x)$):

$$w^{a-1} + (a-1)\left[w^a e^{ax}F(t) + G(x + \ln w)\right] = 0\,,$$

where $F(t)$ and $G(z)$ are arbitrary functions.

3.11 $$\left(\frac{\partial w}{\partial x}+a\right)\frac{\partial^2 w}{\partial t\partial x}=$$

$$\left[\frac{\partial^2 w}{\partial x^2}-2\left(\frac{\partial w}{\partial x}\right)^2-3a\frac{\partial w}{\partial x}-a^2\right]\frac{\partial w}{\partial t}$$

where $a\neq 0$ is a constant.

General solution in implicit form ($w=w(t,x)$):

$$e^w+ae^{-(w+ax)}F(t)+G\left(\frac{w}{a}+x\right)=0\,,$$

where $F(t)$ and $G(z)$ are arbitrary functions.

3.12 $$\left(\frac{\partial w}{\partial x}-(w+a)(2w+a)\right)\frac{\partial^2 w}{\partial t\partial x}=$$

$$\left[\frac{\partial^2 w}{\partial x^2}-(6w+4a)\frac{\partial w}{\partial x}+(w+a)(2w+a)^2\right]\frac{\partial w}{\partial t}$$

where $a\neq 0$ is a constant.

General solution

$$w(t,x)=a\frac{a^2e^{-[W+ax]}F(t)-G'\left(\frac{W}{a}+x\right)}{e^W-a^2e^{-[W+ax]}F(t)+G'\left(\frac{W}{a}+x\right)}\,,$$

where $W=W(t,x)$ is any solution of the following transcendental equation

$$e^W+ae^{-[W+ax]}F(t)+G\left(\frac{W}{a}+x\right)=0\,,$$

and $F(t)$ and $G(z)$ are arbitrary functions.

3.13 $$w\left[\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial t\partial x}+\frac{\partial w}{\partial t}\frac{\partial^2 w}{\partial x^2}\right]+\frac{\partial w}{\partial t}\frac{\partial w}{\partial x}\left[\frac{\partial w}{\partial x}+aw\right]=0\,,$$

where a is a constant.

General solution in implicit form

$$w(t,x)^3=\frac{e^{-ax}}{aW}\left[ae^{ax}G(W)-3W\right]\left[W+F(t)\right]\,,$$

where $W = W(t,x)$ is any solution of the following transcendental equation

$$ae^{ax}W\left[W + F(t)\right]G'(W) - 3W^2 - ae^{ax}F(t)G(W) = 0$$

and $F(t)$ and $G(z)$ are arbitrary functions.

3.14 $\left[\dfrac{\partial^2 w}{\partial x^2} + a\dfrac{\partial w}{\partial x}\right]\left[\dfrac{\partial^2 w}{\partial t \partial x} + a\dfrac{\partial w}{\partial t}\right] = b$

where a, b are constants.

General solution

$$w(t,x) = \left\{\pm \int e^{ax}\sqrt{\frac{[2tW + G(W)][2bx + W]}{W}}\,dx + F(t)\right\}e^{-ax},$$

where $W = W(t,x)$ is any solution of the following transcendental equation

$$W(2bx + W)G'(W) + 2tW^2 - 2bxG(W) = 0$$

and $F(t)$ and $G(z)$ are arbitrary functions.

3.15 $\left[\dfrac{\partial w}{\partial x} + aw\right]\left[\dfrac{\partial^2 w}{\partial x^2} + a\dfrac{\partial w}{\partial x}\right]\left[\dfrac{\partial^2 w}{\partial t \partial x} + a\dfrac{\partial w}{\partial t}\right] = b$

where a, b are constants.

General solution

$$w(t,x) = \left\{\frac{(144b)^{\frac{1}{3}}}{4}\int \frac{[G(W) - xW - t]^{\frac{2}{3}}}{W^{\frac{1}{3}}}e^{ax}\,dx + F(t)\right\}e^{-ax},$$

where $W = W(t,x)$ is any solution of the following transcendental equation

$$2WG'(W) - xW + t - G(W) = 0$$

and $F(t)$ and $G(z)$ are arbitrary functions.

