7a. Systems of nonlinear ODEs with homogeneous right-hand sides

Systems of ODEs with homogeneous polynomial right-hand sides. In general, a system of arbitrarily many, N, first-order ordinary differential equations in $z_1 = z_1(t)$, ..., $z_N = z_N(t)$ with homogeneous polynomial right-hand sides of arbitrary degree, M, reads

$$z_n' = \sum_{m_1+\cdots+m_N=M} c_{nm_1\ldots m_N} z_1^{m_1}\ldots z_N^{m_N}, \qquad n=1,\ldots,N. \tag{1}$$

The sum runs over all nonnegative integer values of the indices m_1, ..., m_N such that $m_1+\cdots+m_N = M$. This indicates that the right-hand sides of system (1) are homogeneous polynomials of degree M, the same for all equations, in the variables z_1, ..., z_N. The numbers N and M are arbitrary positive integers, and the coefficients $c_{nm_1\ldots m_N}$ are t-independent.

The initial value problem for system (1) with initial data $z_1(0)$, ..., $z_N(0)$ admits the particular solution

$$z_n(t) = z_n(0)(1+Kt)^{1/(1-M)}, \qquad n=1,\ldots,N, \tag{2}$$

with K being an arbitrary parameter, provided that the following N explicit algebraic constraints hold:

$$Kz_n(0) = (1-M)\sum_{m_1+\cdots+m_N=M} c_{nm_1\ldots m_N} z_1^{m_1}(0)\ldots z_N^{m_N}(0), \qquad n=1,\ldots,N.$$

Reference

Calogero, F. and Payandeh, F., Explicitly solvable systems of first-order ordinary differential equations with polynomial right-hand sides, and their periodic variants, arXiv:2106.06634v1 [math.DS] 11 Jun 2021 .

A generalization. The more general system of ODEs

$$z_n' = z_n^M F_n\left(\frac{z_2}{z_1},\ldots,\frac{z_N}{z_1}\right), \qquad n=1,\ldots,N,$$

where $F_n(\ldots)$ are arbitrary functions, admits an exact solution of the form (2), where the constants $z_n(0)$ are related by the constraints

$$Kz_n^{1-M}(0) = (1-M)F_n\left(\frac{z_2(0)}{z_1(0)},\ldots,\frac{z_N(0)}{z_1(0)}\right), \qquad n=1,\ldots,N.$$