3.1. Laplace Equation $\Delta w = 0$

The Laplace equation is often encountered in heat and mass transfer theory, fluid mechanics, elasticity, electrostatics, and other areas of mechanics and physics.

The two-dimensional Laplace equation has the following form:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{in the Cartesian coordinate system,}$$
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2} = 0 \quad \text{in the polar coordinate system,}$$

where $x = r\cos\varphi$, $y = r\sin\varphi$, and $r = \sqrt{x^2 + y^2}$.

3.1-1. Particular solutions and methods for their construction.

1°. Particular solutions of the Laplace equation in the Cartesian coordinate system:

$$\begin{aligned}
w(x,y) &= Ax + By + C,\\
w(x,y) &= A(x^2 - y^2) + Bxy,\\
w(x,y) &= A(x^3 - 3xy^2) + B(3x^2y - y^3),\\
w(x,y) &= \frac{Ax + By}{x^2 + y^2} + C,\\
w(x,y) &= \exp(\pm\mu x)(A\cos\mu y + B\sin\mu y),\\
w(x,y) &= (A\cos\mu x + B\sin\mu x)\exp(\pm\mu y),\\
w(x,y) &= (A\sinh\mu x + B\cosh\mu x)(C\cos\mu y + D\sin\mu y),\\
w(x,y) &= (A\cos\mu x + B\sin\mu x)(C\sinh\mu y + D\cosh\mu y),
\end{aligned}$$

where A, B, C, D, and μ are arbitrary constants.

2°. Particular solutions of the Laplace equation in the polar coordinate system:

$$\begin{aligned}
w(r) &= A\ln r + B,\\
w(r,\varphi) &= \left(Ar^m + \frac{B}{r^m}\right)(C\cos m\varphi + D\sin m\varphi),
\end{aligned}$$

where A, B, C, and D are arbitrary constants, and $m = 1, 2, \ldots$.

3°. A fairly general method for constructing particular solutions involves the following. Let $f(z) = u(x,y) + iv(x,y)$ be any analytic function of the complex variable $z = x + iy$ (u and v are real functions of the real variables x and y; $i^2 = -1$). Then the real and imaginary parts of f both satisfy the two-dimensional Laplace equation,

$$\Delta_2 u = 0, \qquad \Delta_2 v = 0.$$

Thus, by specifying analytic functions $f(z)$ and taking their real and imaginary parts, one obtains various solutions of the two-dimensional Laplace equation.

3.1-2. Domain: $-\infty < x < \infty$, $0 \le y < \infty$. First boundary value problem.

A half-plane is considered. A boundary condition is prescribed:

$$w = f(x) \quad \text{at} \quad y = 0.$$

Solution:

$$w(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y f(\xi)\,d\xi}{(x-\xi)^2 + y^2} = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} f(x + y\tan\theta)\,d\theta.$$

3.1-3. Domain: $0 \le x \le a$, $0 \le y \le b$. First boundary value problem for the Laplace equation.

A rectangle is considered. Boundary conditions are prescribed:

$$w = f_1(y) \quad \text{at} \quad x = 0, \qquad w = f_2(y) \quad \text{at} \quad x = a,$$
$$w = f_3(x) \quad \text{at} \quad y = 0, \qquad w = f_4(x) \quad \text{at} \quad y = b.$$

Solution:

$$w(x,y) = \sum_{n=1}^{\infty} A_n \sinh\left[\frac{n\pi}{b}(a-x)\right] \sin\left(\frac{n\pi}{b}y\right) + \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi}{b}x\right) \sin\left(\frac{n\pi}{b}y\right)$$
$$+ \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{a}x\right) \sinh\left[\frac{n\pi}{a}(b-y)\right] + \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi}{a}x\right) \sinh\left(\frac{n\pi}{a}y\right),$$

where the coefficients A_n, B_n, C_n, and D_n are expressed as

$$A_n = \frac{2}{\lambda_n}\int_0^b f_1(\xi)\sin\left(\frac{n\pi\xi}{b}\right)d\xi, \quad B_n = \frac{2}{\lambda_n}\int_0^b f_2(\xi)\sin\left(\frac{n\pi\xi}{b}\right)d\xi, \quad \lambda_n = b\sinh\left(\frac{n\pi a}{b}\right),$$
$$C_n = \frac{2}{\mu_n}\int_0^a f_3(\xi)\sin\left(\frac{n\pi\xi}{a}\right)d\xi, \quad D_n = \frac{2}{\mu_n}\int_0^a f_4(\xi)\sin\left(\frac{n\pi\xi}{a}\right)d\xi, \quad \mu_n = a\sinh\left(\frac{n\pi b}{a}\right).$$

3.1-4. Domain: $0 \le r \le R$. First boundary value problem for the Laplace equation.

A circle is considered. A boundary condition is prescribed:

$$w = f(\varphi) \quad \text{at} \quad r = R.$$

Solution in the polar coordinates:

$$w(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} f(\psi)\frac{R^2 - r^2}{r^2 - 2Rr\cos(\varphi - \psi) + R^2}\,d\psi.$$

This formula is conventionally referred to as the Poisson integral.

3.1-5. Domain: $0 \le r \le R$. Second boundary value problem for the Laplace equation.

A circle is considered. A boundary condition is prescribed:

$$\partial_r w = f(\varphi) \quad \text{at} \quad r = R.$$

Solution in the polar coordinates:

$$w(r,\varphi) = \frac{R}{2\pi}\int_0^{2\pi} f(\psi)\ln\frac{r^2 - 2Rr\cos(\varphi-\psi) + R^2}{R^2}\,d\psi + C,$$

where C is an arbitrary constant; this formula is known as the Dini integral.

Remark. The function $f(\varphi)$ must satisfy the solvability condition $\int_0^{2\pi} f(\varphi)\,d\varphi = 0$.

References

Babich, V. M., Kapilevich, M. B., Mikhlin, S. G., et al., *Linear Equations of Mathematical Physics* [in Russian], Nauka, Moscow, 1964.

Carslaw, H. S. and Jaeger, J. C., *Conduction of Heat in Solids*, Clarendon Press, Oxford, 1984.

Polyanin, A. D., *Handbook of Linear Partial Differential Equations for Engineers and Scientists*, Chapman & Hall/CRC, 2002.