3.16 $\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial t \partial x} = \left[\dfrac{\partial w}{\partial t} + bw\left(\dfrac{\partial w}{\partial x}\right)^2\right]\dfrac{\partial^2 w}{\partial x^2} + aw\left(\dfrac{\partial w}{\partial x}\right)^2,$

where a, and $b \neq 0$ are constants.

General solution in implicit form

$$\int_{s}^{w(t,x)} \frac{d\xi}{W(t,\xi)} = x + F(t)\,,$$

where $W(t,\xi) = W$ is any solution of the following transcendental equation

$$(2a\xi + bW^2)\left[G(2a\xi + bW^2) + t\right] - \ln(-bW^2) + \ln(\xi) = 0$$

and $F(t)$ and $G(z)$ are arbitrary functions, s is any constant.

3.17 $\displaystyle \frac{\partial^2 w}{\partial t \partial x} = \left(\frac{\partial w}{\partial t} + aw\right)\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^{-1} - a\frac{\partial w}{\partial x} + b\left(\frac{\partial w}{\partial t} + aw\right)$,

where a, b are constants.

General solution

$$w(t,x) = e^{-at} G\left(F(t) + e^{-bx}\right)\,,$$

where $F(t)$ and $G(z)$ are arbitrary functions.

3.18 $\displaystyle \frac{\partial^2 w}{\partial t \partial x} = \left(\frac{\partial w}{\partial t} + aw\right)\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^{-1} + c\frac{\partial w}{\partial x} + b\left(\frac{\partial w}{\partial x} + kw\right)$,

where a, b, $c \neq -(a+b)$, and k are constants.

General solution in implicit form

$$\int_{s}^{w(t,x)} \frac{(a+b+c)\,d\xi}{G\left[\xi e^{at}\right] e^{(b+c)t} + bk\xi} + F(t) + x = 0\,,$$

where $F(t)$ and $G(z)$ are arbitrary functions, and s is an arbitrary constant.

3.19 $\displaystyle \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial t \partial x} = \frac{\partial w}{\partial t}\frac{\partial^2 w}{\partial x^2} + aw^m\left(\frac{\partial w}{\partial x}\right)^n$,

where a, m, and $n \neq 2$ are constants.

General solution in implicit form

$$\int_{s}^{w(t,x)} \frac{d\xi}{W(t,\xi)} = x + F(t)\,,$$

where $W(t,\xi) = W$ is any solution of the following transcendental equation

$$W^{(2-n)} + at\xi^m(n-2) = G(\xi)$$

and $F(t)$ and $G(z)$ are arbitrary functions, s is any constant.

3.20 $$\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial t\partial x} = \left[\frac{\partial w}{\partial t} + bw^n\right]\frac{\partial^2 w}{\partial x^2} + aw^m\left(\frac{\partial w}{\partial x}\right)^2,$$

where a, $b \neq 0$, $n \neq 1$ and $m \neq n-1$ are constants.

General solution in implicit form

$$\int_s^{w(t,x)} \exp\left[\frac{a\xi^{(m-n+1)}}{b(m-n+1)}\right] G\left[\frac{\xi^{(1-n)} - bt(n-1)}{b(n-1)}\right] d\xi = F(t) - x\,,$$

where $F(t)$ and $G(z)$ are arbitrary functions, s is any constant.

3.21 $$w\left(c\frac{\partial w}{\partial x} + bw^3\right)\frac{\partial^2 w}{\partial t\partial x} =$$

$$\left[cw\frac{\partial^2 w}{\partial x^2} - c\left(\frac{\partial w}{\partial x}\right)^2 + w(2bw^2 + ac)\frac{\partial w}{\partial x} + abw^4\right]\frac{\partial w}{\partial t}$$

where a, $b \neq 0$, and c are constants.

General solution in implicit form ($w = w(t,x)$):

$$\sqrt{2}\exp\left[\frac{ac}{2bw^2} - ax\right] erf\left[\frac{\sqrt{2ac}}{2w\sqrt{b}}\right] + G\left[\frac{c}{w^2} - 2bx\right] = F(t)\,,$$

where $erf(z)$ is the error function, $F(t)$ and $G(z)$ are arbitrary functions.

