1.5. Heat Equation of the Form $\dfrac{\partial w}{\partial t} = a\left(\dfrac{\partial^2 w}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial w}{\partial r}\right) + \Phi(r,t)$

Nonhomogeneous heat (diffusion) equation with axial symmetry.

1.5-1. Solutions of boundary value problems in terms of the Green's function.

We consider boundary value problems for the nonhomogeneous heat equation with axial symmetry in domain $0 \le r \le R$ with the general initial condition

$$w = f(r) \quad \text{at} \quad t = 0$$

and various homogeneous boundary conditions (the solutions bounded at $r = 0$ are sought). The solution can be represented in terms of the Green's function as

$$w(x,t) = \int_0^R f(\xi)G(r,\xi,t)\,d\xi + \int_0^t\int_0^R \Phi(\xi,\tau)G(r,\xi,t-\tau)\,d\xi\,d\tau.$$

1.5-2. Domain: $0 \le r \le R$. First boundary value problem for the heat equation.

A boundary condition is prescribed:

$$w = 0 \quad \text{at} \quad r = R.$$

Green's function:

$$G(r,\xi,t) = \sum_{n=1}^{\infty} \frac{2\xi}{R^2 J_1^2(\mu_n)} J_0\left(\mu_n \frac{r}{R}\right) J_0\left(\mu_n \frac{\xi}{R}\right) \exp\left(-\frac{a\mu_n^2 t}{R^2}\right),$$

where the μ_n are positive zeros of the Bessel function, $J_0(\mu) = 0$. Below are the numerical values of the first ten roots:

$$\mu_1 = 2.4048,\quad \mu_2 = 5.5201,\quad \mu_3 = 8.6537,\quad \mu_4 = 11.7915,\quad \mu_5 = 14.9309,$$
$$\mu_6 = 18.0711,\quad \mu_7 = 21.2116,\quad \mu_8 = 24.3525,\quad \mu_9 = 27.4935,\quad \mu_{10} = 30.6346.$$

The zeroes of the Bessel function $J_0(\mu)$ may be approximated by the formula

$$\mu_n = 2.4 + 3.13(n-1) \qquad (n = 1,\ 2,\ 3,\ \dots),$$

which is accurate within 0.3%. As $n \to \infty$, we have $\mu_{n+1} - \mu_n \to \pi$.

1.5-3. Domain: $0 \le r \le R$. Second boundary value problem for the heat equation.

A boundary condition is prescribed:

$$\frac{\partial w}{\partial r} = 0 \quad \text{at} \quad r = R.$$

Green's function:

$$G(r,\xi,t) = \frac{2}{R^2}\xi + \frac{2}{R^2}\sum_{n=1}^{\infty} \frac{\xi}{J_0^2(\mu_n)} J_0\left(\frac{\mu_n r}{R}\right) J_0\left(\frac{\mu_n \xi}{R}\right) \exp\left(-\frac{a\mu_n^2 t}{R^2}\right),$$

where the μ_n are positive zeros of the first-order Bessel function, $J_1(\mu) = 0$. Below are the numerical values of the first ten roots:

$$\mu_1 = 3.8317,\quad \mu_2 = 7.0156,\quad \mu_3 = 10.1735,\quad \mu_4 = 13.3237,\quad \mu_5 = 16.4706,$$
$$\mu_6 = 19.6159,\quad \mu_7 = 22.7601,\quad \mu_8 = 25.9037,\quad \mu_9 = 29.0468,\quad \mu_{10} = 32.1897.$$

As $n \to \infty$, we have $\mu_{n+1} - \mu_n \to \pi$.

References

Carslaw, H. S. and Jaeger, J. C., *Conduction of Heat in Solids*, Clarendon Press, Oxford, 1984.

Polyanin, A. D., *Handbook of Linear Partial Differential Equations for Engineers and Scientists* , Chapman & Hall/CRC, 2002.

