1.4. Heat Equation with Axial Symmetry $\dfrac{\partial w}{\partial t} = a\left(\dfrac{\partial^2 w}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial w}{\partial r}\right)$

This is the heat (diffusion) equation with axial symmetry, where $r = \sqrt{x^2 + y^2}$ is the radial coordinate.

1.4-1. Particular solutions of the heat equation with axial symmetry:

$$w(r) = A + B\ln r,$$
$$w(r,t) = A + B(r^2 + 4at),$$
$$w(r,t) = A + B(r^4 + 16atr^2 + 32a^2t^2),$$
$$w(r,t) = A + B\left(r^{2n} + \sum_{k=1}^{n} \frac{4^k[n(n-1)\dots(n-k+1)]^2}{k!}(at)^k r^{2n-2k}\right),$$
$$w(r,t) = A + B\bigl(4at\ln r + r^2\ln r - r^2\bigr),$$
$$w(r,t) = A + \frac{B}{t}\exp\left(-\frac{r^2}{4at}\right),$$
$$w(r,t) = A + B\exp(-a\mu^2 t)J_0(\mu r),$$
$$w(r,t) = A + B\exp(-a\mu^2 t)Y_0(\mu r),$$
$$w(r,t) = A + \frac{B}{t}\exp\left(-\frac{r^2+\mu^2}{4t}\right)I_0\left(\frac{\mu r}{2t}\right),$$
$$w(r,t) = A + \frac{B}{t}\exp\left(-\frac{r^2+\mu^2}{4t}\right)K_0\left(\frac{\mu r}{2t}\right),$$

where A, B, and μ are arbitrary constants, n is an arbitrary positive integer, $J_0(z)$ and $Y_0(z)$ are the Bessel functions, and $I_0(z)$ and $K_0(z)$ are the modified Bessel functions.

1.4-2. Formulas allowing the construction of particular solutions.

Suppose $w = w(r,t)$ is a solution of the heat equation. Then the functions

$$w_1 = Aw(\pm\lambda r,\ \lambda^2 t + C) + B,$$
$$w_2 = \frac{A}{\delta+\beta t}\exp\left[-\frac{\beta r^2}{4a(\delta+\beta t)}\right]w\left(\pm\frac{r}{\delta+\beta t},\ \frac{\gamma+\lambda t}{\delta+\beta t}\right), \qquad \lambda\delta - \beta\gamma = 1,$$

where A, B, C, β, δ, and λ are arbitrary constants, are also solutions of this equation. The second formula usually may be encountered with $\beta = 1$, $\gamma = -1$, and $\delta = \lambda = 0$.

1.4-3. Boundary value problems for the heat equation with axial symmetry.

For solutions of various boundary value problems, see Subsection 1.5.

References

Carslaw, H. S. and Jaeger, J. C., *Conduction of Heat in Solids*, Clarendon Press, Oxford, 1984.

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