3.22 $$\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial t\partial x} - \frac{\partial w}{\partial t}\frac{\partial^2 w}{\partial x^2} + [bV(w) + a]\left(\frac{\partial w}{\partial x}\right)^2 +$$

$$\frac{V(w)[bV(w) + a]}{V'(w)}\frac{\partial^2 w}{\partial x^2} = 0\,,$$

where $a \neq 0$, and b are constants, $V(w) \neq const$ is any function.

General solution in implicit form

$$\int_s^{w(t,x)} V(\xi)G\left\{\frac{e^{at}\left[bV(\xi) + a\right]}{aV(\xi)}\right\} d\xi + x + F(t) = 0\,,$$

where $F(t)$ and $G(z)$ are arbitrary functions, s is any constant.

4 Equations of the Form $\dfrac{\partial^2 w}{\partial t \partial x} = F(w, \dfrac{\partial w}{\partial t}, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial t^2}, \dfrac{\partial^2 w}{\partial x^2})$

4.1 $\left(\dfrac{\partial w}{\partial x}\right)^2 \dfrac{\partial^2 w}{\partial t^2} - \left(\dfrac{\partial w}{\partial t}\right)^2 \dfrac{\partial^2 w}{\partial x^2} = 0$.

General solution

$$w(t,x) = F\left[\frac{(xW - G(W) + t)^2}{W}\right],$$

where $W = W(t,x)$ is any solution of the following transcendental equation

$$G(W) - 2WG'(W) + xW - t = 0$$

and $F(z)$ and $G(z)$ are arbitrary functions.

4.2 $2\dfrac{\partial w}{\partial t}\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial t \partial x} - \left(\dfrac{\partial w}{\partial x}\right)^2 \dfrac{\partial^2 w}{\partial t^2} - \left(\dfrac{\partial w}{\partial t}\right)^2 \dfrac{\partial^2 w}{\partial x^2} = 0$.

General solution in implicit form ($w = w(t,x)$):

$$G(w)\,[F(w) + x] + t = 0\,,$$

where $F(z)$ and $G(z)$ are arbitrary functions.

4.3 $\left[a\dfrac{\partial w}{\partial t} + 2\dfrac{\partial w}{\partial t}\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial x}\right] \dfrac{\partial^2 w}{\partial t \partial x} =$

$$\frac{\partial w}{\partial x}\left(a + \frac{\partial w}{\partial x}\right)\frac{\partial^2 w}{\partial t^2} + \frac{\partial w}{\partial t}\left(b + \frac{\partial w}{\partial t}\right)\frac{\partial^2 w}{\partial x^2}\,,$$

where $a \neq 0$, and b are constants.

General solution in implicit form ($w = w(t,x)$):

$$a\int_s^{\frac{bt+ax}{a}} \frac{d\xi}{aG(w + a\xi) - b} + F(w) + t = 0\,,$$

where $F(z)$ and $G(z)$ are arbitrary functions, s is any constant.

4.4 $$\boldsymbol{\left(\frac{\partial^2 w}{\partial t \partial x}\right)^2 - \frac{\partial^2 w}{\partial t^2}\frac{\partial^2 w}{\partial x^2} + a\frac{\partial^2 w}{\partial t^2} = 0\,,}$$

where a is a constant.

General solution

$$w(t,x) = \frac{at^2}{2} + tG(W) + F(W) + xW\,,$$

where $W = W(t,x)$ is any solution of the following transcendental equation

$$tG'(W) + F'(W) + x = 0$$

and $F(z)$ and $G(z)$ are arbitrary functions.

4.5 $$\boldsymbol{\left(\frac{\partial^2 w}{\partial t \partial x}\right)^2 - \frac{\partial^2 w}{\partial t^2}\frac{\partial^2 w}{\partial x^2} + a\frac{\partial^2 w}{\partial t^2}\frac{\partial w}{\partial x} = 0\,,}$$

where $a \neq 0$ is a constant.

General solution

$$w(t,x) = \frac{1}{a}\left[atW + aG(W) + F(W)e^{ax}\right]\,,$$

where $W = W(t,x)$ is any solution of the following transcendental equation

$$aG'(W) + e^{ax}F'(W) + at = 0$$

and $F(z)$ and $G(z)$ are arbitrary functions.

4.6 $$\boldsymbol{\left(\frac{\partial^2 w}{\partial t \partial x}\right)^2 - \frac{\partial^2 w}{\partial t^2}\frac{\partial^2 w}{\partial x^2} + a\left(\frac{\partial w}{\partial t}\right)^2\frac{\partial^2 w}{\partial x^2} = 0\,,}$$

where $a \neq 0$ is a constant.

General solution

$$w(t,x) = \frac{1}{a}\left\{aF(W) + axW - \ln\left[G(W) + at\right]\right\}\,,$$

where $W = W(t,x)$ is any solution of the following transcendental equation

$$-G'(W) + axG(W) + aF'(W)(G(W) + at) + a^2tx = 0$$

and $F(z)$ and $G(z)$ are arbitrary functions.

$$\textbf{4.7}\quad \left(\frac{\partial^2 w}{\partial t\partial x}\right)^2 + a\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial t\partial x} + \left(b\frac{\partial w}{\partial x} - \frac{\partial^2 w}{\partial x^2}\right)\frac{\partial^2 w}{\partial t^2} = 0\,,$$

where a, and $b \neq 0$ are constants.

General solution

$$w(t,x) = \frac{Te^{bx}}{b} + F(T) + \int_s^t W d\xi\,,$$

where $W = W(T,\xi)$ is any solution of the transcendental equation

$$G(W)e^{-a\xi} + T = 0\,,$$

$T = T(t,x)$ is any solution of the following transcendental equation

$$b\int_s^t \frac{e^{a\xi}d\xi}{G'\left[W(T,\xi)\right]} = bF'(T) + e^{bx}$$

and $F(z)$ and $G(z)$ are arbitrary functions, s is any constant.

$$\textbf{4.8}\quad \frac{\partial w}{\partial x}\left(\frac{\partial^2 w}{\partial t\partial x}\right)^2 + a\frac{\partial^2 w}{\partial t\partial x} + \frac{\partial w}{\partial x}\left(b - \frac{\partial^2 w}{\partial x^2}\right)\frac{\partial^2 w}{\partial t^2} = 0\,,$$

where a, and $b \neq 0$ are constants.

General solution

$$w(t,x) = \pm\frac{(T+2bx)^{\frac{3}{2}}}{3b} + F(T) + \int_s^t W d\xi\,,$$

where $W = W(T,\xi)$ is any solution of the transcendental equation

$$G(W) + 2a\xi + T = 0\,,$$

$T = T(t,x)$ is any solution of the following transcendental equation

$$2b\int_s^t \frac{d\xi}{G'\left[W(T,\xi)\right]} = 2bF'(T) \pm \sqrt{2bx+T}$$

and $F(z)$ and $G(z)$ are arbitrary functions, s is any constant.

$$\textbf{4.9}\quad \left(\frac{\partial^2 w}{\partial t\partial x}\right)^2 + a\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^2 w}{\partial t\partial x} +$$

$$\left[b\left(\frac{\partial w}{\partial x}\right)^2 - \frac{\partial^2 w}{\partial x^2}\right]\frac{\partial^2 w}{\partial t^2} = 0\,,$$

where a, and $b \neq 0$ are constants.

General solution

$$w(t,x) = -\frac{\ln(bx-T)}{b} + F(T) + \int_s^t W\,d\xi\,,$$

where $W = W(T,\xi)$ is any solution of the transcendental equation

$$G(W) - a\xi + T = 0\,,$$

$T = T(t,x)$ is any solution of the following transcendental equation

$$\int_s^t \frac{d\xi}{G'\left[W(T,\xi)\right]} - F'(T) = \frac{1}{b(bx-T)}$$

and $F(z)$ and $G(z)$ are arbitrary functions, s is any constant.

4.10 $\boldsymbol{\left(\frac{\partial^2 w}{\partial t\partial x}\right)^2 + a\frac{\partial^2 w}{\partial t\partial x} - \frac{\partial^2 w}{\partial t^2}\frac{\partial^2 w}{\partial x^2} + b\frac{\partial^2 w}{\partial t^2} + c\frac{\partial^2 w}{\partial x^2} = bc\,,}$

where a, b, and c are constants.

General solution

$$w(t,x) = \frac{ct^2 + bx^2}{2} + xT + F(T) + \int_s^t W\,d\tau\,,$$

where $W = W(T,\tau)$ is any solution of the following transcendental equation

$$G(W) + T + a\tau = 0$$

and $T = T(t,x)$ is any solution of the following transcendental equation

$$\int_s^t \frac{d\tau}{G'[W(T,\tau)]} = F'(T) + x$$

where $F(z)$ and $G(z)$ are arbitrary functions, s is any constant.

4.11 $\boldsymbol{\left[\frac{\partial^2 w}{\partial t^2}\frac{\partial^2 w}{\partial x^2} - \left(\frac{\partial^2 w}{\partial t\partial x}\right)^2\right] V'\left(\frac{\partial w}{\partial x}\right) =}$

$\boldsymbol{a\frac{\partial^2 w}{\partial t\partial x} - b\frac{\partial^2 w}{\partial t^2}\,,}$

where a, b are constants and $V(z) \neq const$ is an arbitrary function.

General solution

$$w(t,x) = \int_s^x W\,d\xi + \int_v^t H\,d\tau + F(T)\,,$$

where $W = W(T,\xi)$ and $H = H(T,\tau)$ are any solutions of the following transcendental equations

$$V(W) - T + b\xi = 0\,,$$

$$G(H) + T + a\tau = 0$$

and $T = T(t,x)$ is any solution of the following transcendental equation

$$\int_s^x \frac{d\xi}{V'[W(T,\xi)]} - \int_v^t \frac{d\tau}{G'[H(T,\tau)]} + F'(T)\,,$$

here $F(z)$ and $G(z)$ are arbitrary functions, s, v are any constants.

4.12 $\left(\frac{\partial^2 w}{\partial t \partial x}\right)^2 - \frac{\partial^2 w}{\partial t^2}\frac{\partial^2 w}{\partial x^2} + b\frac{\partial^2 w}{\partial t^2} + a\frac{\partial w}{\partial x} = ab\,,$

where a, b are constants.

General solution

$$w(t,x) = \frac{at^2 + bx^2}{2} - tG(W) + xW + F(W)\,,$$

where $W = W(t,x)$ is any solution of the following transcendental equation

$$tG'(W) - F'(W) = x$$

and $F(z)$ and $G(z)$ are arbitrary functions.

References

[1] G. Darboux, Sur les equations aux derivees partieles du second ordre. Ann. Sci. Ecole Norm. Sup. 1870, V.7,pp.163-173.

[2] G. Darboux, Lecons sur la theorie generale des surfaces. V.II. Paris: Hermann, 1915.

[3] E.Goursat, Lecons sur l'integration des equations aux derivees partieles du second ordre a deux variables independantes. V.I,II. Paris: Hermann, 1896,1898.

[4] A.R. Forsyth, Theory of differential equations. Part 4. Partial differential equations, vol. 6, Dover Press, New York, 1959.

[5] F. Calogero, A solvable nonlinear wave equation, Studies Appl. Math. 70, pp. 189-199, 1984.

[6] V.M. Boyko, W.I. Fushchych , Lowering of order and general solutions of some classes of partial differential equations, Reports on Math. Phys., V. 41, No. 3, pp. 311-318, 1998.

[7] S.P.Tsarev, On Darboux integrable nonlinear partial differential equations, Proc. Steklov Institute of Mathematics, v. 225, p. 372-381, 1999.

[8] A. V. Zhiber, and V. V. Sokolov, Exact integrable Liouville type hyperbolic equations [in Russian], Uspekhi Mat. Nauk, Vol. 56, No. 1, pp. 64-104, 2001.

[9] A. D. Polyanin and V. F. Zaitsev, Handbook of Nonlinear Partial Differential Equations, Chapman & Hall/CRC Press, Boca Raton, 2004.

[10] Yu.N. Kosovtsov, The decomposition method and Maple procedure for finding first integrals of nonlinear PDEs of any order with any number of independent variables. arXiv:0704.0072v1 [math-ph], http://arxiv.org/abs/0704.0072 , 2007.

5 Appendix. *Maple* procedure *reduce_PDE_order*

```
reduce_PDE_order:=proc(pde,unk)
local B,W,N,NN,ARG,acargs,i,M,pde0,DN,IND,IND2,IND3,IND4,ARGS,SUB,SUB0,
Z0,Bargs,EQS,XXX,WW,BB,PP,pdeI,IV,s,AN,NA;
   option `Copyright (c) 2006-2007 by Yuri N. Kosovtsov. All rights reserved.`;
N:=PDETools[difforder](op(1,[selectremove(has,indets(pde,function),unk)]));
NN:=op(1,[selectremove(has,op(1,[selectremove(has,indets(pde,function),unk)]),diff)]);
ARG:=[op(unk)];
acargs:={};
for i from 1 to nops(ARG) do
if PDETools[difforder](NN,op(i,ARG))=0 then else acargs:=acargs union {op(i,ARG)}
fi; od;
acargs:=convert(acargs,list);
M:=op(0,unk)(op(acargs));
if type(pde,equation)=true then
pde0:=lhs(subs(unk=M,pde))-rhs(subs(unk=M,pde)) else pde0:=subs(unk=M,pde)
fi;
DN:=[seq(seq(i,i=1..nops(acargs)),j=1..N)];
IND:=seq(op(combinat[choose](DN,i)),i=1..N);
IND2:=seq(op(combinat[choose](DN,i)),i=1..N-2);
IND3:=op(combinat[choose](DN,N-1));
IND4:=op(combinat[choose](DN,N));
ARGS:=op(unk),M,seq(convert(D[op(op(i,[IND2]))](op(0,unk))
(op(acargs)),diff),i=1..nops([IND2]));
SUB:={M=W[0],seq(convert(D[op(op(i,[IND]))](op(0,unk))
(op(acargs)),diff)=W[op(op(i,[IND]))],i=1..nops([IND]))};
```

```
SUB0:={W[0]=op(0,unk)(op(ARG)),
seq(W[op(op(i,[IND]))]=subs(M=op(0,unk)(op(ARG)),
convert(D[op(op(i,[IND]))](op(0,unk))(op(acargs)),diff)),i=1..nops([IND]))};
Z0:=B(ARGS,seq(convert(D[op(op(i,[IND3]))](op(0,unk))(op(acargs)),diff),
i=1..nops([IND3])));
Bargs:=op(indets(subs(SUB,Z0),name));
EQS:=convert(subs(SUB,{seq(diff(Z0,op(i,acargs))=0,i=1..nops(acargs))}),diff);
XXX:={seq(W[op(op(i,[IND4]))],i=1..nops([IND4]))};
WW:=select(type,indets(subs(SUB,pde0)), ’name’) intersect
{seq(W[op(op(i,[IND4]))],i=1..nops([IND4]))};
BB:=select(has,combinat[choose](XXX, nops(acargs)),WW);
PP:={}; NA:=0;
pdeI:={seq({subs(subs(solve(EQS,op(i,BB)),subs(SUB,pde0)))},i=1..nops(BB))};
IV:={seq(W[op(op(i,[IND4]))],i=1..nops([IND4]))};
for s from 1 to nops(pdeI) do
try
AN:=[pdsolve(op(s,pdeI),{B},ivars=IV)];
if AN=[NULL] then else NA:=1 fi;
for i from 1 to nops(AN) do
if op(0,lhs(op(i,AN)))=B then
PP:=PP union {rhs(op(i,AN))}
fi;
od;
catch:
end try;
od;
PP:=subs(SUB0,PP);
if NA=1 then if PP= then print(WARNING(”SOLUTION EXISTS”)) fi;fi;
RETURN(PP);
end proc:
```

Calling Sequence: *reduce_PDE_order*(**PDE**, $f(\vec{x})$);

PDE - partial differential equation;
$f(\vec{x})$ - indeterminate function with its arguments.

Notice: The reduced PDE is $B = 0$.